A (VERY) SHORT COURSE IN GENERAL RELATIVITY

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1. Very brief introduction

This is a set of brief lecture notes for a very short course on general relativity delivered at MAPSS 2024 in Les Diablerets.

The notes are mostly based on the excellent differential geometry book *Differential geometry and Lie groups for physicists* by Marián Fecko. Other good literature might be *Geometry, topology and physics* by Mikio Nakahara.

2. Preliminaries/language from differential geometry

- manifold (space) M = nice topological space with a smooth atlas
- tangent vector at $p \in M$ = equivalence class of curves passing through p
- tangent space T_pM = vector space of tangent vectors at p
- cotangent space T_p^*M = dual of T_pM
- (local) frame $\{e_\mu\}$ = collection of bases at each $p \in U \subset M$
- coordinate frame = frame of type $e_\mu = \partial_{x^\mu}$ for some coordinates x^μ
- Einstein summation convention = summation over repeated index assumed
- tensor at p = element of a tensor of type k, l at every point
- tensor field $t \in \mathcal{T}^k_l$ = choice of an element in the above space at every point
- C^∞ = space of smooth functions on M, i.e. $\mathcal{T}^0_0$
- $\mathfrak{X}$ = space of vector fields on M, i.e. $\mathcal{T}^1_0$
- components $t^{ab\cdots}_{cd\cdots}$ of t = coefficients in $t = t^{ab\cdots}_{cd\cdots}(x) e_a \otimes e_b \otimes \cdots \otimes e^c \otimes e^d \otimes \dots$
- differential form = completely antisymmetric tensor field in $\mathcal{T}^0_m$
- Ω^m = space of differential forms of degree m
- de Rham differential = operator d on the space Ω^m of differential forms
- wedge product $\wedge \colon \Omega^m \otimes \Omega^n \to \Omega^{m+n}$ = "multiplication" operation on Ω
- pullback of a tensor field $\alpha \in \mathcal{T}^0_m$ along a map $f \colon M \to N$:

$$f^*\alpha = f^*(\alpha_{\mu\dots\nu}(x) dx^\mu \otimes \cdots \otimes dx^\nu) = \alpha_{\mu\dots\nu}(f(x)) df^\mu(x) \otimes \cdots \otimes df^\nu(x)$$

- Lie derivative $\mathcal{L}_X$ along a vector field X = expresses change along the flow

3. Basic notions in Riemannian geometry

3.1. Metric.

Definition. *A* metric *(or* metric tensor*) is a tensor field $g \in \mathcal{T}^0_2$ which is both*

- *symmetric, i.e. $g(x, y) = g(y, x)$ $\forall x, y \in T_mM$*
- *nondegenerate, i.e. the map*

$$T_mM \to T_m^*M, \quad x \mapsto g(x, \,\cdot\,)$$

  is an isomorphism

at all points $m \in M$.

Definition. *Metric is called* Riemannian *if it is positive definite. Otherwise we call it* pseudo-Riemannian*. A* (pseudo-)Riemannian manifold *is a manifold equipped with a (pseudo-)Riemannian metric.*

In coordinates we have $g_{\mu\nu} = g_{\nu\mu}$, $(g(x,\,\cdot\,))_\mu = g_{\mu\nu}x^\nu$

Convention. *We identify $T_mM \cong T_m^*M$ via $x \mapsto g(x,\,\cdot\,)$. In other words we just write x_μ for $(g(x,\,\cdot\,))_\mu = g_{\mu\nu}x^\nu$.*

This is also the reason why in Riemannian geometry we are careful with the horizontal spacing of indices, e.g. we write $t^{\mu\rho}{}_\nu$ instead of $t^{\mu\rho}_\nu$, for coefficients of a tensor of type $\mathcal{T}^2_1$. One exception is (sometimes) δ^μ_ν.

Exercise. Why don't we care about the horizontal index positioning for δ^μ_ν? ◦

Convention. *We denote the components of the inverse of g by $g^{\mu\nu}$ (instead of $(g^{-1})^{\mu\nu}$), i.e. we have for instance*

$$g^{\mu\nu}g_{\nu\rho} = \delta^\mu_\rho, \qquad g_{\mu\nu}g^{\nu\rho} = \delta^\rho_\mu, \qquad x^\mu = g^{\mu\nu}x_\nu.$$

Definition. *Any submanifold $i\colon N \to M$ of a pseudo-Riemannian manifold (M, g) inherits an* induced metric *i^*g. (Note that if g has indefinite signature then i^*g does not have to be a true metric as it can be degenerate.)*

