Supergeometry - Oddities of the Square

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1 Preface

This is the write up of a nano-course on an heuristic introduction to supergeometry given at the summer school MAPSS in July 2024 in Les Diablerets. Apart from *fuzzy* definitions, if you find any severe mistakes, or worse, *misunderstandings*, as well as typos—or you have an idea that would make this lecture more accessible—please let me know!

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As for the course, it tries to convey the simple idea that many objects we naturally come across in geometry and physics are bilinear of *odd* objects; and that giving these oddities some more attention is well worth the time!

2 Motivation

Let us start with a curious observation:

Principal of taking square roots.(Kapranov [2])
It is useful to take square roots of familiar things.

This seems like a rather bold statement, after all taking square roots always feels a bit unnatural. Let us therefore look at some motivating examples.

1. $i^2 = -1$. The idea of a square root of -1 lead to complex analysis and from there to a myriad of beautiful mathematics. And maybe we could already stop here because this example clearly shows the power of"taking a square root" of a familiar object.

2. Consider A a skew-symmetric matrix. Then $\det A = Pf(A)^2$ where $Pf(A)$ is the Pfaffian of A. Also in this case, the square root—the Pfaffian—has exciting applications in mathematics and physics alike, ranging from characteristic classes (for example, the Euler class of a Rimannian manifold can be defined as the Pfaffian of its curvature) to the theory of differential equations (Pfaffian systems).

3. Quantum mechanics: The probability of finding a particle in a specific state is described by the square (of the modulus) of the wave function, $\rho_\psi = |\psi|^2$, where $\psi \in \mathbb{C}$ is the wavefunction.

4. if (X, g) is a Riemannian manifold, then its space of differential forms $\Omega^\bullet(X)$ admits an inner product $(-,-)_g$. A natural operator to study is the (Beltrami-)Laplacian Δ_g of X. It can be written as

$$\Delta_g = [d, d^*]$$

where d denotes the exterior derivative (de Rham differential) of X and d^* its dual with respect to the inner product $(-,-)_g$.

5. Let A be a gauge field (connection on some principal G-bundle) and ∇_A the associated covariant derivative. Then the field strength F_A, is the square of ∇_A

$$F_A = \nabla_A^2.$$

There is one more example, which we shall use as the main motivation of this text, at least from the point of view of physics, namely supersymmetry. Instead of defining what we mean by supersymmetry right away, let us study an example.

The Quantum Bosonic Harmonic Oscillator The quantum harmonic oscillator can be described in terms of the algebra freely generated by two generators subject to the condition that their commutator is 1:

$$\mathcal{A}_B = \mathbb{C}[a, a^*]/\{[a, a^*] = 1, [a, a] = [a^*, a^*] = 0\} \tag{1}$$

where

$$[a, a^*] = aa^* - a^*a.$$

The generator a is usually called a *bosonic* annihilation operator, a^* a bosonic creation operator. The name comes from the lowest weight representation of $\mathcal{A}$. Let $|0\rangle$ be a lowest weight such that $a|0\rangle = 0$. We now "excite states" (defining a new state) by acting with the creation operator a^* on the vacuum: $|1\rangle = a^*|0\rangle$. Then we can construct a bosonic Hilbert space (the Fock space) by iteratively acting with a^*

$$\mathcal{H}_B = \left\{ |n\rangle = \frac{(a^*)^n}{\sqrt{n}}|0\rangle \mid n \in \mathbb{N} \right\} \tag{2}$$

One defines a number operator

$$N_B = a^*a \tag{3}$$

which is diagonal in the basis $|n\rangle$, namely $N_B|n\rangle = n|n\rangle$.

Exercise 1
Show that

$$[N_B, a] = -a \quad , \quad [N, a^*] = a^*$$

The Hamiltonian of the system is expressed in terms of N as follows

$$H_B = \hbar\omega_B \left(N + \frac{1}{2} \right), \tag{4}$$

which is then also diagonal in the basis $|n\rangle$

$$H_B|n\rangle = E_n|n\rangle \quad , \quad E_n = \hbar\omega_B \left(n + \frac{1}{2} \right) \tag{5}$$

Remark 1. The algebra generated by the bosonic creation and annihilation operators a^* and a admits a nice realization as the Weyl algebra

$$\mathcal{W} = \mathbb{C}[\partial_x, x] \tag{6}$$

generated by $a = \partial_x$ and $a^* = x$. Indeed, one directly finds $[\partial_x, x] = 1$. The bosonic Fock space is then simply given by complex polynomial in one variable x

$$\mathcal{H}_B = \mathbb{C}[x]. \tag{7}$$

The number operator takes the form $N = x\partial_x$—the Euler vector field—and simply extracts the degree. Therefore, the basis $|n\rangle$ diagonalizing N is given by the monomials x^n, with the vacuum state being just $|0\rangle = 1$. $\triangleleft$

The Quantum Fermionic Harmonic Oscillator Instead of the commutation relation

$$aa^* - a^*a = 1$$

we could be bold and study what happens if we flip the sign. As we will see, flipping signs is not so innocent as it might seem, after all $1 + 1$ and $1 - 1$ differ drastically.

Let us thus consider the free algebra generated by generators c and c^* subject to the following relation

$$\{c, c^*\} = cc^* + c^*c = 1 \quad , \quad \{c, c\} = \{c^*, c^*\} = 0. \tag{8}$$

Notice that both, c and c^* are nilpotent elements:

$$0 = \{c, c\} = 2c^2 = 0 \implies c^2 = 0 \tag{9}$$

and similar for c^*. The nil potency of c and c^* feels a bit odd if one has not worked with zero-divisors before. We shall therefore refer to such variables as *odd* or interchangeably as *fermionic*. The fermionic creation and annihilation algebra is therefore defined by

$$\mathcal{A}_F = \mathbb{C}[c, c^*]/\{\{c, c^*\} = 1, \{c, c\} = \{c^*, c^*\} = 0\} \tag{10}$$

Let us again consider the lowest weight representation. Let $|0\rangle$ be the lowest weight, i.e. $c|0\rangle = 0$. In analogy to before we may define the first excited state by $|1\rangle = c^*|0\rangle$. However, this time we see that the next excited state does not exist

$$c|1\rangle = c^2|0\rangle = 0. \tag{11}$$

So, fermions do not like each other much ...

The fermionic Hilbert space (Fock space) is finite dimensional and in this case has just two elements

$$\mathcal{H}_F = \{|0\rangle, |1\rangle\}. \tag{12}$$

Analogously to the bosonic oscillator, one defines a number operator

$$N_F = c^*c \tag{13}$$

which measures the occupation of the state

$$N_F|0\rangle = 0 \quad , \quad N|1\rangle = 1|1\rangle. \tag{14}$$

Exercise 2
Show that
$$[N_F, c] = -c \quad , \quad [N_F, c^*] = c^*$$

The Hamiltonian of the system is defined by

$$H_F = \hbar\omega_F \left(N_F - \frac{1}{2}\right) \tag{15}$$

Remark 2. Nilpotent elements of an algebra are not so scary as they might seem at first. To demystify them a bit, consider the following realization the fermionic creation and annihilation algebra in term of 2×2 matrices:

$$c = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \quad c^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{16}$$

It is then not hard to show that

$$c^2 = (c^*)^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad , \quad \{c, c^*\} = cc^* + c^*c = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{17}$$

In this realization, c, c^* act on $\mathbb{R}^2$ and

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Exercise 3
Compute the number operator N and the Hamiltonian H.

$\triangleleft$

The Supersymmetric Quantum Harmonic Oscillator We mentioned in the beginning that *supersymmetry* is supposed to be a symmetry between bosons and fermions. The idea now is simple, let's try to combine the bosonic and fermionic harmonic oscillator!

