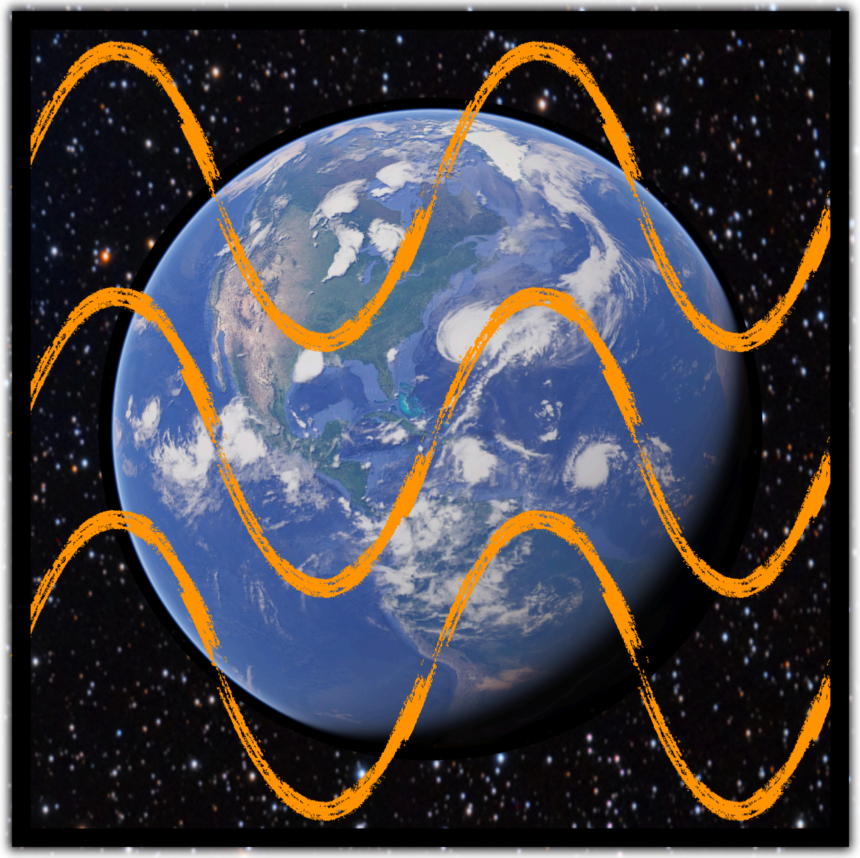

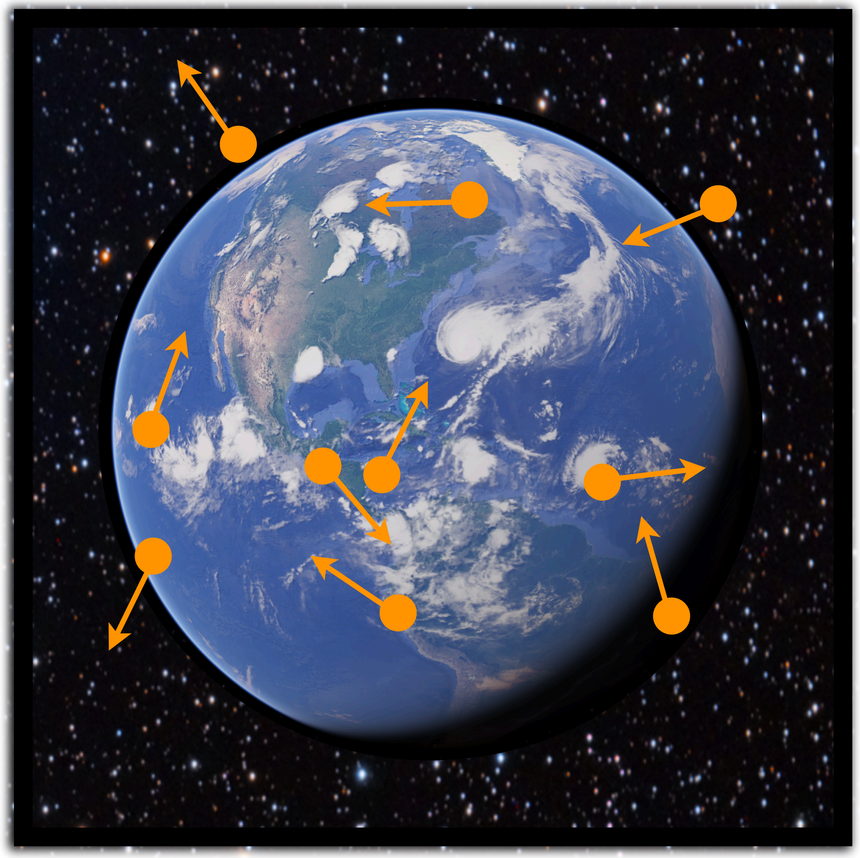

A Quantum Description of Wave Dark Matter

w/ Dhong Yeon Cheong & Lian-Tao Wang

BERKELEY LAB

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Motivation

Establish a more rigorous description of wave DM and the wave-particle boundary

Outline

1. What is the density matrix of dark matter?

2. A rigorous definition of the coherence time

3. A single calculation across the wave-particle boundary

Part I

The Density Matrix of Dark Matter

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Recall, coherent states defined by $\hat{a}\,|\,\alpha\rangle = \alpha\,|\,\alpha\rangle$ are complete (but not orthogonal), so can decompose density matrix as

$$\hat{\rho} = \int d^2\alpha\, P(\alpha)\,|\,\alpha\rangle\langle\alpha\,|$$

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[Glauber 1963],
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See also [Kim, Lenocci 2022]

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Properties of $P(\alpha)$:

$$\hat{\rho}^\dagger = \hat{\rho} \quad \Rightarrow \quad P(\alpha) \in \mathbb{R}$$

$$\mathrm{Tr}[\hat{\rho}] = 1 \quad \Rightarrow \quad \int d^2\alpha\, P(\alpha) = 1$$

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NB: $P(\alpha)$ is not a probability distribution, $P(\alpha) < 0$ allowed

See also [Kim, Lenocci 2022]

The Density Matrix of Dark Matter

[Glauber 1963]: $P(\alpha)$ obeys the central limit theorem
So generally expect (e.g. thermal radiation) that

$$\hat{\rho}_{\mathbf{k}} = \int d^2\alpha_{\mathbf{k}} \underbrace{\left(\frac{1}{\pi \langle N_{\mathbf{k}} \rangle} \exp\left[-\frac{|\alpha_{\mathbf{k}}|^2}{\langle N_{\mathbf{k}} \rangle} \right] \right)}_{P(\alpha_{\mathbf{k}})} |\alpha_{\mathbf{k}}\rangle\langle\alpha_{\mathbf{k}}|$$

Cf. Coherent state: $P(\alpha) = \delta^{(2)}(\alpha - \beta)$