Exercise. Calculate the induced metric on a unit 2-sphere S^2, embedded in the Euclidean 3-dimensional space, parametrised by

$$x = \sin\vartheta\cos\varphi, \quad y = \sin\vartheta\sin\varphi, \quad z = \cos\vartheta.$$

The result should be $i^*g = d\vartheta \otimes d\vartheta + \sin^2\vartheta\, d\varphi \otimes d\varphi$. ◦

Definition. *The* length *of a curve $\gamma\colon [a, b] \to M$ in a pseudo-Riemannian manifold is defined as*

$$\mathrm{length}(\gamma) := \int_a^b \sqrt{|g(\dot\gamma, \dot\gamma)|}d\tau, \qquad \dot\gamma(\tau) := \tfrac{d}{d\tau}\gamma(\tau) \in T_{\gamma(\tau)}M.$$

Definition. *We say that $\xi \in \mathfrak{X}$ is a* Killing vector field *if $\mathcal{L}_\xi g = 0$.*

3.2. **Connection.**

Definition. *A* connection *(or* covariant derivative*) on a manifold M is a bilinear map $\nabla\colon \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$, denoted $\nabla_X Y := \nabla(X, Y)$, satisfying*

$$\nabla_{fX}Y = f\nabla_X Y, \qquad \nabla_X(fY) = f\nabla_X Y + (Xf)Y, \qquad \forall X, Y \in \mathfrak{X},\ f \in C^\infty.$$

Given a connection ∇, we can also define the action of ∇_X on C^∞ and $\mathcal{T}^0_1$ by

$$\nabla_X f := Xf, \qquad \langle\nabla_X\alpha, Y\rangle := \nabla_X\langle\alpha, Y\rangle - \langle\alpha, \nabla_X Y\rangle, \qquad \forall f \in C^\infty, \alpha \in \mathcal{T}^0_1, Y \in \mathfrak{X}$$

and then to any $\mathcal{T}^p_q$ by requiring that for any tensors t, s we have

$$\nabla_X(t \otimes s) = (\nabla_X t) \otimes s + t \otimes \nabla_X s.$$

Definition. *Let x^μ be any local coordinates on M. We define the* Christoffel symbols *$\Gamma^\mu{}_{\nu\rho}(x)$ by*

$$\nabla_{\partial_\nu}\partial_\rho = \Gamma^\mu{}_{\nu\rho}\partial_\mu.$$

We also set $\Gamma_{\mu\nu\rho} := g_{\mu\sigma}\Gamma^\sigma{}_{\nu\rho}$. Note that these are non-tensorial, i.e. they are not coefficients of a tensor.

Exercise. Calculate (in terms of Christoffel symbols) what is $\nabla_{\partial_\mu}dx^\nu$. Check that

$$(\nabla_X Y)^\mu = X^\nu\partial_\nu Y^\mu + X^\nu\Gamma^\mu{}_{\nu\rho}Y^\rho, \qquad \forall X, Y \in \mathfrak{X}.$$

Write down the corresponding formula for $\nabla_X t$ for an arbitrary tensor field t. ◦

Definition. *Fix a connection ∇. For any pair of vector fields X, Y we define the* curvature operator *as the map*

$$R(X, Y)\colon \mathfrak{X} \to \mathfrak{X}, \qquad R(X, Y) := \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}.$$

Exercise. Check that the expression $R(X,Y)W$ is C^∞-linear in all three of X, Y, W, i.e. $R(fX,Y)W = fR(X,Y)W$, etc., for any function $f \in C^\infty$. Thus the curvature operator is in fact a "pointwise" operator

$$T_mM \otimes T_mM \otimes T_mM \to T_mM,$$

i.e. it is a tensor field of type $\mathcal{T}^1_3$. ◦

Definition. *This tensor field is called the* Riemann tensor $R \in \mathcal{T}^1_3$ *(associated to* ∇*). Explicitly, in any frame* e_a

$$R^a{}_{bcd} := \langle e^a, R(e_c, e_d)e_b\rangle = \langle e^a, ([\nabla_{e_c}, \nabla_{e_d}] - \nabla_{[e_c,e_d]})e_b\rangle.$$

Definition. *The* torsion *of a connection* ∇ *is the map*

$$T \colon \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}, \qquad T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y].$$

Exercise. Check that $T(X,Y)$ is C^∞-linear in X, Y, i.e. it defines a tensor $T \in \mathcal{T}^1_2$. Explicitly, in any frame $T^a{}_{bc} = \langle e^a, \nabla_{e_b} e_c - \nabla_{e_c} e_b - [e_b, e_c]\rangle$. ◦

3.3. **Levi-Civita connection.** Let (M,g) be a pseudo-Riemannian manifold.