Consider thus the algebra generated by bosonic operators a, a^* and fermionic operators c, c^* subject to the bosonic and fermionic commutator relations we have studied before

$$\mathcal{A} = \mathcal{A}_B \otimes \mathcal{A}_F \tag{18}$$

where we recall the bosonic and fermionic commutator relations

$$[a, a^*] = 1 \quad , \quad \{c, c^*\} = 1 \quad , \quad c^2 = (c^*)^2 = 0. \tag{19}$$

We again study the lowest weight representation of $\mathcal{A}$. As before, we denote the lowest wight—the vacuum—by $|0\rangle$ which is annihilated by both a and c: $a|0\rangle = c|0\rangle = 0$. The Fock space of the model is given by

$$\mathcal{H} = \mathcal{H}_B \otimes \mathcal{H}_F = \{|n_B, n_F\rangle \mid n_B = 0, 1, 2, \ldots, n_F = 0, 1\} \tag{20}$$

where we usually abbreviate $|0, 0\rangle = |0\rangle$. The Hamiltonian is simply the sum of the bosonic and fermionic Hamiltonians

$$H = H_B + H_F = \hbar\omega_B \left(N_B + \frac{1}{2}\right) + \hbar\omega_F \left(N_F - -\frac{1}{2}\right) \tag{21}$$

where the number operators $N_B \equiv N_B \otimes id$ and $N_F \equiv id \otimes N_F$ only act on the bosonic factor $\mathcal{H}_B$ and the fermionic factor $\mathcal{H}_F$ respectively.

Now, an interesting phenomena happens, if we consider the case where the frequencies are the same, $\omega_B = \omega_F = \omega$. In this case, the Hamiltonian simplifies to

$$H = \hbar\omega(N_B + N_F) \tag{22}$$

which leads to a number of interesting observations:

(i) All energies are *positive* (assuming $\hbar\omega > 0$)

$$E(n_B, n_F) = \hbar\omega(n_B + n_F) \geq 0, \tag{23}$$

where $n_B \in \mathbb{N}_0$ and $n_F = 0, 1$.

(ii) States with non-zero energy come in pairs of opposite parities as $E(n_B + 1, 0) = E(n_B, 1)$.

(iii) There exists a symmetry between these states. Let

$$Q = \sqrt{\hbar\omega}(a \otimes c^*) \quad , \quad \sqrt{\hbar\omega}(a^* \otimes c) \tag{24}$$

Exercise 4
Show that

a) $Q^2 = (Q^*)^2 = 0$

b) $H = \{Q, Q^*\}$

c) $[Q, H] = [Q^*, H] = 0$

The symmetry generated by Q and Q^* is known as *supersymmetry* and indeed relates bosons to fermions. In particular, since Q and Q^* commute with the Hamiltonian, they preserve the energy eigenspaces.

To end this motivational section, let us recall the two main points we learned so far:

1. It is interesting to consider square roots, as they often lead to new and interesting mathematical structures.
2. Just as square roots, nilpotent elements can lead to interesting new phenomena.

In the following we will explore the idea of nilpotent elements from the mathematical point of view, where we will focus on the algebra of functions on some space. If the ring of functions on a manifold admits nilpotent elements, we will call them supermanifolds and we stumbled without noticing into the realm of super geometry. Ultimately, these considerations will lead us to a first construction of supersymmetry in physics.

3 Before Supermanifolds

3.1 Super Vector Spaces

Before we can speak about supermanifolds, we need to lay down the foundations.

Definition 3.1. A *super vector space* is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space, i.e. a vector space V together with a fixed decomposition

$$V = V_0 \oplus V_1. \tag{25}$$

Elements of V_0 are called *even* (bosonic); elements of V_1 are called *odd* (fermionic).

If $\dim V_0 = p$ and $\dim V_1 = q$, we set $\dim V = p \mid q$.

Definition 3.2. We define a *parity* map

$$|\cdot|\colon V \to \mathbb{Z}/2\mathbb{Z} \tag{26}$$

by its action on homogeneous elements

$$|v| = \begin{cases} 0 & \text{if } v \in V_0 \\ 1 & \text{if } v \in V_1 \end{cases} \tag{27}$$

Definition 3.3. If V, W are super vector spaces, a *homomorphism* between V and W is a linear map $f\colon V \to W$ preserving the grading, i.e. such that $f(V_i) \subset W_i$. We denote the space of all homomorphisms by $\mathrm{Hom}(V, W)$.

Let us look at some examples.

1. $\mathbb{R}^{p|q} \equiv \mathbb{R}^p \oplus \mathbb{R}^q$. We call coordinates x^i on $\mathbb{R}^p$ *even coordinates* and θ^i on $\mathbb{R}^q$ *odd coordinates.*

2. If V and W are super vector spaces, then $\mathrm{Hom}(V, W)$ is a super vector space.

> **Exercise 5**
> Show that $f \in \mathrm{Hom}(V, W)$ is equivalent to a pair of maps (f_0, f_1) such that $f_0(V_1) = f_1(V_0) = 0$.

3. Let V be an ordinary vector space, then V defines a super vector space with $V \equiv V_0$. Then one defines the super vector space ΠV by $V \equiv V_1$. The map $\Pi\colon V \to V$ which simply shifts the parity is known as the parity reversal map.

4. Let V be an ordinary vector space. Then $\Lambda^\bullet V = \Lambda^{ev} V \oplus \Lambda^{odd} V$.

   Notice that $\Lambda^\bullet V$ is actually $\mathbb{Z}$-graded: $\Lambda^\bullet V = \bigoplus_k \Lambda^k V$. This $\mathbb{Z}$-grading reduces to the "super"-terminology when considering it modulo 2.

3.2 Functions and Integrals

In understanding supermanifolds, one important tool is to understand its ring of functions. But before we walk we might want to try to crawl and hence we will still focus on super vector spaces in this section.

Definition 3.4. Let $V = V_0 \oplus V_1$ be a super vector space. *Functions* on V are defined by their Taylor expansion in the odd variables $\theta \in V_1$, i.e.

$$\mathcal{C}^\infty(V) = \mathcal{C}^\infty(V_0) \otimes \Lambda^\bullet V_{1^*} \tag{28}$$

Assuming coordinates $x^i inV_0$ and $\theta^i \in V_1$ function $f \in C^\infty(V)$ is therefore of the form

$$f(x, \theta) = f_0(x) + f_i(x)\theta^i + f_{ij}(x)\theta^i\theta^j + \dots \tag{29}$$

where the coefficients $f_0(x)$, $f_i(x)$, $f_{ij}(x)$ etc. are functions. In particular, the $f_{ij}(x)$, $f_{ijk}(x)$ etc. are *completely antisymmetric* by the nature of the commutation relations $\theta^i\theta^j = -\theta^j\theta^i$ of $\Lambda^\bullet V_1^*$.

In particular, if V is an ordinary vector space, then

$$\mathcal{C}^\infty(\Pi V) = \Lambda^\bullet V^*. \tag{30}$$

Now that we have a definition of functions on a super vector space, in order to have a notion of some kind of "super calculus", we would like to differentiate and integrate them.

Differentiation is easy—we simply take derivatives with respect to the odd variables θ^i in the same way we take derivatives with respect to the even variables x^i. However, due to the anti-commuting nature of the θ^i we actually have to be more careful and define a *left* as well as a *right* derivative.

Definition 3.5. Let $V = V_0 \oplus V_1$ be a super vector space with even coordinates x^i and odd coordinates θ^i. The *left derivative* with respect to θ^i are defined on monomials in θ^j by

$$\frac{\overrightarrow{\partial}}{\partial \theta^i} \theta^j = \delta_i^j \tag{31}$$

and is extended to $\mathcal{C}^\infty(V)$ as a (super)derivation, i.e.

$$\frac{\overrightarrow{\partial}}{\partial \theta^i} (\theta^j \theta^k) = \delta_i^j \theta^k - \delta_i^k \theta^j \tag{32}$$

Analogously, the *right derivative* with respect to θ^i are defined on monomials in θ^j by

$$\theta^j \frac{\overleftarrow{\partial}}{\partial \theta^i} = \delta_i^j \tag{33}$$

and is extended to $\mathcal{C}^\infty(V)$ as a (super)derivation, i.e.

$$(\theta^j \theta^k) \frac{\overleftarrow{\partial}}{\partial \theta^i} = \delta_i^k \theta^j - \delta_i^j \theta^k \tag{34}$$

Let us now turn our attention to integration. But what should an integral over a super vector space be? Notice that we have handled the odd variables rather algebraically, and we shall continue to do so.

