Introduction to Symplectic Geometry

Donald R. Youmans*

July 16, 2024

Contents

1 Preface

This is the write up of a nano-course on an introduction to symplectic geometry given at the summer school MAPSS in July 2024 in Les Diablerets. It was aimed at young mathematical physicists who find themselves at any stage of their Master program. If you find any severe mistakes, or worse, *misunderstandings*, as well as typos—or you have an idea that would make this lecture more accessible—please let me know!

As for the course, it is a gentle introduction to the wonderful world of symplectic geometry. Motivated by classical mechanics, we generalize the concept of a *phase space* and develop the first bits of the theory of symplectic vector spaces and symplectic manifolds. The main aim of these lecture notes is the (linear) Darboux theorem which tells us that locally all symplectic manifolds look alike. Put differently, there exist no local symplectic invariants!

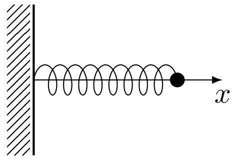
2 Motivation: Classical Mechanics

We start with Newton's law

$$m\ddot{x} = F(x) \tag{1}$$

*donaldryoumansjr@gmail.com

which describes the motion of a particle under the influence of a force F. The most often encountered example in physics is probably the one-dimensional harmonic oscillator, and we shall keep up with this tradition. Neglecting friction and damping, the harmonic oscillator describes the motion of a particle attached to a spring.



In this case, once we have stretch or squeeze the spring together, there exists a force trying to put the particle back to its initial position:

$$F(x) = -m\nu^2 x \tag{2}$$

We can recast Newton's second order ODEs (1) into two first order ODEs

$$\begin{aligned} m\dot{x} &= p \\ \dot{p} &= -\nu^2 x \end{aligned} \tag{3}$$

The above system of ODEs admits the following solution subject to the initial conditions $x(0) = x_0$, $p(0) = p_0$

$$\begin{aligned} x(t) &= x_0 \cos(\nu t) + \frac{p_0}{m\nu} \sin(\nu t), \\ p(t) &= -m\nu x_0 \sin(\nu t) + p_0 \cos(\nu t) \end{aligned} \tag{4}$$

The energy of the system is given by

$$E = E_{kin} + E_{pot} = \frac{p^2}{2m} + \frac{m\nu^2 x^2}{2} = H(x, p) \tag{5}$$

Note that with (4)

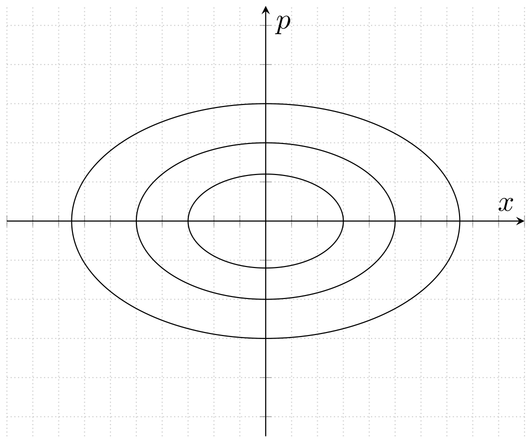
$$H(x(t), p(t)) = \frac{p_0^2}{2m} + \frac{m\nu^2 x_0^2}{2} = E_0 = cst \tag{6}$$

We now define the *phase space* M as the space of all positions and momenta of the particle. A point in M is simply given by (x, p). In M, the motion describe (4) ellipses

$$\frac{x^2(t)}{a^2} + \frac{p^2(t)}{b^2} = 1, \tag{7}$$

where $a^2 = \frac{2E_0}{m\nu^2}$ and $b^2 = 2mE_0$.

Trajectories in Phase Space



In phase space, a solution $(x(t), p(t))$ to (3) is just an integral curve (flow) of the vector field $v(x, p) = p/m\partial_x - \nu^2 x\partial_p$ on phase space.

The vector field $v(p, x)$ is somewhat special: Note that the ODE (3) could be written as

$$\begin{aligned} \dot{x} &= \frac{\partial H(x,p)}{\partial p} \\ \dot{p} &= -\frac{\partial H(x,p)}{\partial x} \end{aligned} \tag{8}$$

where $H(x, p) = \frac{p^2}{2m} + \frac{m\nu^2 x^2}{2}$ as before. So the vector field looks like a "skew-gradient"!

Remark 1. Note that from (8), geometrically $\dot{x}$ denotes the tangent vector of of the curve $x(t)$. But then p must transform oppositely to $\dot{x}$ and hence defines a co-vector. Thus the phase space M with coordinates (x, p) is geometrically speaking a co-tangent space $M = T^*\mathbb{R}$ where the base $\mathbb{R}$ defines the position of the particle. $\triangleleft$

Notice that we can write (8) in matrix notation as

$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial p} \end{pmatrix} \tag{9}$$

The RHS looks like almost like the gradient of H, in fact it looks like some sort of skew-version of it. Indeed, in components we have

$$\dot{m}^i = \omega^{ij} \frac{\partial H}{\partial m^j} \equiv (\mathrm{sgrad}_\omega H)^i \tag{10}$$

where $m = (x, p)$ is a point in phase space. It turns out that not ω^{ij} but rather its inverse ω_{ij} will play a fundamental role in all that follows. We can think of $\omega_{ij} = \varepsilon_{ij}$ the total anti-symmetric symbol ($\varepsilon_{12} = 1 = -\varepsilon_{21}$) as a constant, non-degenerate 2-form

$$\omega = dx \wedge dp \in \Lambda^2 T^* M \tag{11}$$

Remark 2. Note that ω defines a map

$$\omega^\flat \colon T_m M \to T_m^* M \quad , \quad v \mapsto \omega(v, -) \tag{12}$$

so that

$$(\omega^\flat)^{-1} \colon T_m^* M \to T_m M. \tag{13}$$

Exercise 1
Show that $\operatorname{sgrad}_\omega H$ can be defined by

$$\omega(\operatorname{sgrad}_\omega H, -) + dH = 0 \tag{14}$$

◁

Moreover, for consider any other function $f \in \mathcal{C}^\infty(M)$ changes along a solution (4) according to

$$\begin{aligned} \frac{df(p(t), x(t))}{dt} &= \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial p}\dot{p} = \frac{\partial f}{\partial x}\frac{\partial H}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial H}{\partial x} \\ &= \frac{\partial f}{\partial m^i}\omega^{ij}\frac{\partial H}{\partial m^j} \end{aligned} \tag{15}$$