Theorem (Fundamental theorem of pseudo-Riemannian geometry)**.** *There exists a unique connection s.t.* $T = 0$ *and* $\nabla g = 0$.

Definition. *This is called the* Levi-Civita connection*.*

Exercise. Prove the theorem and derive the formula for the Levi-Civita connection:

$$\Gamma^\mu{}_{\nu\rho} = \tfrac{1}{2} g^{\mu\sigma}(\partial_\rho g_{\sigma\nu} + \partial_\nu g_{\sigma\rho} - \partial_\sigma g_{\nu\rho}).$$

In particular, note that $\Gamma^\mu{}_{\nu\rho}$ is symmetric in $\nu\rho$. ◦

From now on, ∇ will always be the Levi-Civita connection, unless otherwise stated.

Exercise. Show that (for a Levi-Civita connection) the Riemann tensor has the following symmetries:

$$R_{abcd} = R_{[ab]cd} = R_{ab[cd]} = R_{cdab}, \qquad R_{a[bcd]} = 0$$ ◦

Exercise. How many independent components does a tensor satisfying the above symmetries have (in d dimensions)? ◦

Definition. *Let* (M,g) *be a pseudo-Riemannian manifold.*

- *The* Ricci tensor $R \in \mathcal{T}^0_2$ *is defined as* $R_{ab} := R^c{}_{acb}$.
- *The* scalar curvature $R \in \mathcal{T}^0_0 \cong C^\infty$ *is defined as* $R := g^{ab}R_{ab}$.

Exercise. Show that the Ricci tensor is symmetric. ◦

Exercise. Why do we take the contraction on first and third index in defining the Ricci tensor? (What do we get if we contract other pairs of indices?) ◦

4. Remark on alternative notation

Let us now briefly comment on an alternative notation, which is very often quite convenient, especially with more complex expressions. Returning to the general setup, let us consider any connection ∇ on M. First, since $\nabla_{fX} t = f\nabla_X t$ for any $f \in C^\infty$, $X \in \mathfrak{X}$, $t \in \mathcal{T}^p_q$, we actually obtain a tensor $\nabla t \in \mathcal{T}^p_{q+1}$, i.e.

$$(\nabla t)(X, \dots) = (\nabla_X t)(\dots).$$

Similarly, iterating this construction we obtain

$$\underbrace{\nabla \dots \nabla}_{n \text{ times}} t \in \mathcal{T}^p_{q+n}.$$

Crucially, we will denote the components of this tensor (in a given frame e_a) by putting *indices* on the ∇'s — for instance if the tensor t has components $t^a{}_{bc}$ then

$$(\nabla\nabla\nabla t)_{abc}{}^d{}_{ef} =: \nabla_a\nabla_b\nabla_c t^d{}_{ef}. \tag{1}$$

This is to be contrasted with an expression like

$$\nabla_{e_a}\nabla_{e_b}\nabla_{e_c} t^d{}_{ef},$$

which simply means considering the components $t^a{}_{bc}$ of the tensor t, and taking the derivative of these (seen as functions) in the direction of e_c, then it the direction of e_b, and then e_a, or with an expression like

$$(\nabla_{e_a}\nabla_{e_b}\nabla_{e_c} t)^d{}_{ef},$$

which means taking the covariant derivative of the *tensor* t in the direction of e_c, then taking the covariant derivative of the resulting *tensor* in the direction of e_b and then finally e_a, and then we take the components of the final result. The big advantage of the notation (1) is that the resulting expression forms components of a tensor, which is not the case with the other two formulas.

Exercise. Check that for any tensor field t

$$\nabla_a\nabla_b t^{\cdots}_{\cdots} = (\nabla_{e_a}\nabla_{e_b} t - \nabla_{[e_a,e_b]} t)^{\cdots}_{\cdots} \qquad \circ$$

Exercise. Check that for a torsion-free connection we have the *Ricci identity*

$$[\nabla_a,\nabla_b]X^c := \nabla_a\nabla_b X^c - \nabla_b\nabla_a X^c = R^c{}_{dab}X^d, \qquad \forall X \in \mathfrak{X}. \qquad \circ$$

5. Geodesics

We start with the observation that on a given pseudo-Riemannian manifold (M,g) the natural notion of acceleration for a curve $\gamma\colon [a,b]\to M$ is $\nabla_{\dot\gamma}\dot\gamma$.