Recall that if V is an ordinary integral, we can think of the integral as a map

$$\int_V \cdot \, dx \colon \mathcal{C}^\infty(V) \to \mathbb{R} \tag{35}$$

which is

(i) Linear

$$\int_V af(x) + g(x) dx = a \int_V f(x) dx + \int_V g(x) dx, \qquad a \in \mathbb{R}, f, g \in \mathcal{C}^\infty(V).$$

(ii) Translation invariant

$$\int_V f(x+y) dx = \int_V f(x) dx.$$

(iii) Total derivatives are mapped to zero

$$\int_V f'(x) dx = 0.$$

The idea is now to define the integral over odd variables in a similar way, namely by a linear, translational invariant map which sends total derivatives to zero.

Definition 3.6. Let V be an ordinary one-dimensional vector space and ΠV be the parity shifted super vector space with odd coordinates θ. We define the *Grassmann-Berezin* integral

$$\int_{\Pi V} \cdot \, d\theta \colon \mathcal{C}^\infty(\Pi V) = \Lambda^\bullet V^* \to \mathbb{R} \tag{36}$$

by

$$\int_{\Pi V} d\theta = 0 \quad , \quad \int \theta d\theta = 1. \tag{37}$$

Exercise 6
Verify the following properties

a) Linearity

$$\int_{\Pi V} af(\theta) + g(\theta)d\theta = a\int_{\Pi V} f(\theta)d\theta + \int_{\Pi V} g(\theta)d\theta.$$

b) Translation invariance

$$\int_{\Pi V} f(\theta + \theta') = \int_{\Pi V} f(\theta)d\theta.$$

c) Total derivatives are mapped to zero

$$\int_{\Pi V} \frac{\overrightarrow{\partial} f(\theta)}{\partial\theta} d\theta = \int_{\Pi V} f(\theta)\frac{\overleftarrow{\partial}}{\partial\theta} d\theta = 0.$$

Hint: Any function $f \in \mathcal{C}^\infty(\Pi V)$ is of the form $f(\theta) = f_0 + f_1\theta$.

The generalization to multiple odd variables is straightforward

Definition 3.7. Let V be an ordinary vector space of dimension n. The Grassmann-Berezin integral over ΠV, with coordinates θ^i is defined by the relations

$$\int_{\Pi V} \theta^n \ldots \hat{\theta}^i \ldots \theta^1 d\theta^1 \ldots d\theta^n = 0 \quad , \quad \int_{\Pi V} \theta^n \ldots \theta^1 d\theta^1 \ldots d\theta^n = 1 \tag{38}$$

where $\hat{\theta}^i$ means omission of the i-th factor θ^i.

Remark 3. In laymen's terms, the Grassmann-Berezin integral selects (up to a sign) the top-part $f_{12\dots n}$ of a function

$$f(\theta) = f_0 + f_i\theta^i + f_{ij}\theta^i\theta^j + \dots + f_{12\dots n}\theta^1 \dots \theta^n$$

$\triangleleft$

The last thing we want to look at is how the "measure" $d\theta^1 \dots d\theta^n$ transforms under a coordinate transformation.

Exercise 7
Verify the following transformation behavior under a change of variables

a) Let $\theta = J\xi$, θ, ξ, $J \in \mathbb{R}$ odd, then $d\theta = \frac{d\xi}{J}$.

b) Let $\theta^i = J^i{}_j\xi^j$, then $d\theta^1 \dots d\theta^n = \frac{d\xi^1 \dots d\xi^n}{\det J}$.

Hint: express $f_{1\dots n}$ as a Grassmann-Berezin integral once over θ^i and once over ξ^i.

Write about Berezinian?

3.3 Super Algebras

Recall that for a super vector space $V = V_0 + V_1$ its space of functions was given by

$$\mathcal{C}^\infty(V) = \mathcal{C}^\infty(V_0) \otimes \Lambda^\bullet V_1^* \tag{39}$$

Now, $\mathcal{C}^\infty(V)$ inherits the structure of a super vector space with

$$\mathcal{C}^\infty(V) - 0 = \mathcal{C}^\infty(V_0) \otimes \Lambda^{ev} V_1^* \quad , \quad \mathcal{C}^\infty(V)_0 = \mathcal{C}^\infty(V_0) \otimes \Lambda^{odd} V_1^*. \tag{40}$$

But just as the (vector) space of functions on an ordinary vector space admits a multiplication turning it into an algebra, so does the space of functions on a super vector space, turning it into a *superalgebra* The multiplication of $\mathcal{C}^\infty(V)$ is given by ordinary point-wise multiplication in $\mathcal{C}^\infty(V_0)$ together with the wedge product in $\mathcal{C}^\infty(V_1)$. If $f(x), g(x) \in \mathcal{C}^\infty(V_0)$ and $\alpha, \beta \in \Lambda^\bullet V_1^*$, then

$$\begin{aligned} \cdot : \mathcal{C}^\infty(V) \times \mathcal{C}^\infty(V) &\to \mathcal{C}^\infty(V) \\ \big(f(x)\alpha, g(x)\beta\big) &\mapsto f(x)\alpha \cdot g(x)\beta = f(x)g(x)\alpha \wedge \beta. \end{aligned} \tag{41}$$

Note that this multiplication is associative, preserves the grading

$$|f(x)\alpha \cdot g(x)\beta| = |f(x)\alpha\beta| + |g(x)\beta| \tag{42}$$

and satisfies

$$f(x)\alpha \cdot g(x)\beta = (-1)^{|f(x)\alpha||g(x)\beta|} g(x)\beta \cdot f(x)\alpha. \tag{43}$$

Such an algebra is known as a (super-)commutative superalgebra. In general we define a superalgebra as follows

Definition 3.8. An *associative superalgebra* (over $\mathbb{R}$) is an associative $\mathbb{Z}/2\mathbb{Z}$-graded algebra $A = A_0 + A_1$ whose multiplication $\cdot\colon A \times A \to A$ preserves the grading, that is for homogeneous elements $a, b \in A$, $|a \cdot b| = |a| + |b|$.
A superalgebra A is called *super-commutative* if $a \cdot b = (-1)^{|a||b|} b \cdot a$.

Let us look at some examples.

1. An ordinary commutative algebra A (like the algebra of functions on a smooth manifold) defines a superalgebra concentrated in even degrees: $A \equiv A_0$.
2. The Grassmann (exterior) algebra of an ordinary vector space V defines a super-commutative superalgebra
$$A = (\Lambda^\bullet V, \wedge)$$
3. The Grassmann algebra of the dual V^* of an ordinary vector space V^* defines the super-commutative superalgebra
$$A = (\Lambda^\bullet V^*, \wedge) = \mathbb{R}[\theta^1, \dots, \theta^n]/\{\theta^i\theta^j = -\theta^i\theta^j\}$$
4. A *super Lie algebra* $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is defined by a super vector space together with a (super) Lie bracket $[\cdot,\cdot]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that for homogeneous elements $x, y, z \in \mathfrak{g}$ the following conditions hold
   (i) symmetry: $[x, y] = -(-1)^{|x||y|}[y, x]$
   (ii) The bracket has degree 0: $|[x, y]| = |x| + |y|$
   (iii) Jacobi identity: $[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]$

Exercise 8
Let $\mathfrak{g}$ be a super Lie algebra. Show that the Jacobi identity is equivalent to

$$(-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||x|}[y, [z, x]] + (-1)^{|z||y|}[z, [x, y]] = 0$$

4 Towards Supermanifolds

4.1 A Heuristic Definition

The question what a *supermanifold* precisely is can be quite intricate and we shall in the following give a rather heuristic definition, a working-definition so to speak. In all what follows we will be guided by the geometers truth — *think globally, work locally.*

Definition 4.1. A *supermanifold* $\mathcal{M}$ of dimension $n|m$ is locally modeled on $\mathbb{R}^{n|m}$ and then patched together to a global object.

We thus treat supermanifolds just as ordinary manifolds, which are locally modeled by $\mathbb{R}^n$ by studying their local models. When we say *locally modeled by* $\mathbb{R}^{n|m}$ we mean that there exists the notion of *local coordinates* which now can be even or odd.

An other approach, which complements our first working-definition, is to define a supermanifold via its structure sheaf, i.e. its ring of functions.

Definition 4.2. A *supermanifold* $\mathcal{M}$ of dimension $n|m$ is a locally ringed space $(M, \mathcal{O}_M)$ which is locally isomorphic to $(U, \mathcal{C}^\infty(U) \otimes \Lambda V^*)$ for $U \subset \mathbb{R}^n$ open and V some m-dimensional vector space.

