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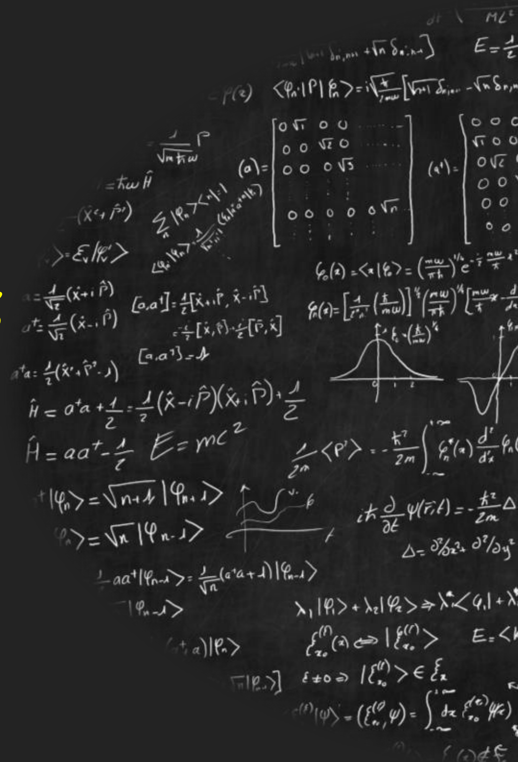

Exact solutions of a three-body problem of Calogero-Marchioro-Wolfes (CMW) type with Coulomb-like confinement in one dimension by the SUSY-QM method.

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Outline:

- Introduction

I. The generalized CMW problem

II. The mathematical formalism of SUSY-QM method.

III. The exact solution of the CMW problem by SUSY-QM method.

- Conclusion

Introduction

There exists a very limited nbr of exactly solvable many-body systems, even in 1dim space.

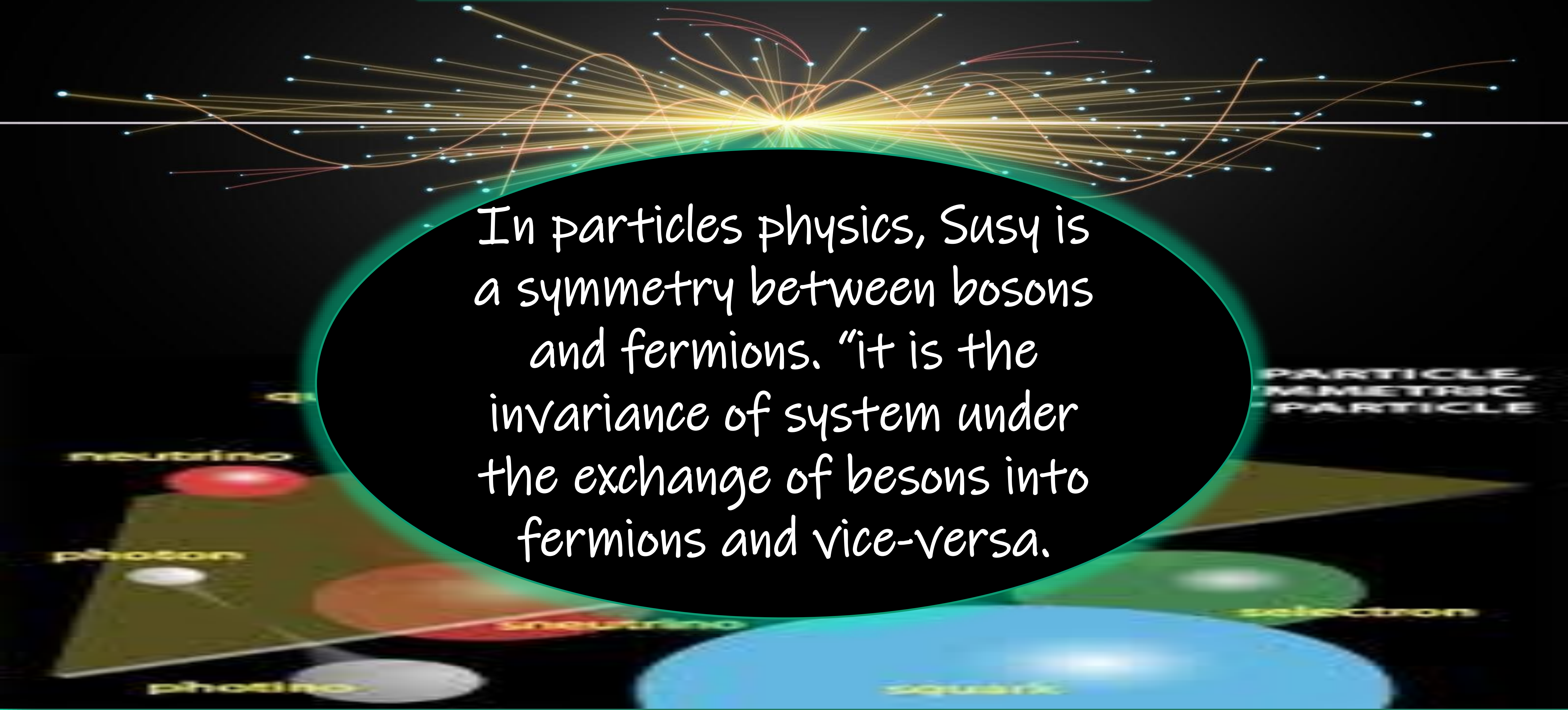
The exact solvability of a quantum problem is related to some kind of intrinsic properties of the problem, such as hidden symmetries.

