Quantum Spacetime Geometry

and

Quantum Relativity

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Prologue : Geometrodynamics

Spacetime Geometry

Noncommutative G. $\Leftrightarrow$ Quantum Gravity

Non-Euclidean G. $\Leftrightarrow$ Classical Gravity

Prologue : Quantum Gravity

(Geometro-)

Dynamics of Quantum Spacetime

rather than

Quantum Dynamics of (Classical)

Spacetime

Lorentz Covariant Quantum Physics :-

• Schrödinger wavefunction $\phi(x^\mu)$

— basic operators x_μ and $-i\hbar\partial_\mu$

• abstract operators as Minkowski four-vectors

— $\hat{X}_i \longrightarrow \hat{X}_\mu$ and $\hat{P}_i \longrightarrow \hat{P}_\mu$

— $[\hat{X}_\mu, \hat{P}_\nu] = i\hbar\eta_{\mu\nu}$

• Heisenberg-Weyl symmetry — $[Y_\mu, E_\nu] = i\hbar\, c\, \eta_{\mu\nu}\, M$

— M **is an effective Casimir element** → **Newtonian mass** m

— $m\hat{X}_\mu \longleftarrow Y_\mu$, different m for different irr. representations

— $\hat{P}_\mu \longleftarrow \frac{1}{c}E_\mu$, constant c (... $c \to \infty$ limit)

Minkowski Metric Operator $\hat{\eta}$ on Krein Space :-

• **Minkowski** nature of proper invariant **inner product**

— effectively, bra as ${}_\eta\langle\cdot| = \langle\cdot|\,\hat{\eta}$

— naive $|\phi(x^\mu)|^2$ **integral** cannot avoid **divergence**

• **observables (pseudo-)Hermitian,** ${}_\eta\left\langle \cdot|\hat{A}^{\dagger^\eta}\cdot\right\rangle = {}_\eta\left\langle \hat{A}\cdot|\cdot\right\rangle$

— $\hat{X}_\mu = \hat{\eta}\hat{X}^\mu\hat{\eta}^{-1}$ and $\hat{P}_\mu = \hat{\eta}\hat{P}^\mu\hat{\eta}^{-1}$

• **noncommutative geometric picture**

— $\hat{X}^\mu$ **and** $\hat{P}^\mu$ **as coordinates**

$H_R(1,3)$ symmetry :-

$$[J'_{\mu\nu}, J'_{\rho\sigma}] = i\hbar\, c \left(\eta_{\nu\sigma} J'_{\mu\rho} + \eta_{\mu\rho} J'_{\nu\sigma} - \eta_{\mu\sigma} J'_{\nu\rho} - \eta_{\nu\rho} J'_{\mu\sigma}\right) \;,$$

$$[J'_{\mu\nu}, Y_\rho] = i\hbar\, c\, (\eta_{\mu\rho} Y_\nu - \eta_{\nu\rho} Y_\mu) \;,$$

$$[J'_{\mu\nu}, E_\rho] = i\hbar\, c\, (\eta_{\mu\rho} E_\nu - \eta_{\nu\rho} E_\mu) \;,$$

$$[Y_\mu, E_\nu] = i\hbar\, c\, \eta_{\mu\nu}\, M$$

- **semi-direct product of $H(1,3)$ and Lorentz symmetry**
- **Casimir elements – M, $\frac{1}{2} T_{\mu\nu} T^{\mu\nu}$, $\frac{1}{4}\epsilon^{\mu\nu\rho\sigma} T_{\mu\nu} T_{\rho\sigma}$**

$$T_{\mu\nu} \equiv M J'_{\mu\nu} - (Y_\mu E_\nu - Y_\nu E_\mu)$$

- **$H_R(1,3)$ irr. reps. as direct products of those of $H(1,3)$ and Lorentz symmetry generated by $T_{\mu\nu}$**

— spin operators $\hat{S}_{\mu\nu} = \frac{1}{mc} \hat{T}_{\mu\nu} = \hat{J}_{\mu\nu} - \hat{L}_{\mu\nu}$

Algebraic Geometry —— Physics :-

Observable Algebra $\mathcal{A} \longleftrightarrow$ **Symplectic Geometry** $\mathcal{S}$

c**-number picture :** Commutative —— Classical

- coordinates as basic observables
- states $[\phi]$ as functionals
- Gel'fand transform : $[\phi](f) = f([\phi]) = f(x,p)|_{[\phi]}$