k: mode of the field

See also [Kim, Lenocci 2022]

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density

$\simeq$ for local DM

Axion: $g_s = 1$
Dark photon: $g_s = 3$

e.g. Standard Halo Model

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$$\langle N_{\mathbf{k}} \rangle \simeq \bar{n} \times V_{\text{coherence}} \simeq \#\text{ of indistinguishable particles}$$

Defines wave-particle boundary (given ρ_{DM} etc)
Axions: $m \simeq 14.4$ eV
Dark photons: $m \simeq 11.0$ eV

Scalar Field Statistics

Let's determine the implications for a scalar field

$$\hat{\phi}(t,\mathbf{x}) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} \left(\hat{a}_{\mathbf{k}} e^{-ik\cdot x} + \hat{a}^{\dagger}_{\mathbf{k}} e^{ik\cdot x} \right)$$

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For a single mode

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$\Rightarrow$ ϕ is a Gaussian random field, with

$$\langle \phi(t,\mathbf{x}) \rangle = 0 \;\&\; \langle \phi^2(t,\mathbf{x}) \rangle \simeq \frac{\rho}{m^2}$$

Also $\partial_t \phi \sim \mathrm{Im}[\alpha]$ is an independent Gaussian random field

$P(\alpha)$ Experimentally Testable

Key assumption: Gaussian $P(\alpha)$

May not be true, e.g. coherent state or Bose-Einstein Condensate

BEC: e.g. [Sikivie, Yang 2009] [Erken, Sikivie, Tam, Yang 2012]

Could resolve with experiment (post discovery of DM): look for non-Gaussianities in the fluctuations of ϕ

Part II

The Coherence Time

Autocorrelation function

Having understood $\langle \phi^n(t, \mathbf{x}) \rangle$, natural to next consider

$$\Gamma(\tau, \mathbf{d}) = \langle \phi(t, \mathbf{x}) \phi(t + \tau, \mathbf{x} + \mathbf{d}) \rangle$$

Assume stationary/homogeneous $\Rightarrow \langle \mathcal{O} \rangle$ independent of $(t, \mathbf{x})$

Intuition: how much does knowledge of the field at one point tell you about it at another?