Exercise. Check that $\nabla_{\dot\gamma}\dot\gamma$ is well-defined, even though $\dot\gamma$ is not a vector field. $\circ$

In the special case of no acceleration we get the following:

Definition. *A* geodesic *is a curve* $\gamma\colon [a,b]\to M$ *satisfying* $\nabla_{\dot\gamma}\dot\gamma = 0$.

Exercise. Show that this condition can be written as

$$\ddot x^\mu + \Gamma^\mu{}_{\nu\rho}\dot x^\nu \dot x^\rho = 0. \qquad \circ$$

Note that here by "curve" we mean the entire map $[a,b]\to M$, not just its image in M.

Exercise. Show that $g(\dot\gamma,\dot\gamma)$ (the square of the length of the velocity vector) is preserved along the geodesic. $\circ$

Exercise. Show that if we parametrise the geodesic differently, i.e. we replace γ by $\gamma' := \gamma\circ\varphi$ for $\varphi\colon [a,b]\to[a,b]$ a diffeomorphism, then

$$\nabla_{\dot\gamma'}\dot\gamma' = f\dot\gamma', \tag{2}$$

where f is a function on $[a,b]$. Conversely, if we start with a solution of (2) we can reparametrise it to obtain $\nabla_{\dot\gamma}\dot\gamma = 0$. $\circ$

Proposition. *The geodesic equation is the Euler–Lagrange equation for*

$$S(\gamma) := \tfrac{1}{2}\int_a^b g(\dot\gamma,\dot\gamma)d\tau.$$

Exercise. Show this. $\circ$

Note that this provides a very easy way to derive the Christoffel symbols.

Exercise. Calculate the Christoffel symbols on S^2 and find out if there are any geodesics with constant φ or constant ϑ. ○

Proposition. *The Euler–Lagrange equation for the (reparametrisation-invariant) functional*

$$\mathrm{length}(\gamma) = \int_a^b \sqrt{|g(\dot\gamma, \dot\gamma)|}d\tau$$

has the form (2)*, i.e. the length-extremising curves are precisely geodesics up to parametrisation.*

Exercise. Show this. ○

Exercise. Are geodesics always local *minima* of the length functional? (hint: consider going from Quito to Campala by taking the "wrong way"). ○

6. Normal coordinates and applications

Fix a pseudo-Riemannian manifold (M, g). Note that for every point $m \in M$ and for every $v \in T_mM$ there exists a unique geodesic γ^v with $\gamma^v(0) = m$ and $\dot\gamma^v(0) = v$ defined on some interval $[-\epsilon, \epsilon]$.

Proposition. *There exists an open set $U \subset T_mM$, containing the origin, such that the* exponential map

$$\exp\colon U \to M, \qquad v \mapsto \gamma^v(1)$$

is a diffeomorphism between U and a neighbourhood of $m \in M$.

Definition. *Fix a point $m \in M$. Let e_μ be any orthonormal basis*[1] *of T_mM, giving (linear) coordinates x^μ on U. The corresponding local coordinates (also denoted x^μ) on M, defined via the exponential map, are called* normal coordinates*.*

Proposition. *In normal coordinates we have $\partial_\mu g_{\nu\rho} = 0$ at the point m. In particular $\Gamma^\mu{}_{\nu\rho}(m) = 0$.*

It is extremely useful to know that these coordinates exist, even without knowing their explicit form. To see this in practice, consider the following coordinate expressions for the Lie derivative and the exterior differential:

$$(\mathcal{L}_XY)^i = X^j\partial_jY^i - Y^j\partial_jX^i, \qquad (d\alpha)_{ij} = 2\partial_{[i}\alpha_{j]}, \qquad X, Y \in \mathfrak{X},\ \alpha \in \mathfrak{T}^0_1. \tag{3}$$

We now claim that in all these expressions we can replace ∂ by ∇, and the equalities will still hold, i.e.

$$(\mathcal{L}_XY)^i = X^j\nabla_jY^i - Y^j\nabla_jX^i, \qquad (d\alpha)_{ij} = 2\nabla_{[i}\alpha_{j]}, \qquad X, Y \in \mathfrak{X},\ \alpha \in \mathfrak{T}^0_1. \tag{4}$$