  $[\phi] : \mathcal{A} \longrightarrow I\!R$ — c-numbers (real numbers) value $[f]_\phi \equiv [\phi](f)$
- value of observable for a state – classical information
- evaluation map (functionals) $[\phi]$ is an algebraic homomorphism
- dynamics given by $\frac{d}{ds} f = \{f, H_s\}$, ($t$ as s)

q-number picture : Noncommutative —— Quantum +

Symplectic Geometry – (projective) Hilbert/Krein space

Observable Algebra $\mathcal{A}$ – operators β

- operator coordinates as basic observables

- states $[\phi]$ as evaluation map : algebraic homomorphism
- $[\beta]_\phi \equiv [\phi](\beta) = [f_\beta(\phi)] = \beta(\hat{x}, \hat{p})|_\phi$

  $[\phi] : \mathcal{A} \longrightarrow \mathbb{Q}$ — q-numbers noncommutative value
- value of observable for a state – quantum information

- dynamics given by $\frac{d}{ds}\beta = \{\beta, \hat{H}_s\}_q \equiv \frac{1}{i\hbar}[\beta, \hat{H}_s]$, ($t$ as s)

The Symplectic Geometry — NC Vs C :-

- **Heisenberg** — $\frac{d}{ds}\alpha(\hat{P}_\mu, \hat{X}_\mu) = \frac{1}{i\hbar}[\alpha(\hat{P}_\mu, \hat{X}_\mu), \hat{H}_s]$

- **Schrödinger** — $\frac{d}{ds} f_\alpha(z^n, \bar{z}^n) = \{f_\alpha(z^n), f_{H_s}\}$

- $f_\alpha(z^n, \bar{z}^n) \equiv \frac{{}_\eta\langle \phi | \alpha(\hat{P}_\mu, \hat{X}_\mu) | \phi \rangle}{{}_\eta\langle \phi | \phi \rangle}$ $\left(|\phi\rangle = \sum_n z^n \, |n\rangle \right)$

— as the pull-back of $\hat{\alpha}$ under $(z^n, \bar{z}^n) \longrightarrow (\hat{P}_\mu, \hat{X}_\mu)$

- $\rightarrow$ **bijective homomorphism between NC Poisson algebras**

— **NC Kähler product** $f_\alpha \star_\kappa f_{\alpha'} = f_{\alpha\alpha'}$ *Cirelli et.al 90*

Heisenberg & Dirac 1925/26 :-

— classical to quantum

only needs a new kinematic

- **H : physical quantities *not* real number variables**
- **Quine : real number as ‘convenient fiction’**
- **D : q-number as the new convenient fiction**

Concept of Numbers (in history) :

- $x + 2 = 0$ $\longrightarrow$ negative numbers
- $2x - 1 = 0$ $\longrightarrow$ rational numbers
- $x^2 - 2 = 0$ $\longrightarrow$ real numbers
- $x^2 + 1 = 0$ $\longrightarrow$ complex numbers
- $xy - x - i = 0$ $\longrightarrow$ (i,2), $(\frac{1}{i-1}$,-i), . . .
- $xy - yx - 1 = 0$ $\longrightarrow$ noncommutative numbers

★ $\hat{x}\hat{p} - \hat{p}\hat{x} - i\hbar = 0$

needs NC/q-number values for the variables

Evaluation as an Algebra Homomorphism :-

— real number is *only* an algebraic system

- classical $\quad [\phi] : f(x_i, p_i) \rightarrow \mathbb{R} \quad$ (observables have real values)

e.g. $\quad E = p^2 + x^2 = pp + xx \quad$ (1-D SHO $m = \frac{1}{2}$, $k = 2$)

$$[\phi](x) = 2,\ [\phi](p) = 3 \quad \Longrightarrow$$

$$[\phi](E) = [\phi](p^2) + [\phi](x^2) = [\phi](p)[\phi](p) + [\phi](x)[\phi](x) = 13$$

$$[\phi](x_i p_i) = [\phi](x_i)[\phi](p_i) = [\phi](p_i)[\phi](x_i) = [\phi](p_i x_i)$$