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Also have results for $\mathbf{d} \neq 0$
Cf. [Derevianko 2018]

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For DM, $\omega \simeq m + \frac{1}{2} m v^2$, with v set by e.g.

$$f(\mathbf{v}) = \frac{1}{\pi^{3/2} v_0^3} e^{-(\mathbf{v} + \mathbf{v}_\odot)^2 / v_0^2}$$

Standard Halo Model

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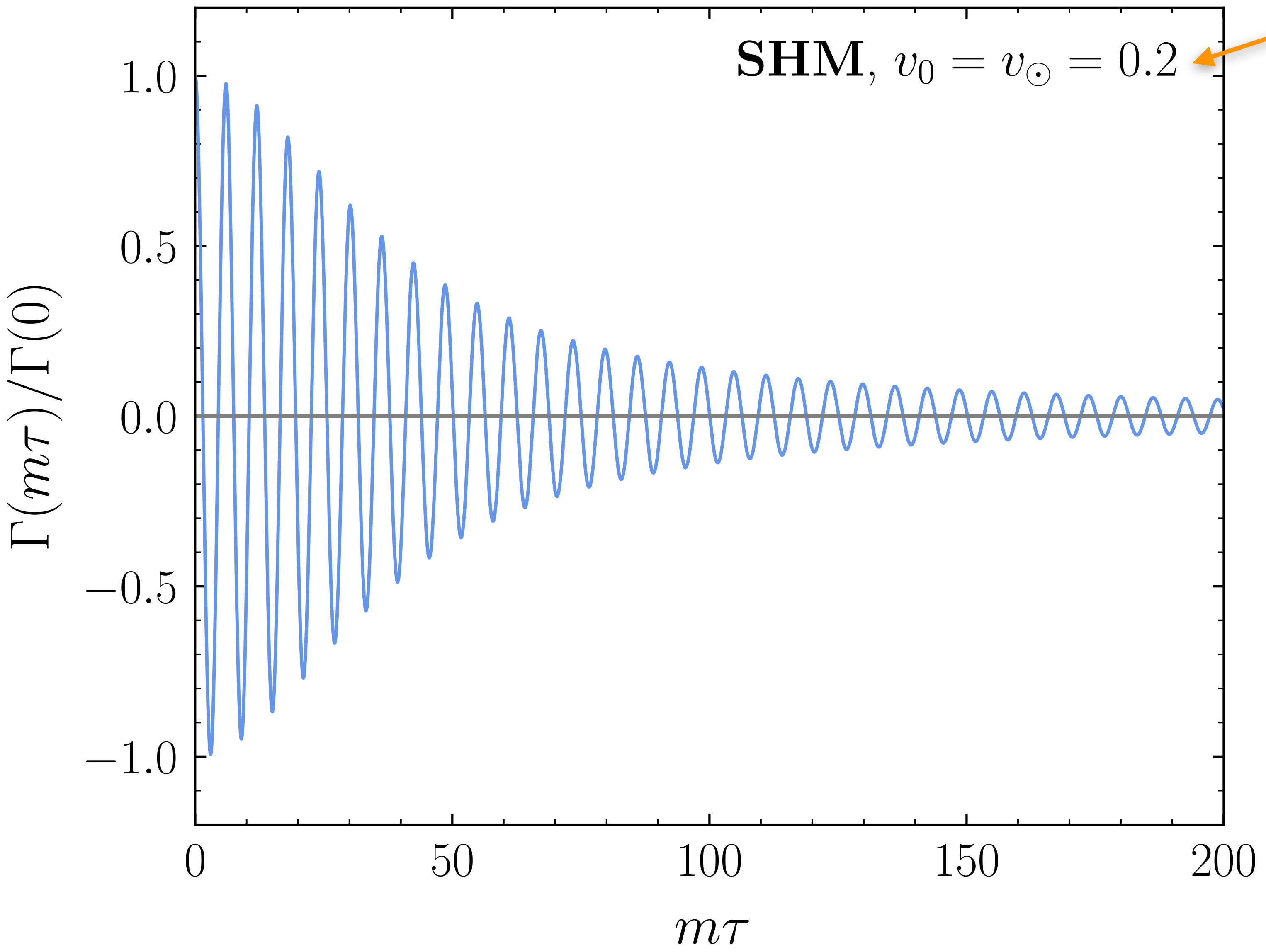