To see this, we reason as follows. First, note that both the expressions

$$X^j\partial_jY^i - Y^j\partial_jX^i \qquad \text{and} \qquad X^j\nabla_jY^i - Y^j\nabla_jX^i$$

are tensorial (i.e. are components of a tensor) — in the first case this is because it is merely the component of the Lie derivative. Second, if we take the normal coordinates around any point m, the two expressions coincide (since $\Gamma^\mu{}_{\nu\rho}(m) = 0$). However, the equality of any two given tensors is independent of any coordinate choice, i.e. the two expressions must coincide in any coordinate system, which establishes the equivalence. The same argument holds for d. In fact, for the Lie derivative of any tensor field, and for d of any differential form, we can always make the replacement

$$\partial_i \to \nabla_i.$$

[1] This is the basis such that all members of the basis are orthogonal to each other, and the inner product of each member with itself is ± 1 (depending on what is the signature of the metric).

Finally, note that in the expression (4) (in contrast to (3)) we can in fact use any frame, not just a coordinate frame (i.e. frame of a type ∂_{x^μ} for some coordinates x^μ).

Exercise. Check that such a reasoning is no longer possible if we have an expression which is second (or higher) order in derivatives. ○

Exercise. Show that in normal coordinates around point m we have at that point:

$$R_{\mu\nu\rho\sigma} = -(\partial_\mu \partial_{[\rho} g_{\sigma]\nu} - \partial_\nu \partial_{[\rho} g_{\sigma]\mu}).$$ ○

Exercise. Use this, together with the above reasoning, to easily derive the symmetries of the Riemann tensor (which hold in any coordinates/frame). ○

Exercise. Show that the Killing equations can be written as

$$\nabla_{(a}\xi_{b)} = 0.$$ ○

7. Einstein equations and Einstein–Hilbert action

7.1. **An aside on the metric volume form.** Consider an oriented (n-dimensional) pseudo-Riemannian manifold. For any point $m \in M$, pick any oriented orthonormal basis e_a of $T_m M$. Define

$$\omega_g := e^1 \wedge \cdots \wedge e^n.$$

Exercise. Show that ω_g is independent of the choice of the particular frame e_a. ○

Definition. *This gives a globally-defined and nowhere vanishing differential form ω_g of the top degree, called the* metric volume form.

Proposition. *In local (oriented) coordinates x^μ*

$$\omega_g = \sqrt{|\det g_{\mu\nu}|}\, dx^1 \wedge \cdots \wedge dx^n.$$

7.2. **Einstein equations.** We wish to write down some equations governing the dynamics of spacetime, and we wish them to take the form

some sort of curvature $= \kappa \times$ something describing the matter content,

where $\kappa \in \mathbb{R}$ is some constant. Physics supplies a pretty good candidate for the RHS, namely the *stress-energy tensor.* This is a symmetric tensor $T_{\mu\nu}$ which is conserved, i.e.

$$\nabla^\mu T_{\mu\nu} = 0.$$

Let's see if we can find a suitable candidate for the LHS. Furthermore, we want to follow the usual physical lore which suggests to find an expression that is at most of second order in derivatives of the physical field, i.e. in this case the metric. A reasonable quantity which has the required form (symmetric tensor $\mathfrak{T}^0_2$) is

$$E_{\mu\nu} := aR_{\mu\nu} + bRg_{\mu\nu} + cg_{\mu\nu},$$

for some constants $a, b, c \in \mathbb{R}$. We can absorb one of the constants in the redefinition of κ; we will use this to set $a = 1$. Next, we need to impose $\nabla^\mu E_{\mu\nu} = 0$. For this we will use the *contracted Bianchi identity*

$$\nabla^\mu R_{\mu\nu} = \tfrac{1}{2}\nabla_\nu R.$$

Exercise. Check that (with $a = 1$) we have $\nabla^\mu E_{\mu\nu} = 0$ if and only if $b = -\frac{1}{2}$. ○

Finally, renaming the constant c to Λ, we obtain the *Einstein equations*

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}.$$

The constants Λ and κ are called the *cosmological constant* and the *Einstein gravitational constant*, respectively.