For two functions $f(x, p), g(x, p)$ on phase space define the bracket

$$\begin{aligned} &\{\cdot, \cdot\} \colon \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M) \\ &f \times g \mapsto \{f, g\} = \partial_i f \omega^{ij} \partial_j g = \omega^{-1}(df, dg) \end{aligned} \tag{16}$$

Exercise 2
Verify the following properties of the bracket

a) anti-symmetry $\{f, g\} = -\{g, f\}$

b) Jacobi
$$\{f, \{g, h\}\} = \{\{f, g\}, h\} + \{g, \{f, h\}\}$$

c) $\{f, g\} = \omega(\operatorname{sgrad}_\omega g, \operatorname{sgrad}_\omega f)$

The bracket (16) is called a *Poisson bracket* and endowes the space of functions $\mathcal{C}^\infty(M)$ with the structure of a Lie algebra. Moreover, notice that once we have fixed the Hamiltonian H, the time-evolution along constant energy hypersurfaces of any other function[1] $f \in \mathcal{C}^\infty(M)$ is given by the Poisson bracket with H

$$\frac{df}{dt} = \{f, H\}. \tag{17}$$

We now come to an important theorem due to Liouville which states that the motion of an ensemble behaves like the flow of an incompressible fluid in phase space. Note that for a phase space $T^*\mathbb{R}^n$, the constant 2-form $\omega = \sum_i dx^i \wedge dp_i$ defines a volume form

$$d\lambda = \frac{\omega^n}{n!} = \prod_i dx^i \wedge dp_i. \tag{18}$$

This volume form is known as the *Liouville volume form.*

Theorem 2.1 (Liouville)**.** The Liouville volume of phase space is preserved along classical trajectories.

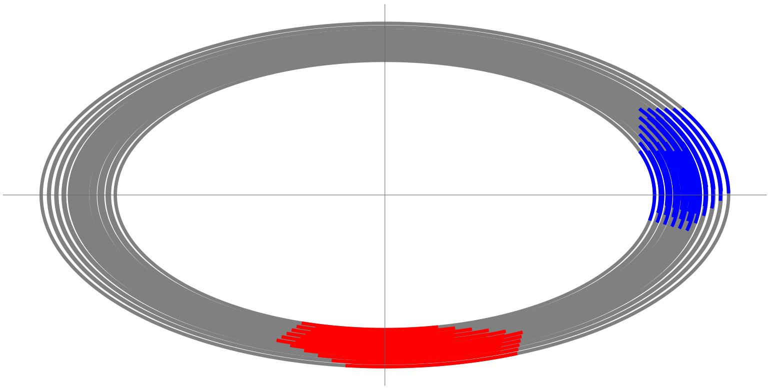


Figure 1: Evolution of Liouville volume of a subset U along classical trajectories

Exercise 3
Show that for $U \subset M$,

$$vol(U) = \int_U d\lambda$$

is invariant along classical trajectories.

We will come back to give a one-line proof of this theorem.

Let us end with some structural obserevations:

[1]Such a function is called an *observable.*

1. The system's motion is described by a motion in a phase space $M = T^*X$ whose coordinates (x, p) are position and momenta respectively.

2. The phase space M is equipped with an extra structure, a *non-degenerate*, *anti-symmetric* matrix ω_{ij} which we use to define a "gradient"

$$\operatorname{sgrad}_\omega f^i = \omega^{ij}\partial_j f$$

for any function $f \in \mathcal{C}^\infty(M)$.

3. The system is defined dy the datum (M, ω, H) where $H \in \mathcal{C}^\infty(M)$ is known as the *Hamiltonian.* The motion of a particle whose energy is given by H is defined by the ω-gradient flow of H, namely by the Hamilton equations

$$\dot{m} = \operatorname{sgrad}_\omega H, \qquad m = (x, p)$$

4. The evolution of any other observable $f \in \mathcal{C}^\infty(M)$ is given by the Poisson bracket with H

$$\frac{df}{dt} = \{f, H\} = \partial_i f \omega^{ij} \partial_j H$$

which means in particular that the energy is conserved (constant) along the phase space trajectories (generated by H)

$$\frac{dH}{dt} = \{H, H\} = 0$$

5. Louville's theorem implies that the volume $vol_\omega(x, p) = \frac{1}{2}\omega_{ij} dm^i \wedge dm^j$ is preserved along the Hamiltonian flow.

As it turns out, the structural observations above are a special instance of what is known as a symplectic (linear) space. In the following we will consider its basic definitions and some of its interesting consequences.

3 Symplectic Linear Geometry

We first consider the linear case.

Let V be a finite dimensional vector space over $\mathbb{R}$ and $\omega \in \Lambda^2 V^*$, i.e.

$$\omega \colon V \times V \to \mathbb{R}, \qquad \text{bilinear, skew-symmetric.} \tag{19}$$

And let

$$\ker(\omega) = \{v \in V \mid \forall u \in V : \omega(v, u) = 0\}.$$

Definition 3.1. The pair (V, ω) is called *symplectic* if $\ker(\omega) = 0$, i.e. if ω is non-degenerate.

Remark 3 (!)**.** We think of ω as a constant 2-form on V, namely $\omega \in \Omega^2(V)$. $\triangleleft$

Note that ω defines a map

$$\begin{aligned} \omega^\flat \colon V &\to V^* \\ v &\mapsto \omega(v, \cdot) \end{aligned} \tag{20}$$

Proposition 3.1. (V, ω) is symplectic iff $\omega^\flat$ is an isomorphism.

