TOPOLOGICAL EFFECTS & THEIR PHENOMENOLOGY

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CRC-TR 211
Strong-interaction matter
under extreme conditions

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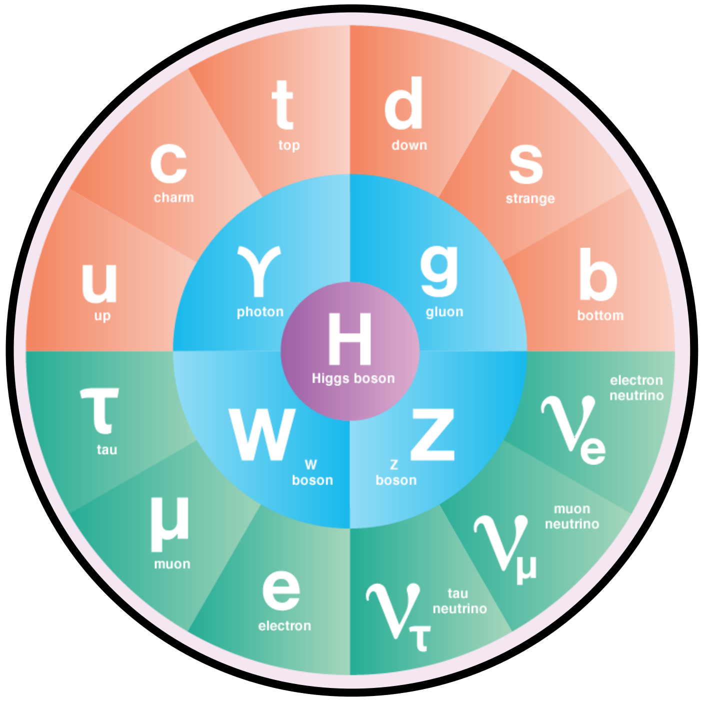
NON-PERTURBATIVE AND TOPOLOGICAL ASPECTS OF QCD

CERN - 30/05/2024

- introduction to topology
- semi-classical analysis
- phenomenology

INTRODUCTION TO TOPOLOGY

THE STANDARD MODEL



$SU(3) \times SU(2) \times U(1)$ gauge theory coupled to fundamental fermions and a Higgs

$$S = -\frac{1}{2g^2}\int d^4x \,\mathrm{tr}\, F^2 + \text{matter}$$

$$D_\mu = \partial_\mu + A_\mu$$

$$F_{\mu\nu} = [D_\mu, D_\nu]$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

Gauge fields arise from demanding invariance of a system of free fermions under local (= gauge) transformations

$$\psi(x) \to g(x)\,\psi(x)\,, \qquad g \in SU(3), SU(2), U(1)\,, \qquad g(x) = e^{i\alpha^a(x)\,T^a}$$