Exercise. Show that in the vacuum, i.e. for $T_{\mu\nu}$ and Λ vanishing, the Einstein equations are equivalent to $R_{\mu\nu} = 0$ (provided the dimension is more than 2). ○

7.3. **Einstein–Hilbert action.**

Definition. *Define the* Einstein–Hilbert action *as*

$$S_{EH}(g) = \frac{1}{2\kappa}\int_M (R - 2\Lambda)\omega_g.$$

Proposition. *Einstein equations arise as the Euler–Lagrange equations for*

$$S_{EH}(g) + S_m(\varphi, g),$$

where S_m is the matter action, and

$$T^{\mu\nu} = -\frac{2}{\omega_g}\frac{\delta S_m}{\delta g_{\mu\nu}}.$$

8. Dictionary between general relativity and differential geometry

8.1. **A brief reminder of special relativity.**

- The spacetime in special relativity is described by the *Minkowski space* $\mathbb{M} := \mathbb{R}^4$, with the following metric of signature $(3,1)$:
$$\eta = \operatorname{diag}(-1, 1, 1, 1).$$
The first direction (denoted t) is the *time*, other directions (denoted x, y, z) correspond to *space*, thus
$$\eta = -dt \otimes dt + dx \otimes dx + dy \otimes dy + dz \otimes dz.$$
This corresponds to a "viewpoint" of a fixed "inertial" observer (in these units we set the speed of light to 1).
- Points of $\mathbb{M}$ are called *events* (they correspond to things that happen at a precise time and in a precise location). Real movement of a particle traces out a curve γ in $\mathbb{M}$, called the *worldline*. The space of vectors $v \in \mathbb{M}$ with $g(v, v) = 0$ is called the *lightcone*. Massive particles move along curves which are contained within this lightcone, i.e. with $g(\dot\gamma, \dot\gamma) < 0$.
- The group of isometries of $\mathbb{M}$ is called the *Poincaré group*. It can be described as $SO(3,1) \ltimes \mathbb{R}^4$, where $\mathbb{R}^4$ corresponds to translations and $SO(3,1)$ is corresponds to transformations fixing the origin in $\mathbb{M}$ and is called the *Lorentz group*. An isometry of $\mathbb{M}$ is interpreted as passing to a coordinate system corresponding to a different observer. More explicitly,
  - elements in $\mathbb{R}^4$ correspond to translated observers
  - elements of $SO(3) \subset SO(3,1)$ correspond to rotated observers
  - observer moving with a constant velocity $\vec{u} = (u_x, u_y, u_z)$ and passing through the origin at time 0 corresponds to a *(Lorentz) boost*; in particular if the velocity is $(u, 0, 0)$ then the transformation is
$$\begin{pmatrix} \gamma & -u\gamma & 0 & 0 \\ -u\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \gamma := \frac{1}{\sqrt{1-u^2}}$$
in other words, boosts in the t, x subspace correspond to hyperbolic rotations
$$\begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix} \in SO(1,1),$$
where θ (defined by $\tanh\theta = u$) is called *rapidity*

– other elements in (the connected component of the identity of) the Poincaré group can be obtained by combining the above
- The time measured by the clock of any observer moving along some curve $\gamma(\tau)$ in $\mathbb{M}$ corresponds to the length (measured using η) of the curve, i.e.

$$\text{time from } a \text{ to } b \ = \int_a^b \sqrt{|\eta(\dot\gamma,\dot\gamma)|}d\tau$$

8.2. **Dictionary in general relativity.**
- Spacetime is a pseudo-Riemannian manifold of signature $(3,1)$, potentially also carrying some "matter" (described in terms of some fields whose dynamics is governed by S_m), satisfying the Einstein equations.
- Test particles[2] move along geodesics.
- Normal coordinates correspond to a freely falling observer — such an observer feels that the surrounding metric looks at first approximation like the Minkowski metric, and then deviates from it in second order, corresponding to the fact that in normal coordinates $g_{\mu\nu}(x) = \eta_{\mu\nu} + \mathcal{O}(x^2)$.
- Time measured by the clock of any observer moving along some curve corresponds to the length of the curve.

9. Cartan structure equations

Let us now try to reformulate the story of (general) connections in terms of differential forms.

Definition. *Consider a connection ∇ on M, and choose a local frame e_a. We then define the associated forms*[3] $\omega^a{}_b \in \Omega^1$, $T^a \in \Omega^1$, $\Omega^a{}_b \in \Omega^2$ *by*

$$(\omega^a{}_b)_c := \Gamma^a{}_{cb}, \qquad (T^a)_{bc} := T^a{}_{bc}, \qquad (\Omega^a{}_b)_{cd} := R^a{}_{bcd}.$$

Exercise. Show that under the change of frame $e'_a = A^b{}_a e_b$ we have

$$\omega'^a{}_b = (A^{-1})^a{}_c \omega^c{}_d A^d{}_b + (A^{-1})^a{}_c dA^c{}_b, \text{ or in short } \omega' = A^{-1}\omega A + A^{-1}dA. \quad \circ$$