> **Exercise 4**
> Prove Proposition 3.1

Remark 4. Since we are working in finite dimensions, $(V^*)^* = V$. Let $\pi \in \Lambda^2 V$. Then π defines a map $\pi^\sharp \colon V^* \to V$ by $V^* \ni \alpha \mapsto \pi^\sharp(\alpha, -)$.

Proposition 3.2. (V, ω) symplectic iff there exists $\pi \in \Lambda^2 V$ such that $(\omega^\flat)^{-1} = \pi^\sharp$.

> **Exercise 5**
> Prove Proposition 3.2.

$\triangleleft$

Examples:

1. $V = \mathbb{R}^{2n} = T^*\mathbb{R}^n$ with basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ and

$$\omega_0(e_i, e_j) = \omega_0(f_i, f_j) = 0 \quad , \quad \omega_0(e_i, f_j) = \delta_{ij} \tag{21}$$

   Often we will consider coordinates (x^i, p_j) for $i, j = 1 \ldots n$ of $\mathbb{R}^{2n}$. Then, seen as a constant 2-form,

$$\omega_0 = \sum_i dx^i \wedge dp_i \tag{22}$$

2. Let L be a n-dimensional vector space over $\mathbb{R}$. Set $V = L \oplus L^*$, and let $x, y \in L$, $\xi, \eta \in L^*$ and define

$$\omega(x + \xi, y + \eta) = \xi(y) - \eta(x). \tag{23}$$

3. Let V be a finite dimensional vector space of $\mathbb{C}$ with hermitian form

$$h \colon V \times V \to \mathbb{C}$$

   such that h is bilinear and

$$h(x, y) = \overline{h(y, x)} \quad , \quad h(x, x) \geq 0 \quad , \quad h(\lambda x, \mu y) = \bar{\lambda}\mu \, h(x, y) \tag{24}$$

   Then define

$$\omega(x, y) = \mathrm{Im}(h(x, y)) \tag{25}$$

Exercise 6
Prove that in all the examples (V, ω) is symplectic.

Once one has defined a new structure, one is usually interested in its symmetries. In our case, the symmetry of a symplectic vector space (V, ω) is known as the *symplectic group*

$$\mathrm{Sp}(V, \omega) = \{g \in \mathrm{GL}(V) \mid g^*\omega = \omega\} \tag{26}$$

Exercise 7
Some exercise about Sp for easy examples?

We know turn our focus to various natural but important subspaces defined by ω. Let $U \subset V$ be a linear subspace.

Definition 3.2. The *symplectic complement* of the linear subspace U is defined as the space

$$U^\omega = \{x \in V \mid \omega(x, y) = 0 \ \forall y \in U\} \tag{27}$$

It is interesting to compare this definition to the annihilator $\mathrm{Ann}(U) \subset V^*$ of U

$$\mathrm{Ann}(U) = \{\alpha \in V^* \mid \langle \alpha, x \rangle = 0 \ \forall x \in U\} \subset V^*. \tag{28}$$

Exercise 8
Show $\omega^\flat(U^\omega) = \mathrm{Ann}(U)$.

Remark 5. Notice that in finite dimensions,

$$\dim U^\omega = \dim \mathrm{Ann}(U) = \dim V - \dim U \tag{29}$$

$\triangleleft$

Proposition 3.3.

(i) $(U^\omega)^\omega = U$.

(ii) $(U_1 \oplus U_2)^\omega = U_1^\omega \cap U_2^\omega$.

Exercise 9
Prove Proposition 3.3.

Now, the symplectic complement is quite different from the orthogonal complement, say. In particular, it allows for the possibility that U *itself* is in U^ω or vice versa!

Definition 3.3. Let (V, ω) be a finite dimensional symplectic vector space and $U \subset V$ a linear subspace.

(i) U is said to be *isotropic* if $U \subset U^\omega$, equivalency if $\omega|_U = 0$.

(ii) U is said to be *coisotropic* if $U^\omega \subset U$.

(iii) U is said to be *Lagrangian* if $U = U^\omega$.

(iv) U is said to be *symplectic* if $U \cap U^\omega = \{0\}$.

> **Exercise 10**
> Show that U is isotropic iff U^ω is coisotropic.

Let us study some examples of this definitions.

1. isotropic subspace: Any linear subspace of dim $= 1$ is isotropic. Let $0 \neq x \in V$. Consider the line $\ell_x = \mathrm{span}_{\mathbb{R}}(x) \subset V$.

> **Exercise 11**
> Show that ℓ_x is isotropic.

2. coisotropic subspace: any linear subspace of codim $= 1$ is co-isotropic.

> **Exercise 12**
> Consider $(V, \omega) = (\mathbb{R}^{2n}, \omega_0)$. Show that
>
> $$U = \{(x, p) \in V \mid x^n = 0\}$$
>
> is coisotropic.

Remark 6. From the point of view of classical mechanics, $V = T^*U$ is the phase space and U is the spatial configuration space. We restrict the motion of the particle in space to the hypersurface defined by $x^n = 0$. This is a general phenomena: constraint motion in phase space T^*X is modeled by hypersurfaces of the form $\{\phi_i(x, p) = 0\}$ for some functions $\phi_i \in \mathcal{C}^\infty(T^*X)$. We will come back to this point in Section 4. $\triangleleft$

3. Lagrangian subspace: Consider $(V = L \oplus L^*, \omega_0)$. Then L, L^* are Lagrangian.

Exercise 13
Consider the symplectic space $(V, \omega) = (\mathbb{R}^{2n}, \omega_0)$. Let $U = \mathbb{R}^n \subset V$ and $f \in C^\infty(U)$. Show that

$$\text{graph}(df) = \left\{(x^i, p_i = \frac{\partial f}{\partial x^i})\right\}$$

is Lagrangian.

Proposition 3.4 (linear reduction lemma)**.** Let (V, ω) be symplectic, $U \subset V$ isotropic. Then $W = U^\omega/U$ is symplectic with symplectic form

$$\bar{\omega}([x], [y]) = \omega(x, y) \tag{30}$$

for any $x, y \in U^\omega$.

Lagrangian subspaces are very special and play an important role in example in quantization.

Proposition 3.5. Let (V, ω) be symplectic.