group generator

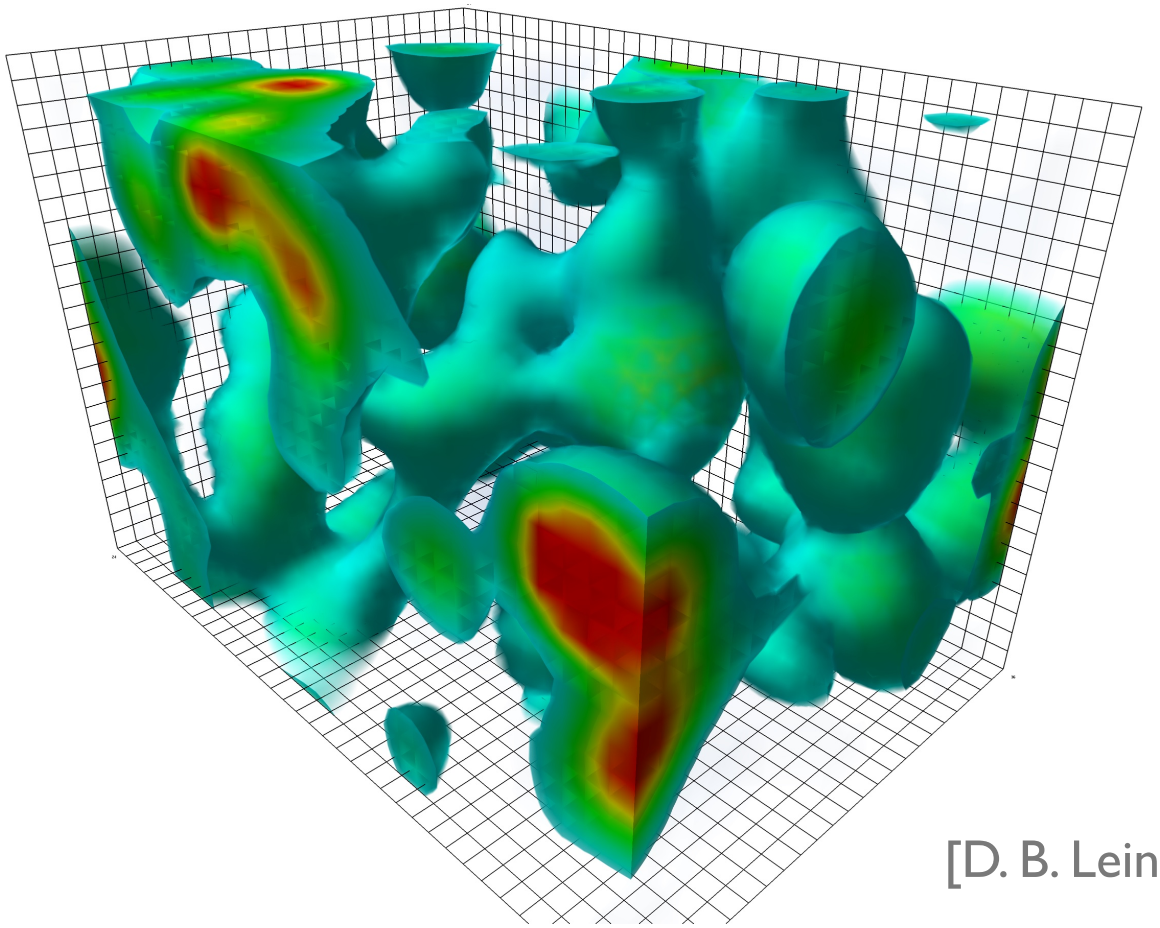
Gauge fields transform non-trivially under gauge transformations

$$A_\mu \to g A_\mu g^{-1} + g\partial_\mu g^{-1}$$

GLUON FIELD CONFIGURATIONS

Ignore matter for now and consider a pure gauge theory

$$S = -\frac{1}{2g^2}\int d^4x \,\mathrm{tr}\, F^2$$



[D. B. Leinweber]

Anyone may ask:
what do gluons look like?
where to begin...

A "classical" physicist may ask:
what is the "classical" ground state of the system in equilibrium?

- go to Euclidean space: path integral → partition function,

$$\int \mathscr{D}A\, e^{iS[A]} \to \int \mathscr{D}A\, e^{-S_E[A]}$$

- largest individual contribution from fields with minimal Euclidean action

GLUON FIELD CONFIGURATIONS

consider the more general case of field configurations with finite Euclidean action:

$$S_E = -\frac{1}{2g^2}\int d^4x_E \,\mathrm{tr}\, F^2$$

- for the action to be finite, A has to fall-off faster than r^{-1} for $|x_E| = r \to \infty$
- "pure gauge" field $A_\mu^{(g)} = g\partial_\mu g^{-1}$: $F(A^{(g)}) = 0$

$$\longrightarrow \quad A_\mu(r,\varphi_i) = g(\varphi_i)\,\partial_\mu g^{-1}(\varphi_i) + \mathcal{O}(1/r^2)$$

polar coordinates $(r, \varphi_{1,2,3})$ of $\mathbb{R}^4$

Desired field configs are defined by gauge trafos $g(\varphi_i)$ that only depend on the angles of $\mathbb{R}^4$

