Supergeometry - Oddities of the Square
Take-Home Exercise

Part I

In this part of the exercise sheet we want to derive the *Chevalley-Eilenberg* differential of a Lie algebra from super geometry.
Let G be a Lie group and consider its odd tangent space

$$\Pi TG = (\mathbb{R}^{0|1} \to G). \tag{1}$$

Let

$$R_g \colon G \to G \quad , \quad h \mapsto hg \tag{2}$$

be the operation of right multiplication on G (which is a smooth map because G is a Lie group). We define the *(right) Maurer-Cartan form* by the map

$$\theta_g := (R_{g^{-1}})_* \colon T_g G \to T_{gg^{-1}} G = T_e G = \mathfrak{g} \tag{3}$$

Notice that θ can be seen as a $\mathfrak{g}$-valued 1-form on G, i.e. as an element of $\Omega^1(G, \mathfrak{g})$

Exercise 1.

1. Assume G is a matrix group. Show that $(R_{g^{-1}})_* = gh$
2. Show that $d\theta = \frac{1}{2}[\theta, \theta]_{\mathfrak{g}}$
3. Argue that $TG = G \times \mathfrak{g}$

Recall that a map $x^i(\theta) \colon \mathbb{R}^{0|1} \to G$ is locally of the form

$$x^i + \theta \psi^i, \tag{4}$$

where x^i is a local coordinate in G and $\psi^i \in \Pi T_x G$. Let

$$c^a = \theta^a(\Psi) = \theta^a{}_i \psi^i \in \Pi\mathfrak{g}. \tag{5}$$

(Here the superscript denotes the component in $\mathfrak{g}$.)

Exercise 2. Recall that $d = \psi^i \partial_i$. Show that

$$dc^a = \frac{1}{2} f^a_{bc} c^a c^b = \frac{1}{2}[c,c]^a. \tag{6}$$

This exercise shows that d acting on

$$\mathcal{C}^\infty(\Pi TG) = \Omega^\bullet(G) \tag{7}$$

can be represented by

$$d \equiv d_{CE} = \frac{1}{2}[c,c]^a \frac{\partial}{\partial c^a} \tag{8}$$

acting on

$$C^\bullet_{CE} = \Lambda^\bullet \mathfrak{g}^* = \Lambda^\bullet[c^1, \ldots, c^{\dim \mathfrak{g}}] \tag{9}$$

The complex

$$(C^\bullet_{CE}, d_{CE}) \tag{10}$$

is called the *Chevalley-Eilenberg* complex and its cohomology computes the cohomology of $\mathfrak{g}$.

Part II

This part of the exercise sheet is about integration over odd variables.
The idea is to define the integral in analogy to usual integration, namely by a linear, translational invariant map which sends total derivatives to zero.

Definition 1. Let $V = V_0 \oplus V_1$ be a super vector space with even coordinates x^i and odd coordinates θ^i. The *left derivative* with respect to θ^i are defined on monomials in θ^j by

$$\frac{\overrightarrow{\partial}}{\partial \theta^i} \theta^j = \delta^j_i \tag{11}$$

and is extended to $\mathcal{C}^\infty(V)$ as a (super)derivation, i.e.

$$\frac{\overrightarrow{\partial}}{\partial \theta^i} \left(\theta^j \theta^k\right) = \delta^j_i \theta^k - \delta^k_i \theta^j \tag{12}$$

Analogously, the *right derivative* with respect to θ^i are defined on monomials in θ^j by

$$\theta^j \frac{\overleftarrow{\partial}}{\partial \theta^i} = \delta^j_i \tag{13}$$

and is extended to $\mathcal{C}^\infty(V)$ as a (super)derivation, i.e.

$$\left(\theta^j \theta^k\right) \frac{\overleftarrow{\partial}}{\partial \theta^i} = \delta^k_i \theta^j - \delta^j_i \theta^k \tag{14}$$

Definition 2. Let V be an ordinary vector space of dimension n. The *Grassmann/Berezin* integral

$$\int_{\Pi V} d\theta_1 \dots d\theta_n \, (-) \colon \mathcal{C}^\infty(\Pi V) \to \mathbb{R} \tag{15}$$

over ΠV, with coordinates θ^i is defined by the relations

$$\int_{\Pi V} d\theta^1 \dots d\theta^n \; \theta^n \dots \hat{\theta}^i \dots \theta^1 = 0 \quad , \quad \int_{\Pi V} d\theta^1 \dots d\theta^n \; \theta^n \dots \theta^1 = 1 \tag{16}$$

where $\hat{\theta}^i$ means omission of the i-th factor θ^i.

Exercise 3 (Gaussian/Berezinian integration)**.** Let V be a one-dimensional vector space and ΠV the super vector space with odd coordinate θ. Verify the following

a) Linearity

$$\int_{\Pi V} af(\theta) + g(\theta) d\theta = a \int_{\Pi V} f(\theta) d\theta + \int_{\Pi V} g(\theta) d\theta.$$

b) Translation invariance

$$\int_{\Pi V} f(\theta + \theta') = \int_{\Pi V} f(\theta) d\theta.$$

c) Total derivatives are mapped to zero

$$\int_{\Pi V} \frac{\overrightarrow{\partial} f(\theta)}{\partial \theta} d\theta = \int_{\Pi V} f(\theta) \frac{\overleftarrow{\partial}}{\partial \theta} d\theta = 0.$$

d) Let $\theta = J\xi$, θ, ξ, $J \in \mathbb{R}$ odd, then $d\theta = \frac{d\xi}{J}$.

e) Let $\theta^i = J^i{}_j \xi^j$, then $d\theta^1 \dots d\theta^n = \frac{d\xi^1 \dots d\xi^n}{\det J}$.