(i) There exists a Lagrangian subspace $L \subset V$ and $\dim V = 2 \dim L$.

(ii) If $L \subset V$ is Lagrangian, then there exists a Lagrangian complement $M \subset V$ such that $V = L \oplus M$.

Proof.

(i) We already have seen that isotropic subspaces exist. Let $L \subset V$ be a maximal isotropic subbspace, in the sense that it is not contained in any other isotropic subspace of strictly greater dimension. Then L is Lagrangian. Indeed, if $L^\omega \neq L$, then $\exists v \in L^\omega \backslash L$ so that $L' = L \oplus \text{span}_{\mathbb{R}}(v)$ is again isotropic: let $x + \alpha v, y + \beta v \in L'$, then

$$\begin{aligned} \omega(x + \alpha v, y + \beta v) &= \omega(x, y) + \alpha\omega(v, y) + \beta\omega(x, v) + \alpha\beta\omega(v, v) \\ &= \alpha\beta\omega(v, v) = 0 \end{aligned}$$

where we used that L is Lagrangian (thus $\omega(x, y) = 0$) and $v \in L^\omega$ (thus $\omega(v, y) = \omega(x, v) = 0$). Now, L' contains L and is of strictly greater dimension which is a contradiction to our starting assumption. It then follows that

$$\dim V = \dim L + \dim L^\omega = 2 \dim L.$$

(ii) Since we are working in finite dimensions, the statement is equivalent to $L \cap M = \{0\}$ with M Lagrangian. Consider an isotropic subspace

U such that $U \cap L = \{0\}$. Such subspaces exist: take any $x \notin L$ (exists for dimensional reasons) and consider the line $\text{span}_{\mathbb{R}}(x)$. Now, we claim that U if U is maximal isotropic with $U \cap L = \{0\}$, then U is Lagrangian.

Assume U is maximal isotropic with $U \cap L = \{0\}$ but U is not Lagrangian. Consider $0 \neq [x] \in U^\omega/U$ and let $U_x = U \oplus \text{span}_{\mathbb{R}}(x)$ which is again isotropic. Since $U \subset U_x$, it is left to show that there exists an $[x] \in U^\omega/U$ such that $U_x \cap L = \{0\}$, since this would contradict our assumption of maximality of U. Note that $[x] = [x+y]$ for any $y \in U$ and since $U \cap L = \{0\}$, $x + y \notin L$. Now, $U_{x+a} \cap L = \{0\}$ for some $a \in U$ and we arrive at our desired contradiction since $U \subset U_{x+a}$.

□

Most importantly, the existence of Lagrangians implies that all finite dimensional linear symplectic spaces are isomorphic. In fact, they are all isomorphic to the model $(\mathbb{R}^{2n}, \omega_0)$.

Theorem 3.6. (Linear Darboux Theorem) Let (V, ω) be a linear symplectic space. Assume $\dim V = 2n$. Then

$$(V, \omega) \cong (\mathbb{R}^{2n}, \omega_0). \tag{31}$$

Proof. Let (V, ω) be symplectic, $\dim V = 2n$. Then by Proposition 3.5, there exists a Lagrangian subspaces $L, M \subset V$ such that $V = L \oplus M$. This gives the following isomorphism

$$M \hookrightarrow V \xrightarrow[\cong]{\omega^\flat} V^* \twoheadrightarrow L^* \tag{32}$$

which implies that $M \cong L^*$. Explicitly, the isomorphism is given by

$$\Phi \colon M \to L^* \quad , \quad x \mapsto \omega(x, -)|_L$$

Exercise 14
Show that the kernel of the above map is empty.
Hint: Show that $\ker \Phi \subset L^\omega$ and use that L is Lagrangian.

Now, choose a basis $\{e_1, \ldots, e_n\}$ in L and a dual basis $\{f_1, \ldots, f_n\}$ in $M \cong L^*$. Since L, M are isotropic, one has

$$\omega(e_i, e_j) = \omega(f_i, f_j) = 0 \quad \text{and} \quad \omega(e_i, f_j) = \delta_{ij}$$

which is just the standard definition of ω_0 on $\mathbb{R}^{2n}$. □

4 Foundations of Symplectic Geometry

We now generalize symplectic structures to symplectic structures on general manifolds.

In the following, to fix notations, let M be a smooth manifold, $\omega \in \Omega^2(M)$. Recall that for $m \in M$, $\omega_m \in \Lambda^2 T^*_m M$, $(T^*_m M, \omega_m)$ is a symplectic vector space iff $\ker \omega_m = \{0\}$, i.e. iff ω_m is non-degenerate. Now we would like to patch these definitions together in a natural way to define the notion of a sympelctic manifold.

Definition 4.1. (M, ω) is *symplectic* iff

(i) ω_m is non-degenerate $\forall m \in M$

(ii) ω is closed: $d\omega = 0$.

Remark 7. The closeness condition generalizes (and relaxes) the assumption that ω is constant. ◁

This definition has some quite non-trivial but interesting consequences, which follow from our discussion of the linear case:

1. If (M, ω) is symplectic, then $\dim T_m M = 2n$ for all $m \in M$ and hence $\dim M = 2n$ is even.

2. If (M, ω) symplectic, $\dim M = 2n$, then $\lambda = \frac{\omega^n}{n!}$ is nowhere vanishing and defines a volume form known as the *Liouville volume form*. This follows directly from the fact that $\lambda_m = \frac{\omega_m^n}{n!}$ is nowhere vanishing $\forall m \in M$. In particular, M is orientable with canonical orientation defined by λ.

3. $vol_\omega(M) = \int_M \lambda > 0$ defines the symplectic volume of M.

4. $\forall m \in M : \exists_! \pi_m \in \Lambda^2 T_m M$ such that $\pi_m^\sharp = (\omega_m^\flat)^{-1}$. The π_m patch together to a smooth bivector $\pi \colon M \to \Lambda^2 TM$ which defines a Lie bracket (the Poisson bracket) on $\mathcal{C}^\infty(M)$ by

$$\{f, g\} := \pi(df, dg) = \pi^{ij} \partial_i f \partial_j g \tag{33}$$

Before we study maps between symplectic manifolds, let us look at some examples.