- **quantum** $\quad [\boldsymbol{\phi}] : \boldsymbol{\beta}(\hat{\boldsymbol{x}}_i, \hat{\boldsymbol{p}}_i) \rightarrow \ ?$

$$[\boldsymbol{\phi}](\hat{\boldsymbol{x}}_i)[\boldsymbol{\phi}](\hat{\boldsymbol{p}}_i) = [\boldsymbol{\phi}](\hat{\boldsymbol{p}}_i)[\boldsymbol{\phi}](\hat{\boldsymbol{x}}_i) + [\boldsymbol{\phi}](\boldsymbol{i\hbar\hat{I}})$$

$\Longrightarrow [\boldsymbol{\phi}](\boldsymbol{\beta}(\hat{\boldsymbol{x}}_i, \hat{\boldsymbol{p}}_i))$ **has to be a noncommutative algebra**

NC values of NC coordinates :-

— NC number as the new convenient fiction

• **state as evaluative homorphism**

— mapping observable algebra to algebra of their NC values

$[\hat{\alpha}]_\phi = \{ f_\alpha|_\phi, V_{\alpha_n}|_\phi \}$ $\quad (V_{\alpha_n} = \frac{\partial f_\alpha}{\partial z^n} = -f_\beta \bar{z}^n + \sum_m \bar{z}^m \langle m|\hat{\alpha}|n\rangle)$

• **Kähler product** — $\quad [\hat{\alpha}\hat{\alpha}']_\phi = [\hat{\alpha}]_\phi \star_\kappa [\hat{\alpha}']_\phi$

$f_{\alpha\alpha'} = f_\alpha f_{\alpha'} + \sum_n V_{\alpha_n} V_{\alpha'_{\bar{n}}}$, $\quad V_{\alpha\alpha'_n} = -f_{\alpha\alpha'} \bar{z}_n + \sum_{m,l} \bar{z}_m \langle m|\hat{\alpha}|l\rangle\langle l|\hat{\alpha}'|n\rangle$

Deutsch & Hayden 00 ; Kong 23

• **locality of quantum information** (**Heisenberg picture**)

Galvão & Hardy 03

• Substituting a **Qubit** for an Arbitrarily Large Number of Classical Bits'

Quantum Mechanics

can and should be seen as

Particle Dynamics on the Quantum Space

rather than

Quantized Dynamics on the Newtonian space

Quantum Relativity Principle :-

- **Penrose : Relativity Principle** $\longrightarrow\otimes\longleftarrow$ **Quantum**

- **Heisenberg picture** – $\hat{x}$ **and** $\hat{p}$ **as coordinates**

— **Noncommutative Geometry for Spacetime**

- **Rel. Sym.** $\longleftarrow$ **Quantum Ref. Frame Transformations**

— ***e.g.*** **translation by the NC value of** $\hat{x}_A - \hat{x}_B$ (ans. Penrose)

- **Quantum Gravity as General Quantum Relativity**

Reference Frame Transformations :-

- $\longleftrightarrow$ **Relativity Symmetry**
  — spacetime symmetry/reference frame

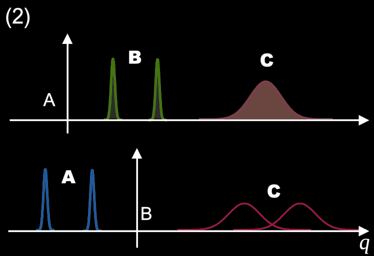
- physical/**quantum frame** Vs absolute/classical frame
  — relative 'uncertainty' / entanglement

- example of **quantum spatial translation**

$$\hat{x}_B^{(A)} \longrightarrow -\hat{x}_A^{(B)} , \qquad \hat{x}_C^{(A)} \longrightarrow \hat{x}_C^{(B)} - \hat{x}_A^{(B)} ,$$
$$\hat{p}_B^{(A)} \longrightarrow -(\hat{p}_A^{(B)} + \hat{p}_C^{(B)}) , \quad \hat{p}_C^{(A)} \longrightarrow \hat{p}_C^{(B)} .$$