In reality, $v_0 \sim v_\odot \sim 10^{-3}$

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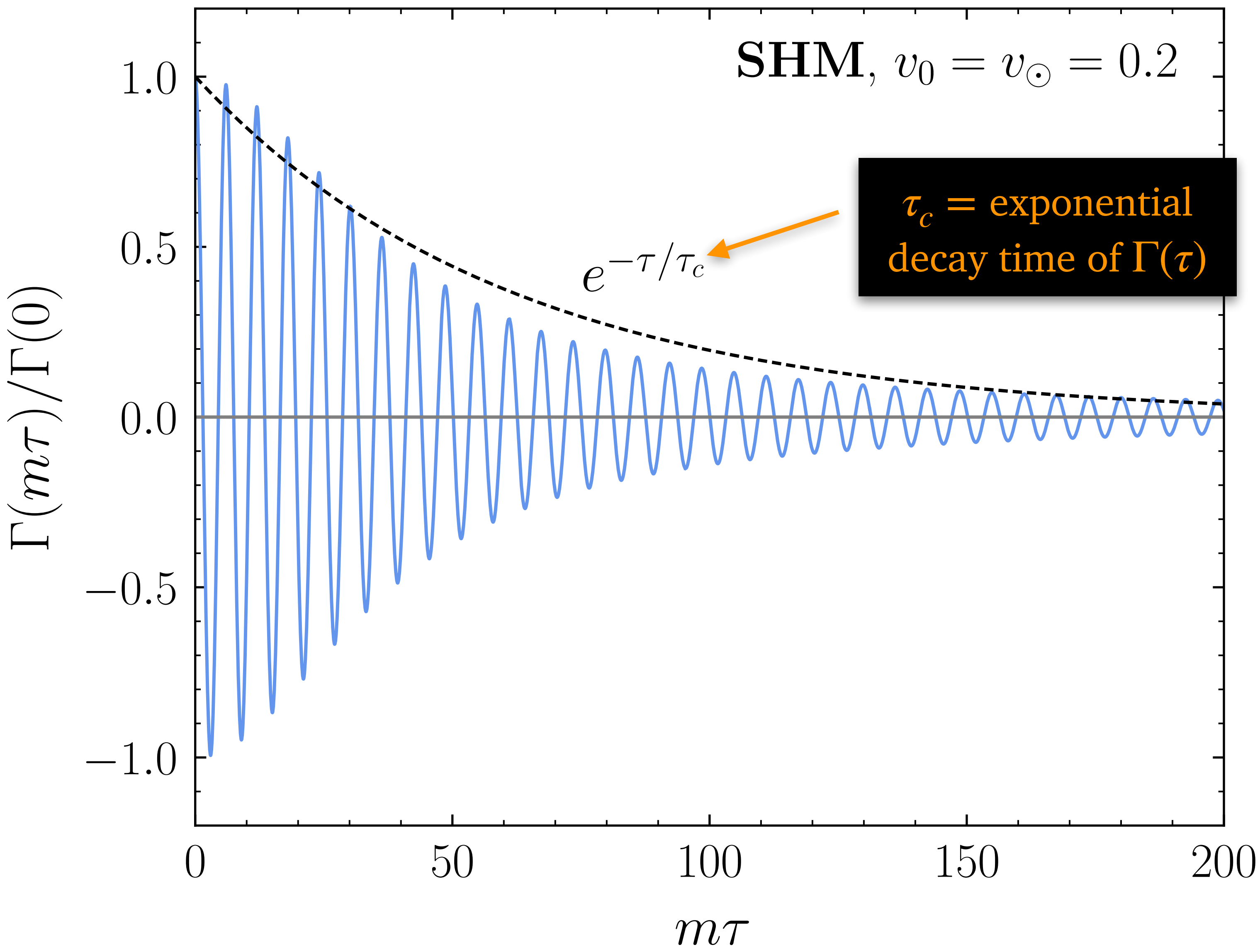

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$$\text{Define: } \tau_c = \int_{-\infty}^{\infty} d\tau \left| \frac{\Gamma(\tau)}{\Gamma(0)} \right|^2$$

Common def. in quantum optics, e.g. [Mandel & Wolf, "Optical Coherence and Quantum Optics"] Cf. [Masia-Roig+ 2023]