A survey of many quantum integrable systems was done by Olshanetky and Perelomov.

The Calogero model belongs to the set of the integrable problems that was studied in numerous works with different extensions.

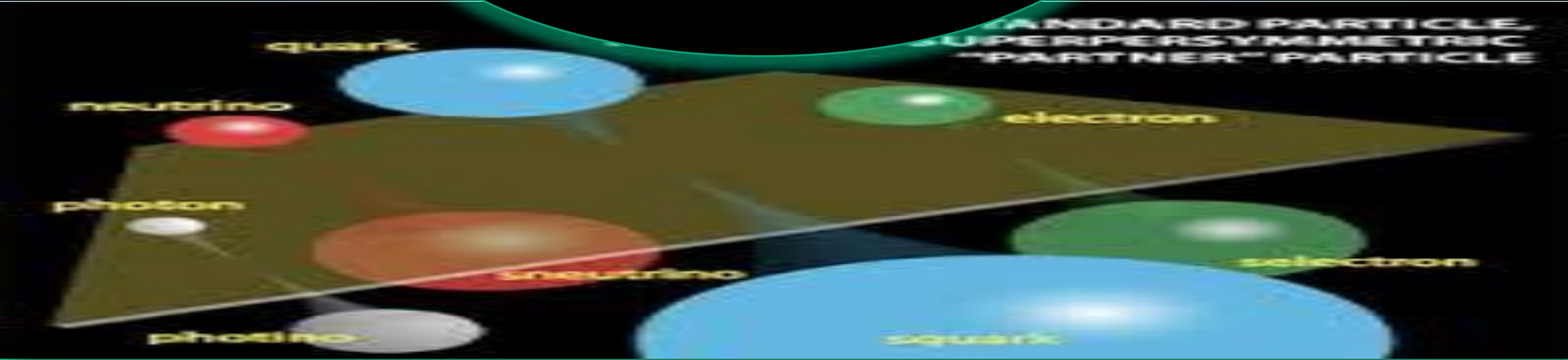
Introduction

In particles physics, Susy is a symmetry between bosons and fermions. "it is the invariance of system under the exchange of besons into fermions and vice-versa.