⟶ defines map from 3-sphere to gauge group: $S^3 \to SU(N)$

⟶ allows for a **topological classification**

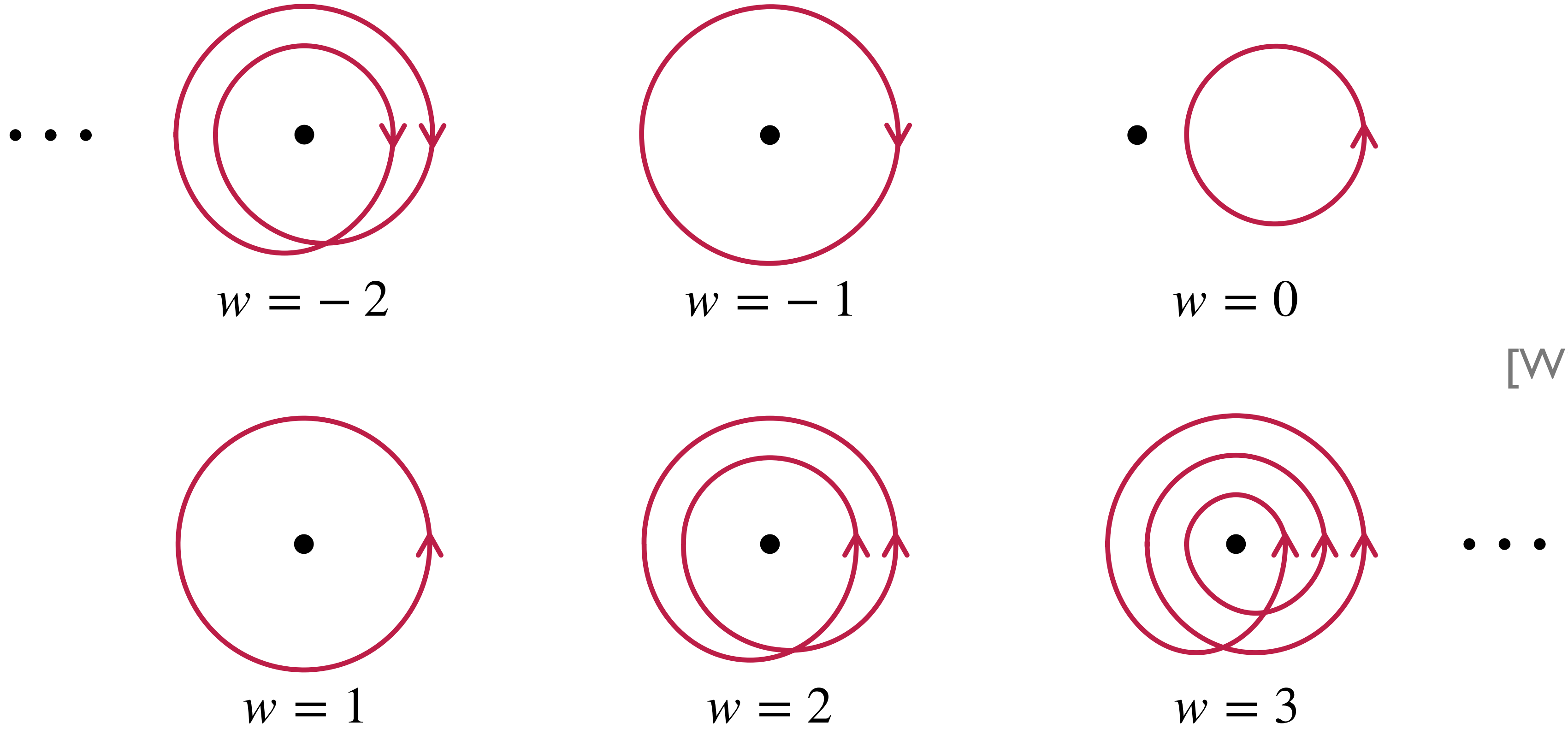
TOPOLOGY


Corporate needs you to find the differences between this picture and this picture.
TOPOLOGIST
They're the same picture.