Notice that

$$\mathcal{C}^\infty(U) \otimes \wedge V^* = \mathcal{C}^\infty(U \times \Pi V) \tag{44}$$

such that we can think of the supermanifold $\mathcal{M}$ locally as $U \times \Pi V$. But this is just the same as saying that $\mathcal{M}$ is locally modeled on $\mathbb{R}^{n|n}$ and hence coincides with our intuitive picture given in Definition 4.1. However, Definition 4.2 is more general.

To really grasp what is going on, it is best to look at examples.

4.2 What's the Fuzz all about?

Definition 4.2 is very algebraic geometric in nature, as it suggests that we should understand a supermanifold by its ring of functions. Let us see how one could approach such an endeavor in practice. We will follow the excellent exposition of Freed [1].

Suppose $\mathcal{P}$ is a supermanifold whose ring of functions is given by

$$\mathcal{C}^\infty(\mathcal{P}) = \mathbb{R}[\theta]/(\theta^2), \tag{45}$$

the ring of polynomials generated by one generator θ subject to the condition $\theta^2 = 0$. We thus think of θ as an odd variable.

Now, given any map $f\colon M \to N$ between two manifolds, super or not, we naturally get a map $\mathcal{C}^\infty(N) \to \mathcal{C}^\infty(M)$ (assigning a manifold to its ring of functions is just a contravariant functor). This is rather intuitive, for if $f\colon M \to N$ and $\phi \in \mathcal{C}^\infty(N)$, we obtain a map $\phi \circ f \in \mathcal{C}^\infty(M)$ by composition.

Let M be an ordinary smooth manifold. As we have just recalled,a map $f\colon \mathcal{P} \to M$ induces a map $\mathcal{C}^\infty(M) \to \mathcal{C}^\infty(\mathcal{P}) = \mathbb{R}[\theta]/(\theta^2)$. Any such map necessarily looks as follows

$$f \mapsto A(f) + B(f)\theta \tag{46}$$

Now, by functoriality, this assignment is a algebra homomorphism, that is for two maps $f, g\colon \mathcal{P} \to M$

$$fg \mapsto A(fg) + B(fg)\theta \overset{!}{=} \Big(A(f) + B(f)\theta\Big)\Big(A(g) + B(g)\theta\Big) \tag{47}$$

expanding the RHS we find

$$A(fg) + B(fg)\theta \overset{!}{=} A(f)A(g) + \big(A(f)B(g) + B(f)A(g)\big)\theta. \tag{48}$$

Since $A(f) \in \mathcal{C}^\infty(\mathcal{P})_0 = \mathbb{R}$, that A defines an algebra homomorphism

$$A\colon \mathcal{C}^\infty(M) \to \mathbb{R}. \tag{49}$$

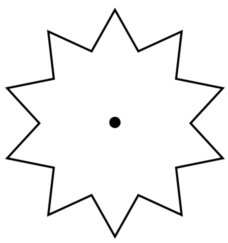
All such homomorphisms are given by evaluation maps with respect to some point $m \in M$ and hence $A = ev_m\colon f \mapsto f(m)$. It then follows that once we identify A with ev_m for some $m \in M$, $B = B_m$ must be a derivation over functions on M, namely

$$B_m(fg) = A(f)B_m(g) + B_m(f)A(g) = f(m)B_m(g) + B_m(f)g(m). \tag{50}$$

Therefore, we can identify B_m with a tangent vector in T_mM. Hence the space of maps $\mathcal{P} \to M$ can be thought of as the tangent bundle of M

$$\mathrm{Maps}(\mathcal{P}, M) \cong TM. \tag{51}$$

This is a remarkable observation. It suggests that points in our supermanifold $\mathcal{P}$ are equivalent to the data of an abstract point and a direction. So, $\mathcal{P}$ looks like a abstract point surrounded by a cloud of directions - a cloud of "fuzz" [1] and we will henceforth refer to it as *the fuzzy point*:



But what about the converse, i.e. what about maps *into* $\mathcal{P}$? Suppose that we are given a map $f\colon M \to \mathcal{P}$. Following the above strategy, we ought to study the resulting map $\mathcal{C}^\infty(\mathcal{P}) \to \mathcal{C}^\infty(M)$. Since $\mathcal{C}^\infty(M)$ does not have any nilpotent elements, any such map must map θ to the constant zero function. This means that any function $M \to \mathcal{P}$ maps the whole of M to a single point in $\mathcal{P}$ — the geometric point of $\mathcal{P}$. Put otherwise: ordinary objects cannot probe fuzz. Fuzz needs to be probed by fuzz. Hence, in order to really probe a supermanifold, we have to consider maps from a supermanifold $\mathcal{M}$ into $\mathcal{P}$. So how to understand maps $M \to \mathcal{P}$ then? Well, if we believe the previous ideas, we need to add some fuzz to M to turn it into a supermanifold. For example, we could simply consider the supermanifold

$$M \times \mathcal{S}$$

for some supermanifold $\mathcal{S}$ and then consider maps

$$M \times \mathcal{S} \to \mathcal{P}. \tag{52}$$

We call such maps *parametrized* by $\mathcal{S}$. If M is just a point, we would call the map $\{pt\} \times \mathcal{S} \to \mathcal{P}$ an $\mathcal{S}$-point of $\mathcal{P}$.

4.3 Three versions of ΠTX

Now that we gained first experience with supermanifolds, let us consider an example where we have more than just one geometric point. A very nice example is the parity shifted tangent bundle of ΠTX of a smooth manifold X. Below, we will give three viewpoints on ΠTX. Our first approach in understanding ΠTX comes from our intuitive definition of supermanifolds, i.e. by gluing local charts. Our second approach will be by studying the functions of ΠTX. Finally, we will investigate ΠTX from the point of view of parametrized maps $\mathbb{R}^{0|1} \to X$.

4.3.1 Gluing Local Charts

Recall that locally, the tangent space T_pX of an n-dimensional smooth manifold X at a point $p \in X$ is given by the collection of pairs

$$T_pX = \{(p, v) \mid v \in \mathbb{R}^n\}. \tag{53}$$

and for any open chart U of X centered at p, we can identify

$$T_UX = U \times \mathbb{R}^n \tag{54}$$

The local tangent spaces T_UX are then glued together according to the following rule: if x denotes a local coordinate on U, and $\tilde{x}$ another coordinate on $\tilde{U}$, and on the overlap $U \cap \tilde{U}$ we have $\tilde{x} = \tilde{x}(x)$, then the transition function from one coordinate patch to the other is given by the Jacobian $\frac{\partial \tilde{x}^i(x)}{\partial x^j}$ and we glue the elements in the fiber according to

$$\tilde{v}^i = \frac{\partial \tilde{x}^i}{\partial x^j} v^j \tag{55}$$

Recall that the parity reversing operator Π simply shifts the parity of a vector and hence we will define locally

$$\Pi T_p X = \{(p, \psi) \mid \psi \in \Pi \mathbb{R}^n = \mathbb{R}^{0|n}\}. \tag{56}$$

and over an open chart U

$$\Pi T_U X = U \times \mathbb{R}^{0|n}. \tag{57}$$

To define ΠTX globally, we now glue the (odd) fibers according to the gluing rule we had before, namely if x is a local coordinate in U $\tilde{x}$ is a coordinate in $\tilde{U}$, and on the overlap $U \cap \tilde{U}$ we can express $\tilde{x} = \tilde{x}(x)$, then we glue the fibers according to

$$\tilde{\psi}^i = \frac{\partial \tilde{x}^i}{\partial x^j} \psi^j. \tag{58}$$

4.3.2 Functions on ΠTX

Let us know look at the functions on ΠTX. We start with the local model. Let U be an open coordinate chart of X centered around a point $p \in X$. By our considerations in the previous section, $\Pi T_U X = U \times \mathbb{R}^{0|n}$ and therefore

$$\mathcal{C}^\infty(\Pi T_U X) = \mathcal{C}^\infty(U \times \mathbb{R}^{0|n}) = C^\infty(U) \otimes \Lambda^\bullet[\psi^1, \ldots, \psi^n] \tag{59}$$

where $\Lambda^\bullet[\psi^1, \ldots, \psi^n]$ denotes the exterior algebra over $\mathbb{R}$ generated by $\psi^1, \ldots, \psi^n$. A function on $\Pi T_U X$ is therefore of the form

$$\alpha(x, \psi) = \sum_k \sum_{i_1 < i_2 < \cdots < i_k} \alpha_{i_1 \ldots i_k}(x) \psi^{i_1} \ldots \psi^{i_k} \tag{60}$$