Introduction

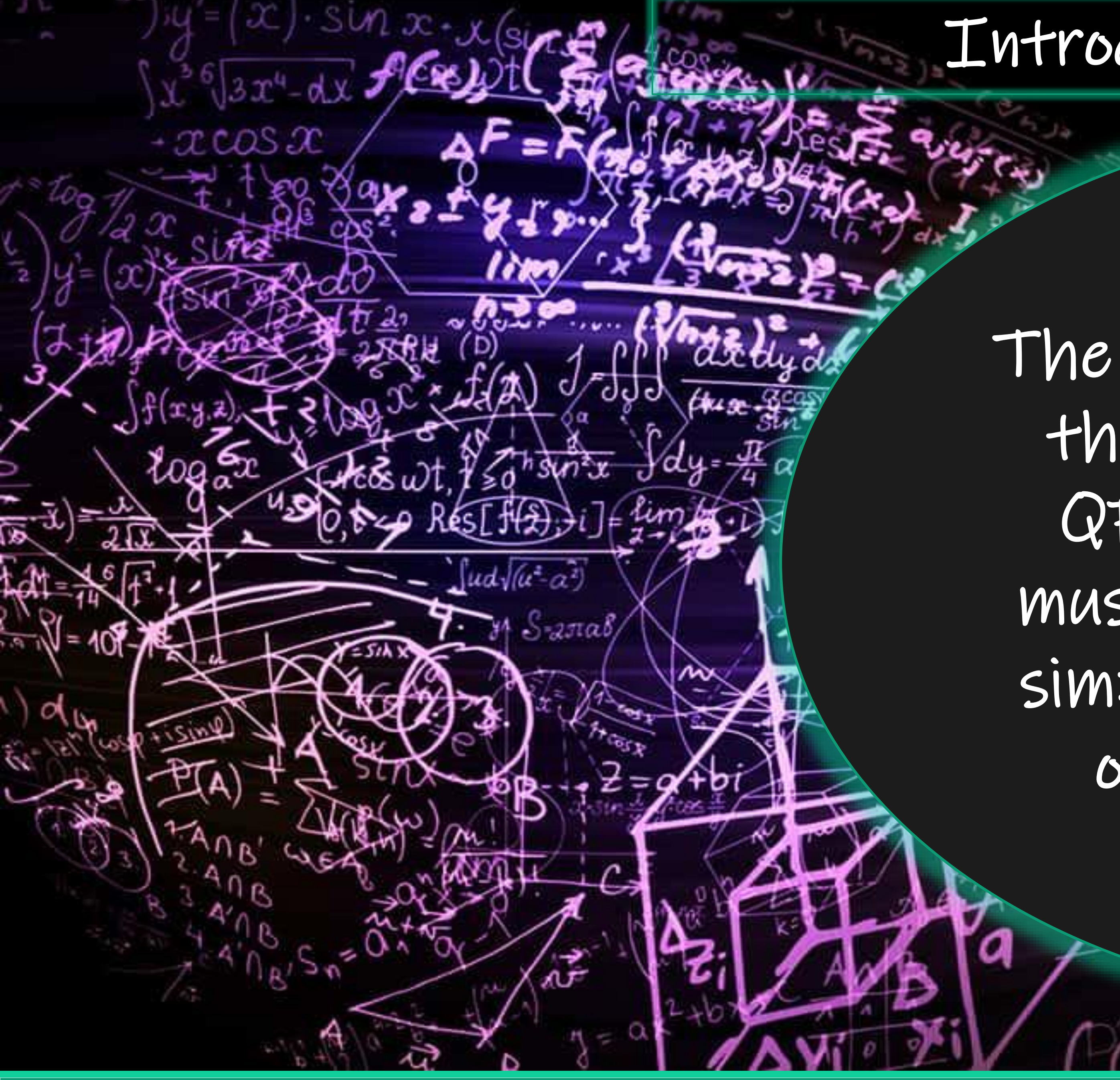
Susy predict to the existence of superpartners to all base nature constituents.



Introduction

There has so far been no experimental evidence of Susy being realized in nature!! which implies that Susy must be spontaneously broken.

Introduction



The difficulty of understand this unusual summetry in QFT implies that Susy must firstly studied in the simplist case i,e in the case of non-relativistic QM

In Quantum Mechanics , Susy is a Mathematical tool allow to find new analytic solvable potentials. It allow to determine both of energy eigenvalues and eigenfunctions with more facility then the operators methods.