Hint 1: Any function $f \in \mathcal{C}^\infty(\Pi V)$ is of the form $f(\theta) = f_0 + f_1\theta$.

Hint to (e)*:* Express $f_{1 \dots n}$ as a Grassmann-Berezin integral once over θ^i and once over ξ^i.

Part III

In this part we want to derive the action functional for $N = 1$ quantum mechanics.

Exercise 4 ($N = 1$ superalgebra)**.** Consider $\mathbb{R}^{1|1}$ with even coordinate t and odd coordinate θ. The space $\mathbb{R}^{1|1}$ has actually the structure of a super group with multiplication defined by

$$(t, \theta)(t', \theta') = (t + t' + \theta\theta', \theta + \theta') \tag{17}$$

a) Show that *left* multiplication inside $\mathbb{R}^{1|1}$ is generated by the vector field

$$Q = \partial_\theta + \theta \partial_t \tag{18}$$

b) Show that *right* multiplication inside $\mathbb{R}^{1|1}$ is generated by the vector field

$$D = \partial_\theta - \theta \partial_t \tag{19}$$

c) Verify the commutation relations

$$[D, D] = -2\partial_t \quad , \quad [Q, Q] = 2\partial_t \quad , \quad [D, Q] = 0 \tag{20}$$

Exercise 5 (Supersymmetry Transformations)**.** We will assume that Q, will generate the supersymmetry. Let

$$\Phi(t, \theta) = x(t) + \theta\psi(t) \tag{21}$$

be a *super field*, i.e. a function on $\mathbb{R}^{1|1}$.
Compute the supersymmetry transformations

$$\delta_\varepsilon \Phi = \varepsilon Q\big(\Phi(t, \theta)\big) \tag{22}$$

component-wise.

Exercise 6. Consider the action functional

$$S[\Phi] = -\frac{1}{2}\int_{\mathbb{R}} dt \int_{\mathbb{R}^{0|1}} d\theta \; \delta_{\mu\nu} \dot{\Phi}^\mu D\Phi^\nu \equiv -\frac{1}{2}\int_{\mathbb{R}} dt \int_{\mathbb{R}^{0|1}} d\theta \; \langle \dot{\Phi}, D\Phi \rangle \tag{23}$$

1. Show that

$$S[\Phi] = \frac{1}{2}\int_{\mathbb{R}} dt \; \delta_{\mu\nu} \left(\dot{x}^\mu \dot{x}^\nu + \psi^\mu \dot{\psi}^\nu \right) \equiv \frac{1}{2}\int_{\mathbb{R}} dt \; \|\dot{x}\|^2 + \langle \psi, \dot{\psi} \rangle \tag{24}$$

2. Show that the model is supersymmetric, i.e.

$$\delta_\varepsilon S[x, \psi] = 0 \tag{25}$$

3. Show (25) directly from (23)

Part IV

We now want to study the construction of $N = 2$ quantum mechanics based on the ideas of the previous exercises.

Exercise 7 ($N = 2$ Superalgebra)**.** Consider $\mathbb{R}^{1|2}$ with coordinates $(t, \theta, \bar{\theta})$ and the super group multiplication

$$(t, \theta, \bar{\theta})(t', \theta', \bar{\theta}') = (t + t' + \theta\bar{\theta}' + \bar{\theta}\theta', \theta + \theta', \bar{\theta} + \bar{\theta}') \tag{26}$$

a) Show that *left* multiplication inside $\mathbb{R}^{1|2}$ is generated by the vector fields

$$Q = \partial_\theta + \bar{\theta}\partial_t \quad , \quad \bar{Q} = \partial_{\bar{\theta}} + \theta\partial_t \tag{27}$$

b) Show that *right* multiplication inside $\mathbb{R}^{1|2}$ is generated by the vector fields

$$D = \partial_\theta - \bar{\theta}\partial_t \quad , \quad \bar{D} = \partial_{\bar{\theta}} - \theta\partial_t \tag{28}$$

c) Verify the commutation relations

$$[D, \bar{D}] = -2\partial_t \quad , \quad [Q, \bar{Q}] = 2\partial_t \tag{29}$$

and show that all other commutators are zero.

Exercise 8 (Supersymmetry Transformations)**.** As in the lecture, $Q, \bar{Q}$ will generate the supersymmetry. Let

$$\Phi(t, \theta, \bar{\theta}) = x(t) + \theta\psi(t) + \bar{\theta}\bar{\psi}(t) + \theta\bar{\theta}F \tag{30}$$

be a super field, i.e. a function on $\mathbb{R}^{1|2}$.
Compute the supersymmetry transformations

$$\delta_\varepsilon\Phi = \varepsilon Q\Phi \quad , \quad \delta_{\bar{\varepsilon}}\Phi = \bar{\varepsilon}\bar{Q}\Phi \tag{31}$$

component-wise.

Exercise 9. Consider the action

$$S[\Phi] = \int_{\mathbb{R}} dt \int_{\mathbb{R}^{0|2}} d\theta d\bar{\theta}\ \frac{1}{2}\delta_{\mu\nu}\bar{D}\Phi^\mu D\Phi^\nu \tag{32}$$

1. Show that

$$S[\Phi] = \int_{\mathbb{R}} dt \left(\frac{1}{2}\|\dot{x}\|^2 - \langle\bar{\psi}, \partial_t\psi\rangle + \frac{1}{2}\|F\|^2\right) \tag{33}$$

2. Show that $S[\Phi]$ is supersymmetric, i.e.

$$\delta_\varepsilon S[\Phi] = \delta_{\bar{\varepsilon}} S[\Phi] = 0 \tag{34}$$

3. Show that (34) directly from (32).