Now, let again x be a coordinate on U, $\tilde{x}$ a coordinate on $\tilde{U}$ and $\tilde{x} = \tilde{x}(x)$ on the overlap $U \cap \tilde{U}$, then

$$\begin{aligned} \alpha(\tilde{x}(x), \tilde{\psi}) &= \sum_k \sum_{i_1 < i_2 < \cdots < i_k} \alpha_{i_1 \ldots i_k}(\tilde{x}) \tilde{\psi}^{i_1} \ldots \tilde{\psi}^{i_k} \\ &= \sum_k \sum_{i_1 < \cdots < i_k} \sum_{j_1 < \cdots < j_k} \alpha_{i_1 \ldots i_k}(\tilde{x}) \frac{\partial \tilde{x}^{i_1}}{\partial x^{j_1}} \cdots \frac{\partial \tilde{x}^{i_k}}{\partial x^{j_k}} \psi^{j_1} \ldots \psi^{j_k} \\ &= \sum_k \sum_{j_1 < \cdots < j_k} \alpha_{i_1 \ldots i_k}(x) \psi^{i_1} \ldots \psi^{i_k} \end{aligned} \tag{61}$$

where

$$\alpha_{i_1\dots i_k}(x) = \sum_k \sum_{i_1<\dots<i_k} \alpha_{i_1\dots i_k}(\tilde{x}) \frac{\partial \tilde{x}^{i_1}}{\partial x^{j_1}} \cdots \frac{\partial \tilde{x}^{i_k}}{\partial x^{j_k}}. \tag{62}$$

But this is just the transformation behavior of a differential form on X! We can therefore conclude that

$$\mathcal{C}^\infty(\Pi TX) \cong \Omega^\bullet(X) \tag{63}$$

where the isomorphism is given by the local assignment

$$\psi^i \mapsto dx^i \tag{64}$$

4.3.3 The Space of Odd Curves in X

Our third approach to ΠTX is in spirit quite similar to how we understood the fuzzy point. This time, however, we want to probe ΠTX by maps *into* ΠTX.

Proposition 4.1. A map $\mathcal{Y} \to \Pi TX$ is the same as a parametrized map $\mathbb{R}^{0|1} \times \mathcal{Y} \to X$, i.e. a parametrized odd curve in X.

Proof. Consider a map $x\colon \mathbb{R}^{0|1} \times \mathcal{Y} \to X$. Let θ be the odd coordinate of $\mathbb{R}^{0|1}$ and y a local coordinate on $\mathcal{Y}$, then locally

$$x^i(\theta, y) = x^i(y) + \theta\psi^i(y) \tag{65}$$

where $x^i(y)$ denotes a local coordinate on X. Moreover, if $\tilde{x}(x)$ is any other local coordinate on X then

$$\tilde{x}^i(x(y,\theta) = \tilde{x}^i(x(y) + \theta\psi) = \tilde{x}^i(x(y)) + \theta \frac{\partial \tilde{x}^i}{\partial x^j}\psi^j = \tilde{x}^i(y) + \theta\tilde{\psi}^i(y) \tag{66}$$

where

$$\tilde{\psi}^i(y) = \frac{\partial \tilde{x}^i}{\partial x^j}\psi^j. \tag{67}$$

But this is just the transformation behavior of an odd vector! Hence, by assigning

$$x^i(y) + \theta\psi^i(y) \mapsto (x^i(y), \psi^i(y)) \tag{68}$$

we can identify

$$\{\mathbb{R}^{0|1} \times \mathcal{Y} \to X\} \cong \{\mathcal{Y} \to \Pi TX\} \tag{69}$$

□

From now on, we will drop the parametrization and simply think of maps $\mathbb{R}^{0|1} \to X$ as elements of ΠTX.

Remark 4. This is of course not quite true. We have already seen that we can only probe fuzz with fuzz and an ordinary smooth manifold is not very fuzzy. What we should have in mind is the simplest possible supermanifold $\mathcal{Y}$ parametrizing our map. Intuitively, we would like to take $\mathcal{Y} = \{pt\}$ just a point since formally we would then indeed obtain

$$\{\mathbb{R}^{0|1} \times \{pt\} \to X\} = \{\mathbb{R}^{0|1} \to X\}.$$

But $\mathcal{Y}$ needs to be super and thus the simplest thing we could do is consider the *super point* $\mathcal{Y} = \mathbb{R}^{0|1}$. $\lhd$

So what is the advantage of viewing ΠTX as the space of odd curves $\mathbb{R}^{0|1} \to X$? Well, for once maps $\mathbb{R}^{0|1} \to X$ inherit an action of $\mathrm{Diff}(\mathbb{R}^{0|1}$. And, as we will see shortly, this has marvelous consequences.

But first, let us briefly say what we mean by $\mathrm{Diff}(\mathbb{R}^{0|1})$. We simply mean the automorphism group of $\mathbb{R}^{0|1}$. Any (parametrized[1]) map $\mathbb{R}^{0|1} \to \mathbb{R}^{0|1}$ is necessarily of the form

$$a\theta + \beta \tag{70}$$

for a even and β odd. This means that we can identify

$$\mathrm{Maps}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1}) = \mathbb{R}^{1|1} \tag{71}$$

and $\mathrm{Diff}(\mathbb{R}^{0|1})$ are just invertible maps.

Exercise 9
Show that a map $a\theta + \beta$ is invertible if and only if $a \in \mathbb{R}^*$. Conclude that $\mathrm{Diff}(\mathbb{R}^{0|1}) = \mathbb{R}^* \times \mathbb{R}^{0|1} \subset \mathbb{R}^{1|1}$ is generated by the two vector fields

$$\partial_\theta \quad , \quad \theta\partial_\theta$$

A diffeomorphism $\phi(\theta) = a\theta + \beta \in \mathrm{Diff}(\mathbb{R}^{0|1})$ acts on a map $x^i + \theta\psi^i$ by

$$\phi\colon x^i + \theta\psi^i \mapsto x^i + (a\theta + \beta)\psi^i \tag{72}$$

that is

$$\phi\colon \begin{cases} x^i \mapsto x^i + \beta\psi^i \\ \psi^i \mapsto a\psi^i \end{cases} \tag{73}$$

Hence, the action of $\mathrm{Diff}(\mathbb{R}^{0|1})$ on the space of odd curves $\mathbb{R}^{0|1} \to X$ is generated by the vector fields

$$d = \psi^i \frac{\partial}{\partial x^i} \quad , \quad E = \psi^i \frac{\partial}{\partial \psi^i} \tag{74}$$

[1]We will be sloppy and do not write the parametrization explicitly.

satisfying the commutation relations

$$d^2 = \frac{1}{2}[d, d] = 0 \quad , \quad [E, d] = d. \tag{75}$$

Observe in particular that d is odd, i.e. augments the parity by 1. We can give a geometric interpretation to both vector fields: First, observe that $\mathfrak{X}(\Pi TX) = \mathrm{Der}(C^\infty(\Pi TX))$. Now, recall that $C^\infty(\Pi TX) \cong \Omega^\bullet(X)$ by $\psi^i \mapsto dx^i$ locally. With this observation we have $d, E \in \mathrm{Der}(\Omega^\bullet(X))$ and

$$d = dx^i \frac{\partial}{\partial x^i} \quad , \quad E = dx^i \frac{\partial}{\partial(dx^i)}. \tag{76}$$

Therefore, the vector field d is nothing else then the de Rham differential (exterior derivative) and E measures the (form) degree. Hence, by studying ΠTX from the point of view of odd curves $\mathbb{R}^{0|1} \to X$, we see that there exists natural vector fields $d, E \in \mathfrak{X}(\Pi TX)$ which endow $(C^\infty(\Pi TX), d, E)$ with the structure of a cochain complex, namely a $\mathbb{Z}$-grading, given by the action of E, and a differential d squaring to zero. This observation is an instance of a more general

Proposition 4.2 (Kontsevich)**.** A supervector space V together with an action of $\mathrm{Diff}(\mathbb{R}^{0|1})$ is the same as the datum of a cochain complex $(V^\bullet, d)$. If $\mathrm{Diff}(\mathbb{R}^{0|1})$ is generated by $\delta = \partial_\theta$ and $\mathcal{E} = \theta\partial_\theta$, then the differential d is induced by the action of δ and the grading is given by the action of $\mathcal{E}$.

