Symplectic reduction - exercises

Exercise 1. *Cotangents as reduction*
Let G be a compact Lie group acting on a manifold M with only one orbit type (meaning that the stabiliser is always conjugated to some fixed subgroup of G). Assume further that M/G is a manifold.
Show the existence of a canonical symplectomorphism

$$T^*(M/G) \cong T^*M//G\,,$$

where the symplectic quotient is carried over the zero coadjoint orbit.

Exercise 2. *Heisenberg algebra*
Let (V, ω) be a finite-dimensional symplectic vector space.

a) Show that the action of V on itself by translations is weakly Hamiltonian, but not Hamiltonian.

Define the Heisenberg group $\mathrm{Heis}(V) = V \oplus \mathbb{R}$ by the group law $(v, \alpha).(w, \beta) = (v + w, \alpha + \beta + \frac{1}{2}\omega(v, w))$, where $v, w \in V$ and $\alpha, \beta \in \mathbb{R}$.

b) Deduce the Lie bracket of the Heisenberg algebra $\mathfrak{heis}(V)$. Make the link to the canonical commutation relations in quantum mechanics.

c) Show that the action of $\mathrm{Heis}(V)$ on V given by $(v, \alpha).w = v+w$ is Hamiltonian and compute its moment map.

Consider the case where $V = \mathbb{R}^2$. We denote $\mathfrak{heis}_3 = \mathfrak{heis}(\mathbb{R}^2)$.

d) Show that $\mathfrak{heis}_3$ can be realized by matrices of the form

$$\begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}.$$

Exercise 3. *Calogero–Moser system*

Consider $\mathrm{SU}(n)$ acting on $T^*\mathfrak{su}(n)$ in a diagonal way via the adjoint action.

a) Show that this action is Hamiltonian and compute the moment map.

Define $\mathcal{O} = \{X \in \mathfrak{su}(n) \mid \exists\, q \in \mathbb{R} \text{ such that } \operatorname{rank}(X - q\,\mathrm{id}) \leq 1\}$.

b) Show that $\mathcal{O}$ is a coadjoint orbit, using the identification $\mathfrak{su}(n) \cong \mathfrak{su}(n)^*$ given by the Killing form.

c) Show that $M = T^*\mathfrak{su}(n)//_{\mathcal{O}}\, \mathrm{SU}(n)$ consists of pairs $(X, Y) \in \mathfrak{su}(n) \oplus \mathfrak{su}(n)$, where X is diagonal with entries $(x_1, ..., x_n)$ and Y is uniquely determined by its diagonal entries $(y_1, ..., y_n)$.

d) On M, consider the Hamiltonian given by $H(X, Y) = \operatorname{tr}(Y^2)$. Express H in terms of $(x_i, y_i)_{1 \leq i \leq n}$.

e) Show that the functions $(X, Y) \mapsto \operatorname{tr}(Y^k)$ for $k = 2, ..., n$ Poisson commute. This proves that the Hamiltonian H on M defines an integrable system.

Reference: Book by Khesin–Wendt "Geometry of infinite-dimensional groups" (chapter I-3 and II-5.4), Springer, 2009.