Proposition. *The following Cartan structure equations hold*

$$de + \omega \wedge e = T, \quad d\omega + \omega \wedge \omega = \Omega,$$

or, written more fully,

$$de^a + \omega^a{}_b \wedge e^b = T^a, \quad d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = \Omega^a{}_b.$$

Exercise. Show this. ∘

Corollary. *Let now (M, g) be a pseudo-Riemannian manifold, ∇ the Levi-Civita connection, and e_a a local orthonormal frame. Then $\omega_{ab} := \eta_{ac}\omega^c{}_b$ are antisymmetric, and we have*

$$de^a + \omega^a{}_b \wedge e^b = 0, \quad d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = \Omega^a{}_b.$$

Let us now look at the particularly interesting case of a Riemannian (i.e. positive-definite) metric in 2 dimensions.

Exercise. Show that in this case, if e_a is an orthonormal frame,[4]

$$R_{abcd} = K\epsilon_{ab}\epsilon_{cd}, \qquad R_{ab} = K\delta_{ab}, \qquad R = \tfrac{1}{2}K, \qquad \Omega_{ab} = K\epsilon_{ab}e^1 \wedge e^2, \tag{5}$$

for some function K (called *Gaussian curvature*). ∘

[2] i.e. small objects whose presence doesn't affect the surrounding spacetime

[3] Here $\Gamma^a{}_{bc}$ is the obvious extension of the definition of Christoffel symbols (defined for a coordinate basis ∂_μ) to an arbitrary frame e_a.

[4] ϵ_{ab} here is the completely antisymmetric tensor normalised such that $\epsilon_{12} = 1$.

Similarly, in an orthonormal basis we have $\omega_{ab} = \epsilon_{ab}\alpha$, for some 1-form α. The Cartan structure equations then give a particularly nice system:

$$de^1 + \alpha \wedge e^2 = 0, \qquad de^2 - \alpha \wedge e^1 = 0, \qquad d\alpha = K e^1 \wedge e^2. \tag{6}$$

This gives the following simple algorithm for determining all the curvature tensors for a given 2d Riemannian space:

- Find an orthonormal coframe e^1, e^2. This can be often quickly read off from the metric, e.g. for S^2 with $g = d\vartheta \otimes d\vartheta + \sin^2\vartheta d\varphi \otimes d\varphi$ we can take $e^1 = d\vartheta$, $e^2 = \sin\vartheta d\varphi$)
- Calculate α from the first two equations in (6).
- Calculate K from the last equation.
- Express the desired curvature tensors using (5).

Exercise. Calculate the curvature tensors for S^2. ◦

Exercise. Consider the 2-dimensional surface embedded in $\mathbb{R}^3$ as

$$x = (a + b\sin\psi)\cos\varphi, \qquad y = (a + b\sin\psi)\sin\varphi, \qquad z = b\cos\psi.$$

What is this? Calculate the Gaussian curvature. ◦

10. Some famous solutions

Let us now very briefly discuss some famous solutions to Einstein equations.

10.1. Maximally symmetric spacetimes.

This subsection applies to any dimension $d \geq 3$. Before listing the options, let us define the pseudo-Riemannian space $\mathbb{M}_{p,q}$ to be $\mathbb{R}^{p+q}$ with constant diagonal metric of signature (p, q).

The maximally symmetric spacetimes of signature $(d-1, 1)$, which solve the Einstein equations with $T = 0$, are now:

- Minkowski space $\mathbb{M}_{d-1,1}$
- *de Sitter space*

$$\mathrm{dS}_d := \left\{ x \in \mathbb{M}_{d,1} \text{ s.t. } |x|^2 = \frac{(d-1)(d-2)}{2\Lambda} \right\}, \quad \Lambda > 0$$

- *anti-de Sitter space*

$$\mathrm{AdS}_d := \left\{ x \in \mathbb{M}_{d-1,2} \text{ s.t. } |x|^2 = \frac{(d-1)(d-2)}{2\Lambda} \right\}, \quad \Lambda < 0$$

The parameter Λ corresponds to the value of the cosmological constant for these solutions. (Minkowski spacetime has $\Lambda = 0$.) We see that both de Sitter and anti-de Sitter spacetimes can be understood as (pseudo)spheres.