1. $M = \mathbb{R}^{2n} = T^*\mathbb{R}^n$ with coordinates $\{x^1, \ldots, x^n, p_1, \ldots, p_n\}$ with $\omega_0 = \sum_i dx^i \wedge dp_i$. In this case the Liouville volume form simply takes the form $d\lambda = \prod_i dx^i \wedge dp_i$.

2. M any two-dimensional orientable manifold (e.g. S^2). Then any volume form on M defines a symplectic form on M.

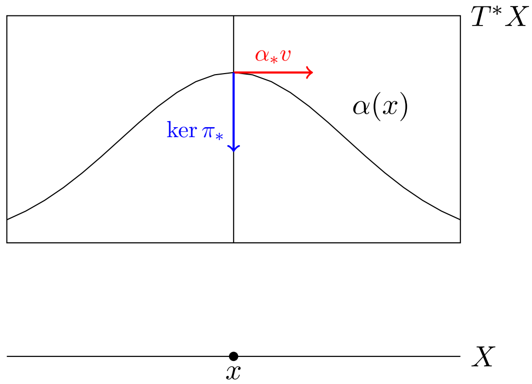
3. The cotangent bundle $M = T^*X$. Let $\pi\colon T^*X \to X$ be the canonical projection.

   Proposition 4.1. There exists a unique 1-form θ on T^*X such that $\alpha^*\theta = \alpha$ for all $\alpha \in \Omega^1(X)$.

   The 1-form $\theta \in \Omega^1(T^*X)$ is known as the *tautological* 1-form. In local coordinates $\{x^1, \dots, x^n\}$ of X and $\{p_1, \dots, p_n\}$ of T_xX,

$$\theta_{(x,p)} = \sum_i p_i dx^i \tag{34}$$

   Now, any $\alpha \in \Omega^1(X)$ can be seen either as a differential form $\alpha_x = \sum_i \alpha_i dx^i$ or as a map $X \ni x \to (x, \alpha(x)) \in T^*_xX$ such that $\pi \circ \alpha = id_X$, cf. (35)



(35)

   If we view $\alpha \in \Omega^1(X)$ as the map $x \mapsto (x, \alpha(x))$, then

$$\alpha^*\theta = \sum_i \alpha_i(x) dx^i = \alpha \tag{36}$$

   where on the RHS, α is interpret as a differential form on X.

   Proof. Let $(x,p) \in T^*X$. For any $v \in T_{(x,p)}(T^*X)$, we define the 1-form $\theta_{(x,p)} \in T^*_{(x,p)}(T^*X)$ by

$$\left\langle \theta_{(x,p)}, v \right\rangle = \langle p, \pi_*(v) \rangle \,. \tag{37}$$

   Note that $\pi_*\colon T_{(x,p)}(T^*X) \to T_{\pi(x,p)}X = T_xX$. Now consider $\alpha\colon X \to T^*X$, $\pi \circ \alpha = id_X$, so that $\alpha(x) \in T^*_xX$. Then

$$\begin{aligned}\left\langle (\alpha^*\theta)_{(x,p)}, v \right\rangle &= \left\langle \theta_{(x,\alpha(x))}, \alpha_* v \right\rangle = \langle \alpha(x), \pi_* \alpha_* v \rangle \\ &= \langle \alpha(x), (\pi \circ \alpha)_* v \rangle = \langle \alpha(x), v \rangle\end{aligned}$$

   which shows that $\alpha^*\theta = \alpha$.

It is left to show that θ is unique. To do so, we will show that it is defined on a dense subset of $T(T^*X)$. Recall that we can view $\alpha \in \Omega^1(X)$ as a graph $\alpha\colon x \mapsto (x, \alpha(x))$ inside T^*X. By definition, for any $v \in TX$ one has

$$\langle \theta, v \rangle = \langle \alpha^* \theta, v \rangle = \langle \theta, \alpha_* v \rangle . \tag{38}$$

Now, the vector $\alpha_* v \in T(T^*X)$ is tangent to the "curve" $\alpha(x) \subset T^*X$. Varying α and v we can actually find *all* of vectors in $T(T^*X)$ except for those who are *vertical*, that is those that lie in $\ker \pi_*$ where $\pi\colon T^*X \to X$ is the canonical projection. Indeed, since $\pi \circ \alpha = id_X$,

$$\pi_*(\alpha_* v) = (\pi_* \alpha_*) v = (\pi \circ \alpha)_* v = v \tag{39}$$

and hence $\alpha_* v$ for $v \neq 0$ cannot belong to the kernel $\ker \pi_*$. □

Proposition 4.2. The cotangent bundle T^*X endowed with $\omega_0 = -d\theta$ is symplectic.

Proof. First of all, by definition, ω_0 is closed: $d\omega_0 = -d^2\theta = 0$. It is left to show that ω_0 is non-degenerate. Locally, $(\omega_0)_{(x,p)} = -d\theta = -d\sum_i p_i dx^i = \sum_i dx^i \wedge dp_i$ which is clearly non-degenerate. □

We now shift our focus to maps between symplectic manifolds.

Definition 4.2. Let (M_1, ω_1), (M_2, ω_2) be symplectic. A map $f\colon M_1 \to M_2$ is a *symplectomorphism* if

(i) f is a diffeomorphism

(ii) $f^*\omega_2 = \omega_1$

"Symmetries" of (M, ω) are then simply symplectoendomorpisms and in analogy with the linear case we define

$$\mathrm{Sp}(M, \omega) = \{ f \in \mathrm{Diff}(M) \mid f^*\omega = \omega \} \tag{40}$$

A diffeomorphism f is thus a symmetry iff it leaves the symplectic structure invariant. In particular, infinitesimally this means that f is generated by a vector field v preserving the symplectic structure.