- **quantum model of space**(time) — the phase space

The 4 scenarios: Case 2



$$
|\emptyset\rangle_A \otimes \frac{1}{\sqrt{2}} \left(|x_1\rangle + |x_2\rangle\right)_B \otimes \int \mathrm{d}y\, \psi(y) |y\rangle_C
$$

$$
\longrightarrow \frac{1}{\sqrt{2}} \Big(|-x_1\rangle_A \otimes |\emptyset\rangle_B \otimes \int \mathrm{d}y\, \psi(y) |y - x_1\rangle_C
$$

$$
+ |-x_2\rangle_A \otimes |\emptyset\rangle_B \otimes \int \mathrm{d}y\, \psi(y) |y - x_2\rangle_C \Big)
$$

Distance Translated (A-frame to B-frame) ?

a Noncommutative Value :-

- classical $e^{i x_B \hat{p}_C}$: translation by x_B

  — say $x^i_B = 2$, $x^f_C = x^i_C - 2$, $x^f_A = -2$

- quantum : $[\hat{x}_B]^i_\phi$ as the value

  $$[\hat{x}_C]^f_{\phi'} = [\hat{x}_C]^i_\phi - [\hat{x}_B]^i_\phi \,, \quad [\hat{x}_A]^f_{\phi'} = -[\hat{x}_B]^i_\phi$$

  — $[\hat{x}_B]^i_\phi$ contains full quantum information of position

- *answering Penrose : Quantum vs Relativity*

Special Quantum Relativity :-

- QRFT as relative position : $\mathcal{P}_{ON} e^{i\hat{x}^\mu_N \hat{p}_\mu}$,

- quantum observables never take zero value

— translation in single coordinate only as approximation

zero eigenstate of $\hat{x}_N$ keeps $\psi(x)$, but $\hat{p}_o = \ldots$

- otherwise : $\mathcal{P}_{N'N} e^{i\hat{x}_N \hat{p}_{\hat{x}}} \mathcal{P}_{ON'} e^{i\hat{y}_{N'} \hat{p}_{\hat{y}}} \neq \mathcal{P}_{NN'} e^{i\hat{y}_{N'} \hat{p}_{\hat{y}}} \mathcal{P}_{ON} e^{i\hat{x}_N \hat{p}_{\hat{x}}}$

- boost and rotation similarly approx.

Quantum Lorentz Boost :-

$$\hat{z}' = c\hat{\tau}\sinh(\hat{\beta} - \hat{\beta}_N) = \hat{z}\cosh\hat{\beta}_N - c\hat{t}\sinh\hat{\beta}_N \ ,$$

$$c\hat{t}' = c\hat{\tau}\cosh(\hat{\beta} - \hat{\beta}_N) = -\hat{z}\sinh\hat{\beta}_N + c\hat{t}\ \cosh\hat{\beta}_N \ ,$$

$$\hat{p}'_{\hat{z}} = \cosh\hat{\beta}_N\hat{p}_{\hat{z}} + \sinh\hat{\beta}_N\hat{p}_{c\hat{t}} \ ,$$

$$\hat{p}'_{c\hat{t}} = \sinh\hat{\beta}_N\hat{p}_{\hat{z}} + \cosh\hat{\beta}_N\hat{p}_{c\hat{t}} \ .$$

- 'unitary' operator $\mathcal{P}_{ON}e^{i\hat{\beta}_N\hat{p}_{\hat{\beta}}}$, $\qquad \hat{p}_{\hat{\beta}} = \frac{\partial\hat{z}}{\partial\hat{\beta}}\hat{p}_{\hat{z}} + \frac{\partial c\hat{t}}{\partial\hat{\beta}}\hat{p}_{c\hat{t}}$

- approx. by $\mathcal{P}_{ON}e^{i\hat{x}^{\mu}_N\hat{p}_{\mu}}$ with $\hat{x}^{\mu} = (c\hat{\tau}, \hat{z}, \hat{x}, \hat{y})$

— metric not preserved

Towards Gravity : on a particle -

- **quantum geodesic equation** (Heisenberg)

— *e.g.* instantaneous frame of free-fall

$$\frac{d^2\hat{x}^\mu}{ds^2} + \frac{d\hat{x}^\nu}{ds}\Gamma^\mu_{\nu\sigma}(\hat{x})\frac{d\hat{x}^\sigma}{ds} = 0$$

