Introduction to Symplectic Geometry
Take-Home Exercise

Exercise 1 (Moser's trick)**.**
The goal of this exercise is to proof the following

Lemma (Moser's trick)**.** Let M be a compact manifold and $\omega_t \in \Omega^2(M)$, $t \in [0,1]$, be a smooth family of symplectic 2-forms. Suppose that

$$\dot{\omega}_t = d\alpha_t \tag{1}$$

for some smooth family $\alpha_t \in \Omega^1(M)$. Then there exists a smooth family of diffeomorphism ψ_t such that $\psi_t^* \omega_t = \omega_0$.

1. Define a smooth family of vector fields v_t by

$$\iota_{v_t} \omega_t = -\alpha_t \tag{2}$$

   Express v_t in terms of α_t.

2. Prove Moser's lemma.

Exercise 2 (Moser's theorem)**.**
Prove the following

Theorem (Moser)**.** Let M, $\dim M = n$, be a compact oriented manifold and $\mu_0, \mu_1 \in \Omega^n(M)$ two volume forms such that

$$\int_M \mu_0 = \int_M \mu_1. \tag{3}$$

Then, there exists a smooth family of diffeomorphisms $\psi_t \in \mathrm{Diff}(M)$ such that $\psi_0 = id_M$ and $\psi_1^* \mu_1 = \mu_0$.

Hint: Modify Moser's trick appropriately.

Exercise 3 (Poincaré homotopy)**.**
Let $U \subset \mathbb{R}^n$ be an open neighborhood of 0. Let $\iota \colon \{0\} \hookrightarrow U$ be the inclusion and $\pi \colon U \to \{0\}$ be the projection. Moreover, let $f_t \colon U \to U$ be the multiplication by t. Show that

$$h \colon \Omega^k(U) \to \Omega^{k-1}(U) \quad , \quad \alpha \mapsto \int_{[0,1]} f_t^* \alpha \tag{4}$$

satisfies

$$(dh + hd)(\alpha) = \alpha \tag{5}$$

Exercise 4 (Darboux theorem)**.**
The goal of this exercise is to prove Darboux's theorem using the techniques we have learned in the previous exercises.

Theorem (Darboux)**.** Let (M, ω), $\dim M = 2n$, be symplectic, then locally we can always find coordinate charts centered at a point $m \in M$ such that there exists a symplectomorphism

$$(U, \omega_m) \cong (\mathbb{R}^{2n}, \omega_0 = \sum_i dx^i \wedge dp_i) \tag{6}$$

Hint: Use Moser's trick for a suitable smooth family ω_t of symplectic forms on V. Use the Poincaré homotopy to find the family of Moser 1-forms α_t explicitly.