We want to end this section by mentioning another natural vector field on ΠTX, namely if $v \in \mathfrak{X}(X)$ is a vector field on X, then

$$\xi_v = v^i(x) \frac{\partial}{\partial \psi^i} \in \mathfrak{X}(\Pi TX). \tag{77}$$

Exercise 10

a) Argue that ξ_v is odd.

b) Compute the commutators $[d, \xi_v]$ and $[E, \xi_v]$. What is the parity of $[d, \xi_v]$?

c) What is the geometric interpretation of ξ_v and $[d, \xi_v]$?

Exercise 11 (*)
Consider $X = G$ a compact connected Lie group. Consider the action of right multiplication $R_g\colon G \to G$ sending $h \to hg$. Then $\omega_g = (R_{g^{-1}})_*\colon T_gG \to T_eG = \mathfrak{g}$. Using ω_g, we can trivialize TG, i.e. realize $TG = G \times \mathfrak{g}$ globally.[a]

a) Show that if G is a matrix group, $\omega_g = dgg^{-1}$ and consequently $d\omega_g = \frac{1}{2}[\omega_g, \omega_g]$

b) Argue that we can use ω_g to trivialize $\Pi TG = G \times \Pi\mathfrak{g}$. Give an example of local coordinates.

c) What is $C^\infty(G \times \Pi\mathfrak{g})$?

d) Compute d as a derivation of $C^\infty(G \times \Pi\mathfrak{g})$.

[a] Note that this trivialization could be done equally well if we would have chosen left multiplication $L_g\colon h \mapsto gh$ instead of R_g. The trivialization is hence not canonical.

5 Towards Supersymmetry

5.1 The $N = 1$ Supersymmetric Model

Before we talk about supersymmetry, let us briefly recall what one understands under a Lagrangian field theory.

A Lagrangian field theory is given by the following data

- a spacetime Σ (source)
- a target space X (usually a smooth Riemannian manifold)
- a space of field $\mathcal{F} = \mathrm{Maps}(\Sigma, X)$
- an action functional $S \in C^\infty(\mathcal{F})$

The action functional is usually not completely arbitrary but subject to certain constraints. For example we usually want S to be invariant under the actions of the symmetry group of Σ (e.g. the Poincaré group if Σ is Riemannian). Once we are equipped with such an action functional, the next step is to look at its extremal points. The resulting equations are known as the Euler-Lagrange (EL) equations of the theory and one often defines the *phase space* of the theory as the solution space of the EL equations. A solution of the EL equation describes a classical field.

Let us describe the Lagrangian field theory of a free particle moving on a manifold. As our spacetime we simply take the real line $\mathbb{R}$ with coordinate t.

We think of t as time. As our target space, we consider (X, g) a Riemannian manifold. The space of fields are then paths $x\colon \mathbb{R} \to X$, that is we set

$$\mathcal{F} = \mathrm{Maps}(\mathbb{R}, X) \tag{78}$$

Now, our space time $\mathbb{R}$ has a symmetry group given by the group structure on $\mathbb{R}$ via addition, or translation. Finite translations are generated by the vector field ∂_t which induces a vector field on $\mathcal{F}$ which acts by

$$x(t) \mapsto \dot{x}(t) \tag{79}$$

If our target manifold X is curved, then $\dot{x}$ describes the tangential map of x, i.e. its linearization and therefore it takes values in the pullback bundle x^*TX. We can then use the metric g on X to construct an action functional X which is invariant under time translations (the symmetry group of our spacetime):

$$S[x] = \frac{1}{2}\int_{\mathbb{R}} g(\dot{x}, \dot{x})dt. \tag{80}$$

Exercise 12
Show that the EL equation associated to (80) is given by the geodesic equation

$$\nabla_{\dot{x}}\dot{x} = 0$$

where ∇ is the Levi-Civita connection on defined by the metric g.

Classical particles thus move on geodesic of X and all such geodesics can be uniquely described by a starting point x_0 and a starting direction $\dot{x}_0$. The phase space of the free particle is therefore in one-to-one correspondence with the tangent space TX. In fact, it is more useful to use the Riemannian metric g on X to describe the phase space as the cotangent space $T^*X \cong_g TX$. This is done by simply going from velocities $\dot{x}^\mu$ to momenta $p_\mu = g_{\mu\nu}\dot{x}^\nu$.

To sum up, we describe a free particle by a map $\mathbb{R} \to X$ and a theory by choosing an action functional invariant under the induced action of the symmetry group of the spacetime $\mathbb{R}$. It seems now natural to add some fuzz, i.e. to define the *super particle* by a map $\mathbb{R}^{1|1} \to X$.

Let us fix our target manifold to be a Riemannian manifold (X, g) as before. Consider the super spacetime $\mathbb{R}^{1|1}$ with even coordinates t and odd coordinates θ. The space of super fields is then simply the space of maps from our super spacetime into our target manifold

$$\mathcal{F} = \mathrm{Maps}(\mathbb{R}^{1|1}, X). \tag{81}$$

As before, let us determine the symmetry group of our spacetime. The super vector space $\mathbb{R}^{1|1}$ admits the following group law

$$(t, \theta)(t', \theta') = (t + t' + \theta\theta', \theta + \theta') \tag{82}$$

Exercise 13

a) Show that(82) is generated by either one of the vector fields

$$D = \partial_\theta - \theta\partial_t \quad , \quad Q = \partial_\theta + \theta\partial_t.$$

b) Show that D generates right translations and Q generates left translations.

c) Show that D is a left-invariant and Q is right invariant.

d) Compute $D^2 = \frac{1}{2}[D, D]$.

Either of the vector fields D or Q generate the induced action of the symmetry group of $\mathbb{R}^{0|1}$ on $\mathcal{F}$. Let us focus on D. As usual, we define a $x(t, \theta) \in \mathcal{F}$ by its Taylor expansion in the odd variable θ

$$x(t, \theta) = x(t) + \theta\psi(t) \tag{83}$$

and analogously to our discussion of ΠTX one can show that ψ takes values in Πx^*TX. It follows that

$$D\colon x(t, \theta) \mapsto -\theta\dot{x}(t) + \psi \tag{84}$$

that is

$$D\colon \begin{cases} x(t) \mapsto \psi(t) \\ \psi(t) \mapsto -\dot{x}(t) \end{cases} \tag{85}$$

Likewise, one can show that

$$Q\colon \begin{cases} x(t) \mapsto \psi(t) \\ \psi(t) \mapsto \dot{x}(t) \end{cases} \tag{86}$$

Let us postpone the construction of an action functional invariant under (85) to Exercise 14. For the time being, we focus on the result:

$$S[x, \psi] = \frac{1}{2}\int_{\mathbb{R}} g(\dot{x}, \dot{x}) + g(\psi, \nabla_{\partial_t}\psi)dt \tag{87}$$

Exercise 14

Consider $x(t, \theta) \in \mathcal{F}$. Show that (87) is equivalent to

$$\frac{1}{2}\int_{\mathbb{R}} g\big(\dot{x}(t, \theta), Dx(t, \theta)\big)dtd\theta$$

Exercise 15
Show that the EL equations of (87) are given by

$$\nabla_{\partial_t}\dot{x} = R(\psi,\psi)\dot{x} \quad , \quad \nabla_{\partial_t}\psi = 0$$

where $R(\psi,\psi)\dot{x}^\mu = R^\mu_{\nu\alpha\beta}\psi^\alpha\psi^\beta\dot{x}^\nu$.

Exercise 16
Show that (87) is invariant under (85).
Hint: Use the results of Exercise 15.