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Example 2: DM with the SHM

$$\tau_c = \frac{\sqrt{2\pi}\mathrm{Erf}\left[\sqrt{2}v_\odot/v_0\right]}{mv_0v_\odot}\left(1 + \frac{3v_0^2}{4} - \frac{v_0v_\odot e^{-2v_\odot^2/v_0^2}}{\sqrt{2\pi}\mathrm{Erf}\left[\sqrt{2}v_\odot/v_0\right]} + \mathcal{O}(v^4)\right)$$

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$$\simeq 2.8 \text{ s} \left(\frac{1 \text{ neV}}{m}\right)$$

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Frequency Domain

By the Wiener-Khinchin theorem,

$$S(\omega) = \int_{-\infty}^{\infty} d\tau\, \Gamma(\tau) e^{i\omega\tau}$$

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Cf. [Dror, Murayama, NLR 2021]

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Further, width of $S(\omega)$ is $\Delta\omega = 1/\tau_c$

Intuition: τ_c measures how long
$\phi(t) = \phi_0 \cos(mt)$ is a good approximation
See also [Dror, Gori, Leedom, NLR 2023]

Part III

Wave-Particle Boundary

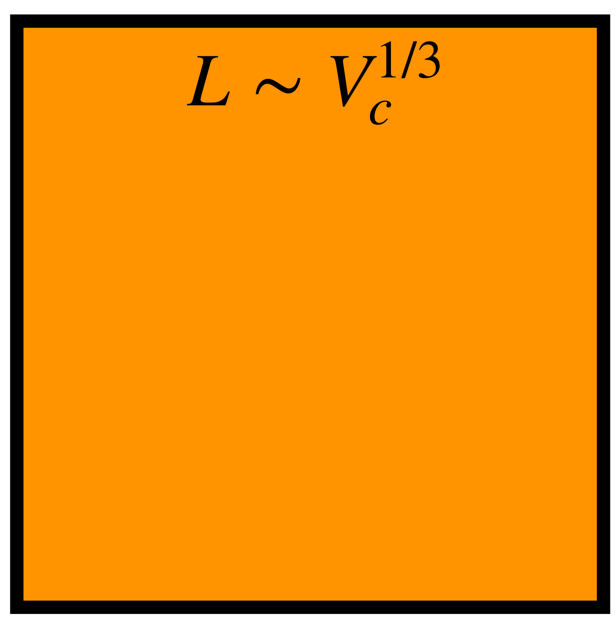
Wave-Particle Boundary

So far $[\hat{a}, \hat{a}^\dagger] \simeq 0$ (justify by $N \gg 1$)

Now $[\hat{a}, \hat{a}^\dagger] = 1$, but for simplicity take a single mode ($\omega = m$)

Question: what is the energy in a box of volume V_c?

Similar result holds for calculation in a finite physical volume



Wave-Particle Boundary

Rewrite Gaussian $\hat{\rho}$ in the number basis

$$\hat{\rho} = \int d^2\alpha \, \frac{e^{-|\alpha|^2/\langle N \rangle}}{\pi \langle N \rangle} \, |\alpha\rangle\langle\alpha|$$

$$= \frac{1}{1+\langle N \rangle} \sum_{k=0}^{\infty} \left(\frac{\langle N \rangle}{1+\langle N \rangle} \right)^k |k\rangle\langle k|$$

Here $k \in \mathbb{N}$, not wavevector!

Can use to show $\mathrm{Tr}[\hat{\rho}^2] = (1+2\langle N \rangle)^{-1}$

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Probability of seeing k quanta in V_c is

$$p(k) = \frac{1}{1+\langle N \rangle} \left(\frac{\langle N \rangle}{1+\langle N \rangle} \right)^k$$

For a single mode: $E = m \times k$, so we can just study k

Wave-Particle Boundary

The mean and standard deviation of k:

$$\mu_k = \langle k \rangle = \langle N \rangle$$

$$\sigma_k^2 = \langle k^2 \rangle - \langle k \rangle^2 = \langle N \rangle (1 + \langle N \rangle)$$

