DIFFERENTIAL GEOMETRY IN 1, 2, 3 AND MORE DIMENSIONS

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1. Metric tensor

First step into the space-time: generalize scalar product to smooth manifolds.

Exercise 1. Show that tensors transform as tensors. Do it for tensors of rank 0, (0,1), (1,0), (2,0), (0,2), (1,1) and (m,n).

Exercise 2. Consider S^2 embedded in $(\mathbb{R}^3, \delta)$ by the inclusion map

$$f : (\theta, \phi) \mapsto (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).$$

Find the induced metric on S^2.
The answer: $g_{\mu\nu}dx^\mu \otimes dx^\nu = d\theta \otimes d\theta + \sin^2\theta d\phi \otimes d\phi$.

Exercise 3. Let (M, g_M) and (N, g_N) be Riemannian manifolds of dimension 2.
Let g_M be defined by

$$ds^2 = (1 + u^2)du^2 + (1 + 4v^2)dv^2 + 2(2v - u)dudv$$

and g_N by

$$ds^2 = (1 + u^2)du^2 + (1 + 2v^2)dv^2 + 2(2v - u)dudv.$$

One of them is flat, another one is curved. Find out, which is which.

2. Connection

Connection connects. Connection parallel transports.
Connection covariantly differentiates.

Exercise 4. Show that for covariant derivative of covector fields one finds

$$\nabla_X(\alpha)_\nu = X^\mu \partial_\mu \alpha_\nu - X^\mu \Gamma^\lambda_{\mu\nu} \alpha^\lambda.$$

Generalize this result to arbitrary $(2, 0)$, $(1, 1)$, $(0, 2)$ tensor fields.

Exercise 5. Show that connection coefficients are (in general) not tensors.
Hint: in the transformation formula, recover an extra term of the form:

$$\frac{\partial^2 x^\nu}{\partial y^\alpha \partial y^\beta} \frac{\partial y^\gamma}{\partial x^\nu}$$

Note that under affine and linear coordinate transformations the connection coefficients do behave as tensors.

Exercise 6. Verify that the transformation rule for $\Gamma^\gamma_{\alpha\beta}$ makes $\nabla_X Y$ a vector (independently of the chosen coordinate system), if Y is a vector:

$$\tilde{X}^\alpha(\tilde{\partial}_\alpha \tilde{Y}^\gamma + \tilde{\Gamma}^\gamma_{\alpha\beta} \tilde{Y}^\beta) f_\gamma = X^\lambda(\partial_\lambda Y^\nu + \Gamma^\nu_{\lambda\mu} Y^\mu) e_\nu$$

Exercise 7. Show that for a connection compatible with a metric the operations of lowering and raising indeces commute with covariant derivatives.

Exercise 8. Show that a connection is compatible with a metric if and only if for any triple of vector fields η, ξ_1, ξ_2,

$$\partial_\eta g(\xi_1, \xi_2) = g(\nabla_\eta \xi_1, \xi_2) + g(\xi_1, \nabla_\eta \xi_2)$$

3. Various guises of curvature

Where dimension may make a difference.

Exercise 9. For the curvature tensor of a symmetric connection compatible with a metric deduce the following Bianchi identity:

$$\nabla_m R^n_{ikl} + \nabla_l R^n_{imk} + \nabla_m R^n_{ilm} = 0$$

Exercise 10. From the previous exercise deduce the following property of the divergence of the Ricci tensor:

$$\nabla_l R^l_m = \frac{1}{2} \frac{\partial R}{\partial x^m}$$

Exercise 11. Show that for 2-dimensional surfaces embedded in 3-dimensional Euclidean space scalar curvature is equal to twice the Gaussian curvature.

Exercise 12. Quantize gravity.

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