Calogero-Marchioro-Wolfes generalistion of 3-body problem in 1-dim (CMW):

The Hamiltonian of the system is:

$$H = \sum_{i=1}^{3}\left(-\frac{\partial^2}{\partial X_i^2} - \frac{\alpha}{(X_1^2 + X_2^2 + X_3^2)^{\frac{1}{2}}}\right) + \lambda.\sum_{i<j}^{3}\frac{1}{(X_i - X_j)^2} + \frac{\mu}{\sum_{i=1}^{3} X_i^2} + 3f.\sum_{\substack{i<j \\ i,j\neq k}}^{3}\frac{1}{(X_i + X_j - 2X_k)^2}$$

Coulombian-like field

two-body inverse square potentials of Calogero type

Non-translationally invariant 3-body potential

3-body potential translationally invariant

we must to make two changes of a coordinates system to solve the Sch eqt corresponding to the above H separately :

1st: from $(x_1, x_2, x_3) \rightarrow (s, t, u)$, 2nd : $(s, t, u) \rightarrow (r, \theta, \varphi)$

The Sch equation corresponds to the above Hamiltonian, in the sphrical coords:

$$\left\{\left[\frac{\partial^2}{\partial r^2}-\frac{2}{r}-\frac{\alpha}{r}+\frac{\mu}{r^2}\right]+\frac{1}{r^2}.\left[-\frac{\partial^2}{\partial\theta^2}-cot\theta\frac{\partial}{\partial\theta}+\frac{M}{sin^2\theta}\right]\right\}\Psi(r,\theta,\varphi)=E.\Psi(r,\theta,\varphi)$$

With: $M=-\frac{\partial^2}{\partial^2\varphi}+\frac{9\lambda}{2cos^2(3\varphi)}+\frac{9f}{2sin^2(3\varphi)}$

Then, we consider the factorization of the wave function as:

$$\Psi(r,\theta,\varphi)=\frac{F(r)}{r}.\frac{\Theta(\theta)}{\sqrt{sin\theta}}.\Phi(\varphi)$$

Which allow to get three differentials eqts of one variable:

$$\left\{\begin{array}{l}\left(-\frac{d^2}{d\varphi^2}+\frac{9\lambda}{2cos^2(3\varphi)}+\frac{9f}{2sin^2(3\varphi)}\right)\Phi(\varphi)=B.\Phi(\varphi)\\ \left(-\frac{d^2}{d\theta^2}+\frac{B-\frac{1}{4}}{sin^2\theta}\right)\Theta(\theta)=D.\Theta(\theta)\\ \left(-\frac{d^2}{dr^2}-\frac{\alpha}{r}+\frac{\mu-\frac{1}{4}+D}{r^2}\right)F(r)=E.F(r)\end{array}\right.$$

Mathematical Formalism of the SUSY-QM Method:

$$H_{\pm} = A^{\pm}A^{\mp} + E_0$$

Super-potential

$$A^{+} = \frac{d}{dx} + W(x,a);\ A^{-} = -\frac{d}{dx} + W(x,a)$$

For the choice of E_0=0;

$$H_{\pm} = -\frac{d}{dx} + v_{\pm}(x,a)$$

Superpartners-potential

With: $v_{\pm}(x,a) = w^{2}(x,a) \pm \frac{dw(x,a)}{dx}$

$H_{\pm}$: are superpartners Hamiltonians, components of supersymmetry H, they are semi-positif different Hamiltonians ($E_n^{\pm} \geq 0$), and they are also characterized by the iso-spectrality property.

Shape invariance property:

If the superpotential of the problem obeys a further constraint « Shape Invariance», then for either H we can derive all the eigenvalues and step-by-step construct all the Eigen functions, i.e the problem become « Exactly Solvable ».