Definition 4.3. A $v \in \mathfrak{X}(M)$ is called *symplectic* if $\mathcal{L}_v \omega = 0$, where $\mathcal{L}_v$ denotes the Lie derivative along the vector field v. We denote the space of all symplectic vector fields by

$$\mathfrak{X}(M, \omega) = \{ v \in \mathfrak{X}(M) \mid \mathcal{L}_v \omega = 0 \} \tag{41}$$

Aside: The Flow of a Vector Fields and the Lie Derivative Let M be a smooth manifold and $v \in \mathfrak{X}(M)$ be a vector field. Following the vector field along its integral curves, we get a curve Φ_t whose tangent vectors are given by the vector field v everywhere: for $m \in M$ a point,

$$\frac{d\Phi_t(m)}{dt}\Big|_{t=0} = v_m. \tag{42}$$

The curve Φ_t is called the *flow* of the vector field v and we sometimes say that Φ_t *integrates* v. For example, Hamilton's equation describing the movement of a particle of a system governed by the Hamiltonian H define a flow equation cf. (10): if $(x, p) \in M = (\mathbb{R}^2, \omega_0)$

$$\dot{x}(t) = \frac{\partial H}{\partial p} \quad , \quad \dot{p}(t) = -\frac{\partial H}{\partial x}. \tag{43}$$

The flow $\Phi_t = (x(t), p(t))$ integrates the vector field

$$\operatorname{sgrad}_{\omega_0} H = \partial_p H \partial_x - \partial_x H \partial_p \in \mathfrak{X}(M) \tag{44}$$

Now, often times we are interested in how geometric objects like vectors and co-vectors (or in general differential forms) change under the flow Φ_t. To measure the infinitesimal change, we push the vector forward or pull the co-vector back respectively along Φ_t and then compute the derivative with respect to t evaluated at time $t = 0$. Examples:

1. Infinitesimal change of a scalar (function).

   Given $v \in \mathfrak{X}(M)$, let Φ_t^v be its flow. Let $f \in \mathcal{C}^\infty(M)$ be smooth function. Then f changes infinitesimally along Φ_t as follows:

   $$\mathcal{L}_v f := \frac{d}{dt}\Big|_{t=0} (\Phi_t^v)^* f(x) = \frac{d}{dt}\Big|_{t=0} f(\Phi_t^v(x)) = v(x)\partial_x f(x). \tag{45}$$

   A function thus changes infinitesimally as we would expect, namely by the derivative in the direction of the vector field v.

2. Infinitesimal change of a vector.

   Given vectors $v, u \in \mathfrak{X}(M)$, let Φ_t^v be the flow of v. Then u changes along the Φ_t^v infinitesimally as follows:

   $$\begin{aligned}\mathcal{L}_v u := \frac{d}{dt}\Big|_{t=0} (\Phi_t^v)_* u^i(x)\partial_x &= \frac{d}{dt}\Big|_{t=0} u(\Phi_t^v(x))\partial_{\Phi_t^v(x)} \\ &= (v(x)u'(x) - u(x)v'(x))\partial_x \\ &= [v, u]\end{aligned} \tag{46}$$

3. Infinitesimal change of differential forms.

Exercise 15
Let $\alpha \in \Omega^1(M)$.

a) Show that $\mathcal{L}_v\alpha = (d\iota_v + \iota_v d)\alpha$.

b) Deduce that $\mathcal{L}_v = d\iota_v + \iota_v d$ on $\Omega^\bullet(M)$.

Hint: Consider general homogeneous elements $\alpha = \alpha_1 \wedge \cdots \wedge \alpha_k \in \Omega^k(M)$.

The operator $\mathcal{L}_v$ defining the infinitesimal change is known as the *Lie derivative*. The representation of the Lie derivative on differential forms

$$\mathcal{L}_v = d\iota_v + \iota_v d \tag{47}$$

is known as *Cartan's magic formula*.

Let us state some important properties of $\mathcal{L}_v$

(i) $\mathcal{L}_v$ is a derivation on $\mathcal{C}^\infty(M)$: $\mathcal{L}_v(fg) = \mathcal{L}_v(f)g + f\mathcal{L}_v(g)$

(ii) $\mathcal{L}_v$ is a derivation on $\mathcal{C}^\infty(M)\otimes\mathfrak{X}(M)$: $\mathcal{L}_v(f\otimes u) = \mathcal{L}_v(f)\otimes u + f\otimes\mathcal{L}_v(u)$

(iii) $\mathcal{L}_v$ is a derivation on $\mathfrak{X}(M)$: $\mathcal{L}_v([u,w]) = [\mathcal{L}_v(u), w] + [u, \mathcal{L}_v(w)]$

(iv) Acting on differential forms, one has

- $\mathcal{L}_v(\alpha\wedge\beta) = \mathcal{L}_v(\alpha)\wedge\beta + \alpha\wedge\mathcal{L}_v(\beta)$
- $[\mathcal{L}_v, \mathcal{L}_u]\alpha = \mathcal{L}_{[v,u]}\alpha$
- $[\mathcal{L}_v, \iota_u]\alpha = [\iota_u, \mathcal{L}_v]\alpha = \iota_{[u,v]}\alpha$

After this aside, recall that for classical mechanics (and for linear symplectic geometry in general) there exists an important class of vector fields—known as *Hamiltonian vector fields*—which were associated to a function, namely

$$X_f(m) \equiv \mathrm{sgrad}_{\omega_m} f = \pi_m^{-1} \in T_mM. \tag{48}$$

Equivalently, we have defined $X_f(m)$ as

$$\iota_{X_f}\omega_m = \omega_m(X_f(m), -) = df. \tag{49}$$

These vector fields patch together to define a global vector field

Definition 4.4. Let (M,ω) be symplectic and $f \in \mathcal{C}^\infty(M)$. Then the *Hamiltonian vector field* (symplectic gradient) X_f is defined by

$$\iota_{X_f}\omega + df = 0. \tag{50}$$

We denote the space of Hamiltonian vector fields by $\mathfrak{X}_{ham}(M,\omega)$.