Exercise. What is the topology of de Sitter and anti-de Sitter space? ◦

Exercise. Calculate the scalar curvature of AdS and dS (in an easy way) in terms of Λ, using the fact that the scalar curvature is constant. ◦

Proposition. *A pseudo-Riemannian manifold of signature (p, q) is locally isometric*[5] *to $\mathbb{M}_{p,q}$ iff the Riemann tensor vanishes.*

[5] i.e. isomorphic as a pseudo-Riemannian manifold

10.2. **Black hole solutions.** Let us now discuss black hole solutions in 4d (i.e. with signature $(3,1)$). The simplest black hole solution was found by Schwarzschild in 1916. It has $\Lambda = T = 0$ and it look as follows:

$$g = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2, \qquad d\Omega^2 = d\vartheta^2 + \sin^2\vartheta d\varphi^2.$$

Here ϑ, φ parametrise a 2-sphere, $r \in (2M, \infty)$, $t \in \mathbb{R}$, and $M > 0$ is a parameter (mass of the black hole). Note that something wrong seems to be happening as $r \to 2M$. However, one can check that this is merely a problem of the coordinates, and nothing terrible happens to the metric itself — although the subspace $r = 2M$ is actually physically interesting and corresponds to the *black hole horizon.* In fact, there are other coordinates, such as *Kruszkal–Szekeres coordinates*, which extend the Schwarzschild ones beyond the horizon and give an expression for g which is nonsingular at the horizon. Still, one can show that there is a point hidden under the horizon, where the curvature goes to infinity — this is a true *singularity.*

Theorem (Birkhoff)**.** *The Schwarzschild solution is the most general spherically symmetric solution of Einstein equations in vacuum (i.e. with $\Lambda = 0$ and $T = 0$).*

Assuming the presence of the electromagnetic field, i.e. allowing a nonzero $T_{\mu\nu}$ of a specific form, one has the more general *Reissner–Nördstrom solution*

$$g = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}dr^2 + r^2 d\Omega^2,$$

describing the charged black hole of mass M and electric charge Q.

Even more generally, one has the *Kerr–Newman* solution, describing a rotating charged black hole with a mass.

10.3. **Cosmological spacetimes.** Observational data suggests that on very large scales our universe is, to a very good approximation, spatially homogeneous and isotropic. The most general such metric is the *FLRW (Friedmann–Lemaître–Robertson–Walker) metric*

$$g = -dt^2 + a^2(t)(dr^2 + S^2(r)d\Omega^2), \qquad S(r) = \begin{cases} \sin r \\ r \\ \sinh r \end{cases}$$

the three options corresponding to a spatially spherical, flat, and hyperbolic universe, respectively. We see that this metric has only one parameter $a(t)$, depending on a single variable. Plugging this into Einstein equations (with a suitable choice of matter content and Λ) we obtain the *Friedmann equations* for the parameter a and the matter parameters.

11. Gravitational waves

Let us now look at perturbation of the flat Minkowski metric. Starting with the Minkowski spacetime $\mathbb{M} = \mathbb{M}_{3,1}$, we perturb the metric to

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$

with $h_{\mu\nu}$ very small.

Exercise. Show that the Ricci tensor takes the form

$$R_{\mu\nu} = -\tfrac{1}{2}\partial^\rho\partial_\rho h_{\mu\nu} - \tfrac{1}{2}\partial_\mu\partial_\nu h^\rho{}_\rho + \partial_\sigma\partial_{(\mu}h^\sigma{}_{\nu)} + \mathcal{O}(h^2),$$

where the indices are raised/lowered using the flat constant metric η. $\circ$

Now note that after a change of coordinates the metric stays physically the same but we can get a different expression for $h_{\mu\nu}$. If the infinitesimal diffeomorphism which performs this change of coordinates is given by a vector field ξ, we should therefore consider the equivalence

$$h \sim h + \epsilon \mathcal{L}_\xi \eta, \text{ i.e. } h_{\mu\nu} \sim h_{\mu\nu} + 2\epsilon \partial_{(\mu}\xi_{\nu)}.$$

Using this freedom, we can "fix a *(harmonic/Lorenz) gauge*"

$$\partial_\mu h^\mu{}_\nu = \tfrac{1}{2}\partial_\nu h^\rho{}_\rho,$$

i.e. in each class of physically equivalent metric perturbations we choose a representative h satisfying this condition.

Exercise. Show that, after imposing the harmonic gauge condition, the vacuum (i.e. $\Lambda = T = 0$) Einstein equations $R_{\mu\nu} = 0$ reduce (up to $\mathcal{O}(h^2)$) to

$$\partial^\rho \partial_\rho h_{\mu\nu} = 0,$$

which we recognise as the wave equation. ○