A superpartners potentials are said to be shape invariant (SIP) if they satisfy:

$$V_+(x, a_0) = V_-(x, a_1) + R(a_1) \qquad \text{with: } a_1 = f(a_0)$$

With the eigenenergies of $V_-(x, a_1)$: $E_n^- = \sum_{k=1}^{n} R(a_k)$

$a_k = f^k(a_0)$ $\qquad E_{n=0}^- = 0$

Resolution of the angular equation of the polar angle (φ) :

We suppose the superpotential: $\mathbf{W(A, B, \varphi) = A. \tan(3\varphi) + B. \cot(3\varphi)}$

We calcul the superpartners potentials:

$$v_{\pm}(A, B, \varphi) = \frac{A.(A \pm 3)}{cos^2 3\varphi} + \frac{B(B \mp 3)}{sin^2 3\varphi} - (A - B)^2$$

Verification of the shape invariance property of the superpartners pots:

$$\boldsymbol{v_+(A, B, \varphi) - v_-(A+3, B-3, \varphi) = 12.[a_0 - b_0 + 3] \equiv R(a_1, b_1)}$$

With : $\boldsymbol{a_0} = A$; $\boldsymbol{a_1} = A + 3$, $b_0 = B$; $b_1 = B - 3$

So: $\boldsymbol{v_\pm}((A, B, \varphi)$ are Shape invariant potentials.

Calcul the spectrum of $\boldsymbol{v_-(A, B, \varphi)}$: after conclude the recurrence relation of the shape invariance property $\boldsymbol{R(a_k, b_k) = 12.[a_0 - b_0 + 3.(2k-1)]}$;

$$\boldsymbol{E_n^- = \sum_{k=1}^{n} R(a_k) = 12n(a_0 - b_0 + 3n)\ ;\ n = 0, 1, 2 \ldots\ldots E_{n=0}^- = 0}$$

Rewrite the ang pot included in the polar diff eqt according to the superpartner pot:

$$V(\lambda, f, \varphi) = v_-(A, B, \varphi) + (A - B)^2$$

with: $\frac{9\lambda}{2} = A(A-3)$; $\frac{9f}{2} = B(B+3)$

Calcul of the spectrum of the polar diff eqt according to the spectrum of the superpartners pot: $B_n = E_n^- + (A - B)^2 = [(A - B) + 6n]^2$

To ensure the positvity of E_n^-; we take only the positive value of A, and negative value of B.

As a result: $$B_n = 9.(2n + a + b + 1)^2$$

Where: $a = \frac{1}{2}\sqrt{1 + 2\lambda}$; $b = \frac{1}{2}\sqrt{1 + 2f}$

With: $\lambda > -\frac{1}{2}$; $f > -\frac{1}{2}$

Resolution of the angular equation of the azimuthal angle (θ) :

We suppose the superpotential: $W(A,\theta) = A.\cot\theta$

We calcul the superpartners potentials:

$$v_{\pm} = \frac{A.(A\mp 1)}{\sin^2\theta} - A^2$$

Verification of the shape invariance property of the superpartners pots:

$v_+(\theta, A) - v_-(\theta, A-1) = -2a_0 + 1 \equiv R(a_1)$

With : $a_0 = A$; a_1 =A-1 So: $v_{\pm}(\theta, A)$ are Shape invariant potentials.

Calcul the spectrum of $v_-(\theta, A)$: after conclude the recurrence relation of the shape invariance property $R(a_l) = -2a_0 + (2l-1)$;

where: $a_0 = A\,, a_1 = A-1$

$$E_l^- = \sum_{k=1}^{l} R(a_k) = l(l-2a_0)\,;\ \ l = 0,1,2\ldots..E_{l=0}^- = 0$$

Rewrite the ang pot included in the azimuthal diff eqt according to the superpartner pot:

$$V(b_n, \theta) = V_-(A, \theta) + A^2$$

with: $\mathbf{b_n} = \mathbf{A(A+1)} + \frac{1}{4}$

Calcul of the spectrum of the azimuthal diff eqt according to the spectrum of the superpartners pot: $D_l = E_l^- + A^2 = (l - A)^2$

To ensure the positivty of E_l^-; we take only the negative value of A.