Exercise 16
Show that

a) $\mathfrak{X}_{ham}(M,\omega) \subset \mathfrak{X}(M,\omega)$ is a Lie subalgebra.

b) $\{g,f\} := \omega(X_f, X_g) = \pi(dg, df) = \mathcal{L}(X_f)g$ is a Lie bracket (namely the Poisson bracket)

c) $\mathcal{C}^\infty(M) \to \mathfrak{X}_{ham}(M,\omega)$ with $f \mapsto X_f$ is a Lie algebra (anti-) homomorphism, i.e.

$$[X_f, X_g] = X_{\{g,f\}} \tag{51}$$

d) Proof of Liouville's theorem: show that the Liouville volume $d\lambda = \frac{\omega^n}{n!}$ is preserved under Hamiltonian flows.

Hint: Show that $\mathcal{L}_{X_f}\lambda = 0$.

Theorem 4.3 (Darboux)**.** Let (M,ω), $\dim M = 2n$, be symplectic, then locally we can always find coordinate charts centered at a point $m \in M$ such that there exists a symplectomorphism

$$(U, \omega_m) \cong (\mathbb{R}^{2n}, \omega_0 = \sum_i dx^i \wedge dp_i) \tag{52}$$

In words, the theorem tells us that locally *all* symplectic manifolds look like a cotangent bundle T^*X. This means in particular that there exists no local invariants!

Sketch of proof. The proof is an application of the following

Lemma 4.4 (Moser's Trick)**.** Let M be a smooth manifold and $\omega_t \in \Omega^2(M)$, $t \in [0,1]$ a smooth family of symplectic forms. Assume that $\dot{\omega}_t = d\alpha_t$ is exact for some smooth family $\alpha_t \in \Omega^1(M)$. Define the vector field v_t such that $\iota_{v_t}\omega_t = -\alpha_t$ and assume that v_t integrates to $\Phi_t \in \mathrm{Diff}(M)$ with $\Phi_0 = id_M$ (which always holds if M is compact). Then

$$\Phi_t^*\omega_t = \omega_0 \tag{53}$$

Exercise 17
Show $v_t = -\pi_t^\sharp(\alpha_t)$.

Proof of lemma. The proof goes by direct calculation:

$$\begin{aligned}\frac{d}{dt}\Phi_t^*\omega_t &= \dot{\Phi}_t^*\omega_t + \Phi_t^*\dot{\omega}_t \\ &= \Phi_t^*(\mathcal{L}_{v_t}\omega_t + d\alpha_t) \\ &= \Phi_t^*(d\iota_{v_t}\omega_t + d\alpha_t) \\ &= \Phi_t^*(-d\alpha_t + d\alpha_t) = 0\end{aligned}$$

□

Hence $\Phi_t^*\omega_t$ is constant in t and thus $\Phi_t^*\omega_t = \omega_0$.

Now, choose a chart $\phi\colon T_mM = \mathbb{R}^{2n} \supset V \to U \subset M$ such that $\phi(0) = m \in M$ and $d\varphi_0 = id_{TM}$. Note that both (V, ω_0) and (U, ω_m) are symplectic. Moreover, there exist two symplectic forms on V, namely ω_0 and $\phi^*\omega_m$. Define the family

$$\omega_t = (1-t)\omega_0 + t\phi^*\omega_m \tag{54}$$

for $t \in [0,1]$. It is easy to see that $d\omega_t = 0$, however its non-degeneracy for all values of t is not so clear. However, since the statement of Darboux's theorem is local, we can always shrink V so that ω_t is in fact non-degenerate. Let us thus assume that ω_t defines a family of symplectic forms (for V small enough). Define the Moser 1-forms

$$\alpha_t = h_P\dot{\omega}_t \tag{55}$$

where h_P denotes the Poincaré homotopy operator on V. In particular,

$$dh_P - h_Pd = id_V. \tag{56}$$

Exercise 18
Show that $\dot{\omega}_t = d\alpha_t$.

Assume further that V is so small that v_t integrates to a flow Φ_t on V (otherwise, shrink V further). By Moser's lemma,

$$\Phi_1^*\omega_m = \Phi_1^*\omega_1 = \Phi_t^*\omega_t = \Phi_0^*\omega_0 = \omega_0 \tag{57}$$

□

We end this short and vastly non exhaustive introduction to the foundations of symplectic geometry by the study of some important submanifolds of symplectic manifolds. As in the linear case, one distinguishes between isotropic, coisotropic and Lagrangian submanifolds. Notice that a natural way to define these submanifolds is to mimic the definition of the linear case applied point-wise to the tangent spaces (which are indeed linear!).

Definition 4.5. Let (M, ω) be symplectic and $S \subset M$ a submanifold. Moreover let $\iota\colon S \hookrightarrow M$ be the inclusion. We say that

- S is *isotropic* if $T_xS \subset (T_xM, \omega)$ is isotropic $\forall x \in S$, equivalently $\omega|_S = \iota^*\omega = 0$.
- S is *coisotropic* if $T_xS \subset (T_xM, \omega)$ is coisotropic $\forall x \in S$.
- S is *Lagrangian* if $\dim S = \frac{1}{2}\dim M$ and $T_xS \subset (T_xM, \omega)$ is isotropic $\forall x \in S$ (equivalently $\omega|_S = \iota^*\omega = 0$).

We would like to give an important example of coisotropic submanifolds which arise in the study of constrained Hamiltonian systems.

Consider a symplectic Hamiltonian (M, ω, H) system where (M, ω) is a symplectic manifold—the phase space—and $H \in \mathcal{C}^\infty(M)$ the Hamiltonian governing the dynamics of the system. Assume that there is a submanifold $C \subset M$ cut out by functions $\phi_i \in C^\infty(M)$:

$$C = \{x \in M \mid \phi_i(x) = 0 \ \forall i\} \tag{58}$$

and assume further that all the constraints ϕ_i Poisson-commute with H to ensure that the time evolution of a point in C stays in C

$$\dot{\phi}|_C = \{\phi, H\}|_C = 0 \tag{59}$$

If we restrict the dynamics to C, we speak of a *constrained* Hamiltonian system.