Remark 5. The phase space of the model governed by the action (87) is the solution space of the EL equations computed in Exercise 15. This solution space can be parametrized by the initial conditions $(x_0, \dot{x}_0, \psi_0)$ and geometrically corresponds to the parity shifted pullback bundle

$$\Pi p^* TX \to TX$$

over TX where $p\colon TX \to X$ denotes the canonical projection. $\lhd$

Let us now shift our attention from the left invariant vector field D, to the right invariant vector field Q. It is not hard to see that

$$Q^2 = \frac{1}{2}[Q,Q] = \partial_t \tag{88}$$

so that Q is actually a *square root* of the generator of time translations! But the generator of time translation, once we quantize the theory, should be the Hamiltonian and hence we expect that the Hamiltonian of our quantum theory is actually a square of two odd operators. Moreover, it follow immediately from the (super) Jacobi identity that

$$[Q, \partial_t] = 0 \tag{89}$$

and hence we would expect that in the quantum theory Q is an odd symmetry, as it commutes with the Hamiltonian. This symmetry is called a *super symmetry* since it maps bosons (even) to fermions (odd) and vice versa, cf. (86). In this context, the action (87) of the super particle is known as an 1D $\mathcal{N} = 1$ supersymmetric model.

Exercise 17 (*)
Construct the 1D $\mathcal{N} = 2$ supersymmetric model by considering the super spacetime $\mathbb{R}^{1|2}$ with an even coordinates t and two odd coordinates $\theta, \bar{\theta}$. This model is called *supresymmetric quantum mechanics.*

5.2 SUSY QM

Quantization of the model $\mathcal{N}=1$ supersymmetric model is rather involved. In general, quantization is an art form and we would like to not discuss it here in detail. We may simply think of the quantization procedure as a black box machinery which we can feed a classical theory and which gives us back an associated Hilbert space and a Hamiltonian acting on said Hilber space. The Hamiltonian is the quantization of the generator of time translations and as we have seen it may happen that the theory admits an extra odd symmetry (or more!) such that the Hamiltonian can be written as a square of these symmetries. As we have seen way back in the motivation when we studied the supersymmetric quantum harmonic oscillator, the fact that the Hamiltonian is a square of odd operators has far reaching consequences. And in this section we want to explore heuristically some of these consequences. We will focus on the overall structure of supersymmetric quantum mechanics, i.e. of the 1D $\mathcal{N}=2$ supersymmetric model which we constructed in Exercise 17, as defined by Witten [4].

Definition 5.1. Supersymmetric quantum mechanics is defined by the following data:

1. a graded Hilbert space $\mathcal{H} = \bigoplus_{p\geq 0} \mathcal{H}^p$ which splits into a super Hilbertspace $\mathcal{H} = \mathcal{H}^{ev} \oplus \mathcal{H}^{odd}$ via the reduction of the grading $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$. Elements of $\mathcal{H}^{ev}$ are called *bosons*, elements of $\mathcal{H}^{odd}$ fermions
2. an odd (degree 1) vector field Q with dual Q^* such that
   (i) $Q^2 = \frac{1}{2}[Q, Q] = 0$
   (ii) $H = \frac{1}{2}[Q, Q^*]$

Let us discuss first consequences of this definition. Firstly, one can show that the H is positive

$$\langle\psi|H|\psi\rangle = \frac{1}{2}\langle\psi|[Q, Q^*]|\psi\rangle = \frac{1}{2}\left(\|Q|\psi\rangle\| + \|Q^*|\psi\rangle\|\right) \geq 0. \tag{90}$$

Secondly, since $[H, Q] = 0$, non-zero energy eigenstates must come in pairs of opposite parity: If $|n\rangle$ is an eigenstate of H with non-zero energy $E_n \neq 0$, then $Q|n\rangle$ has also energy E_n but opposite parity. This allows us to conclude that

$$\dim \mathcal{H}_n^{ev} = \dim \mathcal{H}_n^{odd} \tag{91}$$

whenever $E_n \neq 0$. This observation allows us to compute the partition function of the model purely in terms of the dimensions of the zero-energy eigenspaces

$$
\begin{aligned}
Z(\beta) &= Tr_{\mathcal{H}}\left((-1)^F e^{-\beta H}\right) \\
&= \sum_{n\geq 0} e^{-\beta E_n}\left(\dim \mathcal{H}_n^{ev} - \dim \mathcal{H}_n^{odd}\right) \\
&= \dim \mathcal{H}_0^{ev} - \dim \mathcal{H}_0^{odd} \\
&= \sum_p (-1)^p \dim \mathcal{H}_0^p
\end{aligned}
\tag{92}
$$

Here $(-1)^F$ denotes the parity operator which simply detects whether a space is bosonic (even) or fermionic (odd). It acts by $+1$ on $\mathcal{H}^{ev}$ and by -1 on $\mathcal{H}^{odd}$. A remarkable fact about (92) is that the RHS is *independent* of β! In physics, β is the inverse temperature and we may think of it as the length of the "thermal circle"—the compactification of our naive spacetime. Of importance is that β introduces a (geometric) scale. However, since the result is independent of β the theory is independent of that scale! One says that supersymmetric quantum mechanics is *topological*.

Now, since Q is a symmetry, we expect that it leaves physical states invariant $Q|\psi\rangle_{phys} = 0$. Such states are called Q-closed. Since $Q^2 = 0$, takes of the form $|\psi\rangle = Q|\chi\rangle$ trivially satisfy this property. Such states are called Q-exact. However, expectation values of physical observables involving those states vanish: if $Q(A)$ and B are two (physical) observables, then

$$\langle (QA)B\rangle \equiv TrQ(A)Be^{-\beta H} = TrQ\left(ABe^{-\beta H}\right) = 0 \tag{93}$$

Since Q is a symmetry and hence $TrQ(\dots) = 0$.

Hence, the space of physical states is equivalent to the *cohomology* of Q, namely to Q-closed states modulo Q-exact states. Let us compute the Q-cohomology in the basis of energy eigenstates. Suppose that $|n\rangle$ is an energy eigenstate of with energy $E_n \not= 0$. Suppose further that $Q|n\rangle = 0$. It then follows that

$$|n\rangle = \frac{1}{E_n}H|n\rangle = \frac{1}{2E_n}\left(QQ^* + Q^*Q\right)|n\rangle = Q\left(\frac{1}{2E_n}Q^*|n\rangle\right). \tag{94}$$

Therefore, $|n\rangle$ is Q-exact and hence zero in the Q-cohomology. That means that the whole cohomology of Q is concentrated in the ground states

$$H_Q^\bullet \subset \mathcal{H}_0. \tag{95}$$

Now, we know that $H \geq 0$. It then follows that $H|\psi\rangle = 0$ if and only if $Q|\psi\rangle = Q^*|\psi\rangle = 0$ which shows that

$$H_Q^\bullet \supset \mathcal{H}_0. \tag{96}$$

Since we are dealing with finite dimensional objects, we can conclude that in fact

$$H_Q^\bullet \cong \mathcal{H}_0. \tag{97}$$

This allows a very elegant interpretation of the partition function in homological algebraic terms, namely as the Euler characteristic of the cochain complex $(\mathcal{H}, Q)$

$$Z = \sum_{p\geq 0} (-1)^p \dim \mathcal{H}_0^p = \sum_{p\geq 0} (-1)^p \dim H_Q^p = \chi(\mathcal{H}, Q). \tag{98}$$

To digest these abstract notions, let us end with some examples.