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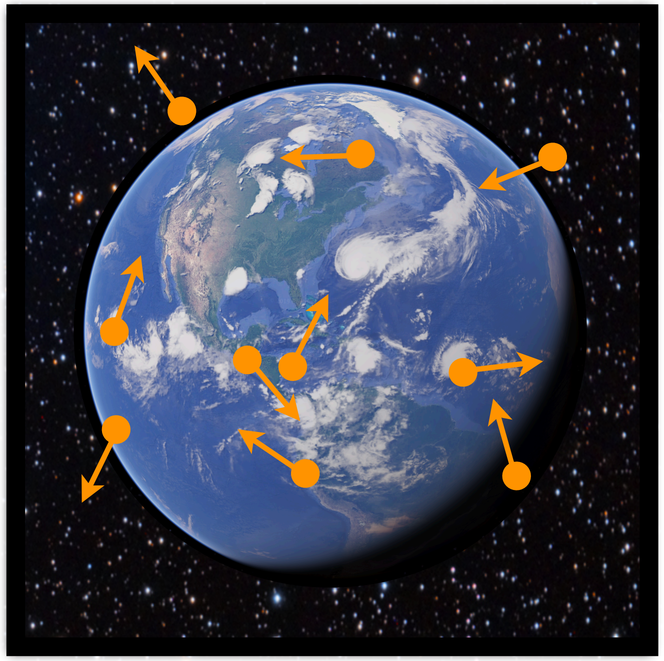
Poisson distributed



Holds for all higher moments

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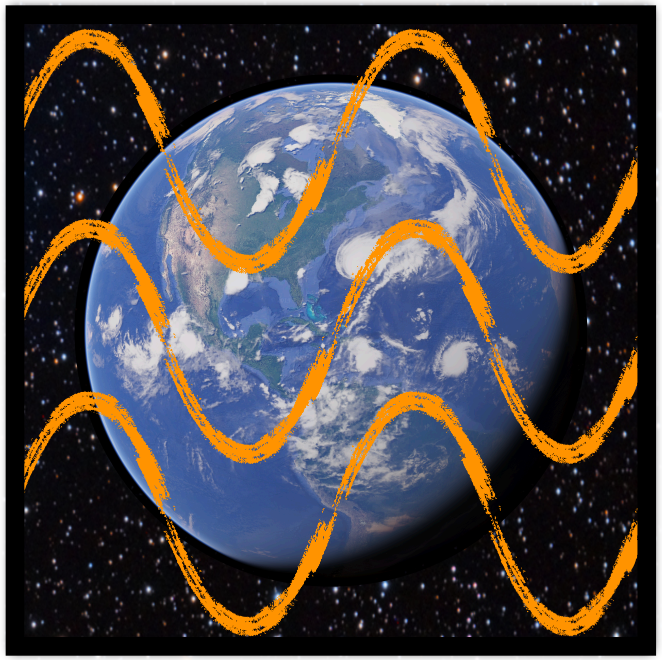
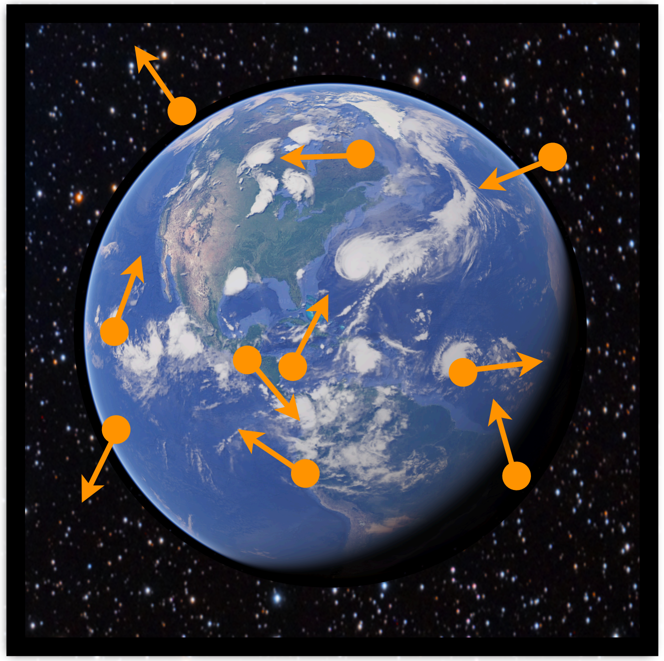
For $\langle N \rangle \gg 1$, $\sigma_k^2 = \mu_k^2$
Exponentially distributed

For $\langle N \rangle \ll 1$, $\sigma_k^2 = \mu_k$
Poisson distributed





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For $\langle N \rangle \sim 1$ neither Poisson nor exponential

Conclusion

The quantum approach opens a path to a rigorous description of wave dark matter

Open questions:

- Determine the exact $P(\alpha)$ of DM
- Interface with experiment (quantum measurement theory)
- Resolve the distribution of polarizations for dark photons
- …

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