As a result:

$$D_{n,l} = \left(l + \frac{1}{2} + b_n\right)^2, \; l, n = 0, 1, 2 \ldots\ldots$$

With: $b_n = \sqrt{B_n} = 3(2n + a + b + 1)$, $a = \frac{1}{2}\sqrt{1 + 2\lambda}$; $\lambda > -\frac{1}{2}$;

$b = \frac{1}{2}\sqrt{1 + 2f}$; $f > -\frac{1}{2}$

Resolution of the radial equation r :

We suppose the superpotential: $W(r, A, C) = \frac{A}{2.C} - \frac{C}{r}$

We calcul the superpartners potentials:

$$v_{\pm} = -\frac{A}{r} + \frac{C.(C \pm 1)}{r^2} + \frac{A^2}{4.C^2}$$

Verification of the shape invariance property of the superpartners pots:

$$v_{+}(r, A, c) - v_{-}(r, A, c-1) = \frac{A^2}{4}\left(\frac{1}{a_0^2} - \frac{1}{a_1^2}\right) \equiv R(a_1)$$

With : $a_0 = c$; $a_1 = c\text{-}1$ So: $v_{\pm}(r, A, c)$ are Shape invariant potentials.

Calcul the spectrum of $v_{-}(r, A, c)$: after conclude the recurrence relation of the shape invariance property $R(a_k) = \frac{A^2}{4}\left(\frac{1}{a_{k-1}^2} - \frac{1}{a_k^2}\right)$ where: $a_0 = c\,, a_1 = c - 1$

$$E_k^-(A, C) = \sum_{j=1}^{k} R(a_j) = \frac{A^2}{4}.\left(\frac{1}{C^2} - \frac{1}{(C+k)^2}\right); \quad k = 0, 1, 2 \ldots\ldots E_{k=0}^- = 0$$

Rewrite the radial pot included in the radial diff eqt according to the superpartner pot:

$$V(r,\alpha,\mu+D_{n,l}) = v_-(r,A,C) - \frac{A^2}{4C^2}$$

with: $\alpha = A$; $\mu + D_{n,l} - \frac{1}{4} = C(C-1)$

Calculte the spectrum of the radial diff eqt according to the spectrum of the superpartners pot: $E = E_k^-(A,C) - \frac{A^2}{4C^2} = -\frac{A^2}{4(C+k)^2}$

To ensure the positivity of E_k^-; we take only the positive value of C.

As a result:

$$E_{n,l,k} = -\frac{A^2}{4\left(\frac{1}{2}+c_{n,l}+k\right)^2}, \qquad n,l,k = 0,1,2\ldots\ldots$$

With: $c_{n,l} = \sqrt{\mu + D_{n,l}}$, $\mu + D_{n,l} > 0$

Results

As resume : The eigenvalues of the Hamiltonian of the generalized CMW problem 'for binding states' are obtained by:

$$E_{n,l,k} = -\frac{\alpha^2}{\left(1+2c_{n,l}+k\right)^2}$$

$$c_n = \sqrt{\mu + D_{n,l}}$$

$$D_{n,l} = \left(l + \frac{1}{2} + b_n\right)^2$$

$$b_n = 3.\left(2n + a + b + 1\right)$$

$$a = \frac{1}{2}\sqrt{1+2\lambda} \quad ; \quad b = \frac{1}{2}\sqrt{1+2f}$$

$$\lambda > -\frac{1}{2} \quad ; \quad f > -\frac{1}{2}$$

$$\forall (n,l,k) \geq 0 \; ; \; \mu > -\frac{49}{4}$$

Conclusion

- *The entire Spectrum Can be determined Algebraically, without ever referring to underlying differential equations, through the mathematical formalism of SUSY-QM and Shape Invariance property.*

- The Shape Invariance property is sufficient condition to ensure the exact solvability of the problem.

So we can say that: SUSYQM and shape invariance provide an excellent formalism to determine the entire spectrum of solvable quantum systems through a step-by-step algebraic procedure, without any need to solve a differential equation.

The same prb is treated by another algebraic approach which is SGA method, where it based also on the SI condition to generate the algebra and the spectrum of the prb, and we find the same results. Which confirmed once again that SI condition is a sufficient condition for the exact solvability.

This work is represented as poster in the poster sessions.

Thank you for your Attention

Teşekkür ederim