Example 1. Let (X, g) be a Riemannian manifold and $\mathcal{H} = \Omega^\bullet(X)$, $Q = d$, $Q^* = d^*$. Then the Hamiltonian is given by the Laplacian on X: $H = \frac{1}{2}[d, d^*] = \frac{1}{2}\Delta_g$. The space of ground states of degree p is thus given by the kernel of the Laplacian acting on p-forms. It is known that this space is isomorphic to the p-th de Rham cohomology of X

$$\mathcal{H}_0^p = \ker \Delta_g|_{\Omega^p(X)} \cong H_{dR}^p \tag{99}$$

Moreover, evoking another deep theorem, the partition function then calculates the Euler characteristic of X

$$Z = \sum_p (-1)^p \dim H_{dR}^p = \chi(X) \tag{100}$$

◀

Example 2. Let (X, g, f) be a Riemannian manifold together with a specified Morse function $f \in \mathcal{C}^\infty(X)$. Consider again the Hilbert space $\mathcal{H} = \Omega^\bullet(X)$. This time, define

$$\begin{aligned} Q_s &= e^{-sf} d e^{sf} = d + s df\wedge \\ Q_s^* &= \left(e^{-sf} d e^{sf}\right)^* = d^* + s\iota(\mathrm{grad}\, f) \end{aligned} \tag{101}$$

Exercise 18
1. Show that $H_{Q_s}^\bullet \cong H_{Q_0}^\bullet$ and conclude that $Z_s = \chi(X)$ is independent of s.
2. Show that

$$H_s = \frac{1}{2}[Q_s, Q_s^*] = \frac{1}{2}\left(\Delta + s(\mathcal{L}_{\mathrm{grad}\, f} + \mathcal{L}_{\mathrm{grad}\, f}^*) + s^2 \|\mathrm{grad}\, f\|_g^2\right).$$

The parameter s introduces a scale. However we have seen that the partition function is independent of that scale. Hence we can study the model perturbatively, namely in the limit $s \to \infty$. Classically, the ground states (lowest energy eigenstates) now sit at the minima of the potential term in the Hamiltonian $V = s^2\|\mathrm{grad}\, f\|_f^2$, i.e. at points where $\mathrm{grad}\, f = 0$. But these points are nothing else than the critical points of f!

It is a theorem that locally there exists coordinates y^i centered around a fixed critical point p such that f takes the form $f(y) = f(p) + \sum_i \lambda_i^2 (y^i)^2 + \dots$, wheree λ_i are the eigenvalues of the Hessian $\mathrm{Hess}_p(f)$ of f at p. In these coordinates, and in the limit $s \to \infty$, the Hamiltonian as an operator on $\mathcal{H} = \Omega^\bullet(X)$ takes the form

$$H_s \sim s \sum_i -\frac{\partial^2}{(\partial y^i)^2} + \lambda_i^2 (y^i)^2 + \lambda_i [dy^i \wedge, \iota(\partial_{y_i})] + \mathcal{O}\left(\sqrt{s}\right) \tag{102}$$

One can now construct zero-energy eigenstates as follows: Observe first that it follows from the term linear in λ_i that if $\lambda_i < 0$, the presence of dy^i in a state $\ker \psi \in \Omega^\bullet(X)$ lowers the energy; if $\lambda_i > 0$, the presence of dy^i raises the energy. Hence, our lowest energy states $|\psi\rangle$ should be a differential form of degree $\mu(p) = |\{\lambda_i < 0\}|$. The number $\mu(p)$ depends on the cricital point p and is known as the *Morse index* of f at p, i.e.

$$|\psi(p)\rangle \in \Omega^{\mu(p)}(X) \tag{103}$$

Exercise 19
Define $|\psi_i\rangle$ by the rule

$$\begin{cases} \lambda_i < 0 : |\psi_i\rangle = e^{\lambda_i (y^i)^2/2} dy^i \\ \lambda_i > 0 : |\psi_i\rangle = e^{-\lambda_i (y^i)^2/2} \end{cases}$$

Show that $|\psi\rangle = \wedge_i |\psi_i\rangle$ has zero-energy to leading order in s.

Exercise 19 shows that the pertrubative ground states defined above are not necessarily zero-energy states. However, they have zero-energy in leading order in s. In particular, this shows that since $H^\bullet_{Q_s} = H^\bullet_{Q_0}$, there exists at least $b_k = \dim H^k_{Q_0}$ true (i.e. $E = 0$) ground states. Let m_k the number of perturbative ground states of form-degree k, or equivalently the number of critical points of f if index $\mu(p) = k$. From the above considerations it follows that

$$b_k \leq m_k. \tag{104}$$

These inequalities are known as *weak Morse inequalities*. They are a marvelous set of inequalities in particular because they relate analytical properties of the function f with toplogical properties of the manifold X.

Remark 6. From a physical perspective, the phenomena that some perturbative groundstates may have non-zero energy is related to the phenomena of quantum tunneling. If we are facing a classical theory with a degenerate ground state, then in its quantization the groundstate might split into a true ground sate and a slightly excited state. A perturbative ground state as discussed above can be such an excited state and thus might have non-zero energy. ◁

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Example 3. Let X be a smooth manifold and G a Lie group acting on X. The action of G is generated by vector fields $v_a \in \mathfrak{X}(X)$ associated to a basis e_a of the Lie algebra $\mathfrak{g}$ of G. These vector fields are known as the fundamental vector fields, defined as follows: let $x \in \mathfrak{g}$ and $f \in \mathcal{C}^\infty(X)$. Consider the flow e^{tx} which acts (as an element in G) on X by $e^{tx} \cdot x$. Then we define the fundamental vector field $x^\sharp \in \mathfrak{X}(X)$ by

$$x^\sharp(f) = \frac{d}{dt} f(e^{tx} \cdot x)\Big|_{t=0} \tag{105}$$

Then the $v_a = e_a^\sharp$.

Now, consider as Hilbert space

$$\mathcal{H} = (W(\mathfrak{g}) \otimes \Omega^\bullet(X))_{\text{basic}} \tag{106}$$

where

$$W(\mathfrak{g}) = \Lambda^\bullet \mathfrak{g}^* \otimes \operatorname{Sym}^\bullet \mathfrak{g} \tag{107}$$

is the Weyl algebra associated to $\mathfrak{g}$ with odd c^a generating the factor $\Lambda^\bullet \mathfrak{g}^*$ and even coordinates u^a generating the factor $\operatorname{Sym}^\bullet \mathfrak{g}$. Then take

$$Q = d_W + d_{dR} = -\frac{1}{2}[c,c]^a \frac{\partial}{\partial c^a} - u^a \frac{\partial}{\partial c^a} + dx^\mu \frac{\partial}{\partial x^\mu} \tag{108}$$

and

$$Q^* = c^a \frac{\partial}{\partial u^a} + d^* \tag{109}$$

Exercise 20

a) Show that $Q^2 = 0$

b) Show that $H = \frac{1}{2}\left(\Delta_g + u^a \frac{\partial}{\partial u^a} + c^a \frac{\partial}{\partial c^a}\right)$

The cohomology of Q is known as the Weyl model of the *equivariant* cohomology of X. Let us briefly sketch the idea of equivariant cohomology. Ideally, the action of G on X is nice enough (most of all fixed-point free) so that the orbit space X/G is again a smooth maniofld and we can compute its de Rham cohomology. But sometimes, the G-action is not quite so nice and X/G is not a smooth manifold. However, for a topologist, only the homotopy type of the *topological space* X/G matters. After all, the cohomology of a topological manifold is a topological invariant. Hence, a topologist would simply replace X by some other space $\tilde{X}$ which admits a nice G-action such that on one hand $X_G = \tilde{X}/G$ is again smooth manifold and on the other hand $\tilde{X}/G$ and X/G are homotopically the same. It turns out that this can be achieved by setting

$$\tilde{X} = X \times EG$$

where EG is the total space of the universal bundle over the classifying space BG. Roughly speaking, EG is a G-bundle over a space BG calssifying all principal G-bundles over X: Any G-bundle $P \to X$ over X can be constructed as the pull back of $EG \to BG$ by some map $f: X \to BG$. Another important property is that EG is simply connected and therefore X and $X \times EG$ have the same homotopy type. One then defines the equivariant cohomology by

$$H^\bullet_G(X) = H^\bullet(X_G).$$

Let us come back to our model. The summand $u^a \frac{\partial}{\partial u^a} + c^a \frac{\partial}{\partial c^a}$ of the Hamiltonian is the Euler vector field and simply counts the degree in u and c. Hence, the ground states are given by

$$\ker(H) \cong \ker \Delta_g. \tag{110}$$

Now, on one hand

$$Z = \sum_p (-1)^p \dim H^p(\mathcal{H}, Q) = \sum_p (-1)^p \dim H^p((M \times EG)/G) = \chi(M_G). \tag{111}$$

On the other hand

$$z = \sum_p (-1)^p \dim \mathcal{H}_0^p = \sum_p (-1)^p \dim \ker \Delta_g|\Omega^p(X) = \sum_p (-1)^p \dim H^p_{dR} = \chi(X) \tag{112}$$

and we can conclude that the Euler characteristic of X_G is the same as the Euler characteristic of X

$$\chi(X_G) = \chi(X) \tag{113}$$

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