$\longrightarrow$ maintaining (Weak) **Equivalence Principle**

— $\{\hat{x}^a, \hat{p}_b\} = \delta^a_b, \quad \{\hat{x}^a, \hat{x}^b\} = 0 = \{\hat{p}_a, \hat{p}_b\}$

$$\frac{\partial}{\partial \hat{x}^a} \equiv \{\cdot, \hat{p}_a\}, \qquad \frac{\partial}{\partial \hat{p}_a} \equiv -\{\cdot, \hat{x}^a\}$$

Quantum Rindler Frame :-

$$\hat{x} = \hat{\rho}\cosh\frac{\hat{a}_N\hat{\tau}}{c}, \qquad c\hat{t} = \hat{\rho}\sinh\frac{\hat{a}_N\hat{\tau}}{c}$$

- eigenstates : $\hat{x}$ & $\hat{t}$ $\longrightarrow$ $\hat{\rho}$ & $\hat{a}_N\hat{\tau}$

  entanglement between $\hat{\tau}$ and $\hat{a}_N$
- metric : $\hat{g}_{c\hat{\tau},c\hat{\tau}} = \frac{\hat{a}_N^2\hat{\rho}^2}{c^4}$
- quantum geodesic equations :

$$\frac{d^2c\hat{\tau}}{ds^2} + \frac{dc\hat{\tau}}{ds}\frac{1}{\hat{\rho}}\frac{d\hat{\rho}}{ds} + \frac{d\hat{\rho}}{ds}\frac{1}{\hat{\rho}}\frac{dc\hat{\tau}}{ds} = 0\ ,$$

$$\frac{d^2\hat{\rho}}{ds^2} + \frac{dc\hat{\tau}}{ds}\frac{\hat{a}_N^2\hat{\rho}}{c^4}\frac{dc\hat{\tau}}{ds} = 0\ .$$

Quantum Mech. in Curved Spacetime :-

- $\hat{H}_{\rm free} = \frac{1}{2m}\,\hat{p}_{A}\,g^{Ab}(\hat{x})\,\hat{p}_{b}$, $\hat{p}^{a} = g^{ab}(\hat{x})\hat{p}_{b}$

— **Schrödinger representation fails**

- **Hamilton's Eqs.** $\longrightarrow$ **mass-indep. E.O.M.**

- $\hat{x}^{a}$ **and** $\hat{p}^{a}$ **as** $\hat{g}$**-Hermitian within the ref. frame**

⋆ Relativity Principle

...

- all positions coordinates, and their functions, Hermitian

- four vectors : $V'^a = V^i \frac{\partial x'^a}{\partial x^i}$, $W'_a = \frac{\partial x^i}{\partial x'^a} W_i$,
$V'^A \equiv V'^{a\dagger} = \left(\frac{\partial x'^a}{\partial x^i}\right)^\dagger V^{i\dagger} \equiv \left(\frac{\partial x^A}{\partial x'^I}\right) V^I$, $W'_A = W_I \left(\frac{\partial x^I}{\partial x'^A}\right)$.

- p^i and $\frac{dx^i}{ds}$ (not $p_i = g_{iJ} p^J$) $g(x^j)$-Hermitian : RF -dep.

- quantum geodesic equation : $\frac{d^2 x^i}{dt^2} =$

$$\frac{dx^h}{ds} \frac{\partial_J g_{hK}}{2} \frac{dx^K}{dt} g^{Ji} - \frac{dx^h}{ds} \frac{\partial_K g_{hJ}}{2} g^{Ji} \frac{dx^K}{ds} - \frac{dx^k}{ds} g_{kM} \frac{dx^h}{ds} g^{Ml} \frac{\partial_h g_{lJ}}{2} g^{Ji}$$

— s a Hamiltonian evolution parameter

— proper time a quantum observable

Future (H & D $\rightarrow \ldots$) :-

- **q-number physics – Q gravity as GQR**

  more NC physics : $[\hat{x}^i, \hat{x}^j] \neq 0$?

- **q-number geometry – NC Geo. as symplectic coordinate picture : 'Euclidean NC Geo.'**

- **q-number theory – algebra beyond algebra**

- **q-number technology – q-information**

  Quine: To be is to be the (q-number) value of a (physical) variable.

THANK YOU !