Definition 4.6. We call the constraint ϕ_i *first-class* if it Poisson-commutes with all other constraints on the constraint surface

$$\{\phi_i, \phi_j\}|_C = 0 \qquad \forall j. \tag{60}$$

If the constraint ϕ_i does ot Poisson-commute with at least one of the other constraints, we say that ϕ_i is *second-class.*

The geometric meaning of first-class constraints stems form the following fact. Let $I_C = \{f \in \mathcal{C}^\infty(M) \mid f|_C = 0\}$ be the vanishing ideal of C. The tangent bundle $TC \subset TM$ and its annihilator have the following algebraic description: for $x \in C$

$$T_xC = \{v \in T_xM \mid v(f) = 0 \ \forall f \in I_C\} \subset T_xM \tag{61}$$

and

$$\mathrm{Ann}(T_xC) = \{\alpha \in T_x^*M \mid \alpha = df_x \text{ for some } f \in I_C\} \subset T_x^*M. \tag{62}$$

Since ω is a symplectic form, $\omega_x^\flat\colon T_xM \to T_x^*M$ is an isomorphism and by (49) $\omega^\flat(X_f) = df$ and hence

$$T_xC^\omega = (\omega^\flat)^{-1}\Big(\mathrm{Ann}(T_xC)\Big) = \{X_f(x) \mid f \in I_C\}. \tag{63}$$

Proposition 4.5. Then the following statements are equivalent

(i) the Hamiltonian vector field X_f is tangent to C for all $f \in I_C$

(ii) $\{I_C, I_C\} \subset I_C$

(iii) C is coisotropic

Put differently, the Hamiltonian vector fields of first-class constraints are tangent to the constrained surface!

Recall that we would like to study the dynamics on the constraint surface C. Since Hamiltonian dynamics is essentially the flow of Hamiltonian flows, we hence would like to define a symplectic form on C. A first anstaz is to simply restrict ω to C: $\omega_C = \iota^*\omega$ for $\iota\colon C \hookrightarrow M$ the inclusion. The problem is that if at least one of the constraints, say ϕ is first-class, then ω_C is degenerate: since $X_\phi \in TC$,

$$\iota_{X_\phi}\omega_C = \iota^*\omega(X_\phi, -) = \iota^* d\phi = d(\iota^*\phi) = 0 \tag{64}$$

First-class constraints generate a symmetry (since they commute each other and with the Hamiltonian). Due to this symmetry, ω_C is degenerate and (C, ω_C) is not symplectic. However, there exsits a way to quotient out these symmetries resulting into a *reduced phase space* which *is* symplectic.

As an example consider $X = \mathbb{R}^2$ with coordinates (x, y) and the phase space $(M = T^*X, \omega_0)$ with coordinates (p_x, p_y) in the fiber. Then

$$\omega_0 = dx \wedge dp_x + dy \wedge dp_y. \tag{65}$$

Assume that we have a constraint

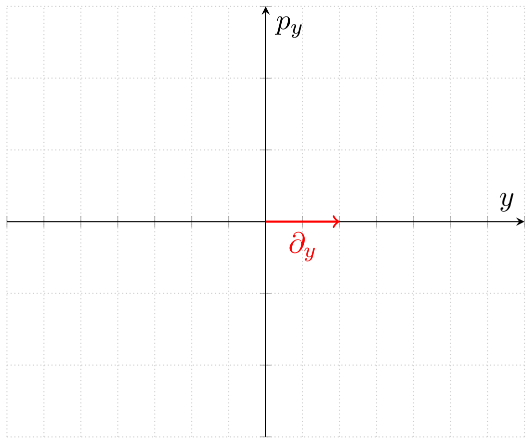
$$\phi_{p_y}\colon M \to \mathbb{R} \quad , \quad \phi_{p_y}(x, y, p_x, p_y) = p_y \tag{66}$$

defining the constraint surface $C = \mathrm{zeros}(\phi_{p_y}) = \{p_y = 0\}$. We would like to study the dynamics on C. Since p_y is the momentum in the y-direction, the constraint $\phi_{p_y} = 0$ restricts all motion to the x-direction. The Hamiltonian vector field of ϕ_{p_y} is readily computed as

$$X_{\phi_{p_y}} = \partial_y \tag{67}$$

and is indeed tangent to the constraint surface C.

Cross section of $C \subset T^*X$



However, if we restrict the symplectic form to C

$$\omega|_C = dx \wedge dp_x \tag{68}$$

we see that

$$\ker \omega|_C = \text{span}(\partial_y) = \text{span}(X_{\phi_{p_y}}) \tag{69}$$

and thus $\omega|_C$ is degenerate! This is intuitively clear, since the constraint does indeed restrict all motion to the x-direction, however it does not fix the value of y. We could introduce another constraint $\phi_y(x, y, p_x, p_y) = y - y_0$ and consider the constraint surface

$$C = \{\phi_y = \phi_{p_y=0}\} = \{p_y = 0, y = y_0\} \tag{70}$$

Note that

$$\{\phi_y, \phi_{p_y}\} = 1 \tag{71}$$

and hence the constraints are all second-class (since they do not Poisson-commute). Moreover, the Hamiltonian vector fields

$$X_{\phi_{p_y}} = \partial_y, \qquad X_{\phi_y} = -\partial_{p_y} \tag{72}$$

are not anylonger tangent to C (they move any point on C off C). In this case, the restriction

$$\omega_C = \omega|_C = dx \wedge dp_x \tag{73}$$

is in fact symplectic and we could study dynamics on (T^*C, ω_C).

References

[1] V. I. Arnold, *Mathematical methods of classical mechanics.* Vol. 60. Springer Science & Business Media, 2013.

[2] A. C. Da Silva. *Lectures on symplectic geometry.* Vol. 3575. Berlin: Springer, 2001.

[3] D. McDuff, and D. Salamon. *Introduction to symplectic topology.* Vol. 27. Oxford University Press, 2017.

[4] A. Alekseev. *Symplectic geometry and moment maps.* Lecture notes (unpublished), 2018.

[5] M. Blau. *Symplectic geometry and geometric quantization.* Lecture notes (1992)

[6] E. Meinrenken. *Symplectic Geometry, Lecture Notes.* University of Toronto, 2000.