TOPOLOGY

Properties of geometric objects that are preserved under continuous deformations of this object

- Example: string winding around a hole in a plane
- described by a topological invariant, the winding number w



[Wikipedia]

- can be characterized by all possible ways to map circles onto circles, $S^1 \to S^1$

→ winding number ~ element of the homotopy group $\pi_1(S^1) = \mathbb{Z}$

In general, topological invariants for objects living in a space X are given by elements of the homotopy group $\pi_n(X)$, characterizing maps $S^n \to X$

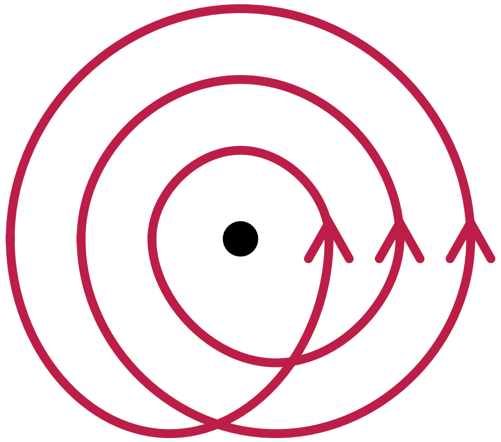
GAUGE FIELDS & TOPOLOGY

Finite action gauge fields can be classified by homotopy group $\pi_3(SU(N))$

- $\pi_3(U(1)) = \pi_3(S^1) = 0$: photons are topologically trivial
- $\pi_3(SU(N)) = \pi_3(SU(2)) = \pi_3(S^3) = \mathbb{Z}$: **non-Abelian gauge fields wind!**

How to define the winding number Q, also called topological charge?

- illustration: $\pi_1(S^1)$, defined via map $g(\varphi) = e^{iQ\varphi}$



$$Q_\infty = \frac{i}{2\pi}\int_0^{2\pi} d\varphi\, g(\varphi)\, \partial_\varphi\, g^{-1}(\varphi)$$

GAUGE FIELDS & TOPOLOGY

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How to define the winding number Q, also called topological charge?

- for $\pi_3(SU(2))$ this is (use, e.g., $g = [(x_0 - i\vec{\sigma}\vec{x})/r]^Q$)

$$Q_\infty = -\frac{1}{24\pi^2}\int d\varphi_1\, d\varphi_2\, d\varphi_3\, \epsilon^{ijk}\, \mathrm{tr}\Big[\big(\mathrm{g}\,\partial_{\varphi_\mathrm{i}}\,\mathrm{g}^{-1}\big)\big(\mathrm{g}\,\partial_{\varphi_\mathrm{j}}\,\mathrm{g}^{-1}\big)\big(\mathrm{g}\,\partial_{\varphi_\mathrm{k}}\,\mathrm{g}^{-1}\big)\Big]$$

In terms of gauge fields, this can be expressed as:

$$\boxed{Q = -\frac{1}{16\pi^2}\int d^4x\, \mathrm{tr}\, F\tilde{F} \equiv \int d^4x\, q(x) \;\; \in \mathbb{Z}}$$

dual field strength

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$$

topological charge density

Note: Q_∞ is the winding for $r \to \infty$, while Q (the 2nd Chern number) works for all x.

LARGE GAUGE TRANSFORMATIONS

Since $SU(N)$ is a Lie group, we are tempted to think that gauge transformations are continuously connected to the identity

- $h(x) = e^{i\alpha^a(x)\,T^a} \sim 1$ for sufficiently small α and $r \to \infty$
- such trafos can only have $Q = 0$ ($Q_\infty = 0$ actually...): small gauge trafos

Gauge trafos with $Q \neq 0$ cannot be connected to the identity for $r \to \infty$:

⟶ large gauge transformations

$A_\mu^{(Q)}$ is a field configuration with top. charge Q, and g_Q and h are large and small trafos, then

$$A_\mu^{(Q)} \xrightarrow{r\to\infty} g_Q\,\partial_\mu\,g_Q$$

- $\tilde{A}_\mu^{(Q)} = hA_\mu^{(Q)}\,h^{-1} + h\partial_\mu\,h^{-1}$ also has topological charge Q
- $\tilde{A}_\mu^{(Q_1+Q_2)} = g_{Q_2}A_\mu^{(Q_1)}\,g_{Q_2}^{-1} + g_{Q_2}\partial_\mu\,g_{Q_2}^{-1}$ has topological charge $Q_1 + Q_2$

⟶ $\pi_3(SU(N))$ describes equivalence classes of gauge fields with different Q

MANY VACUA

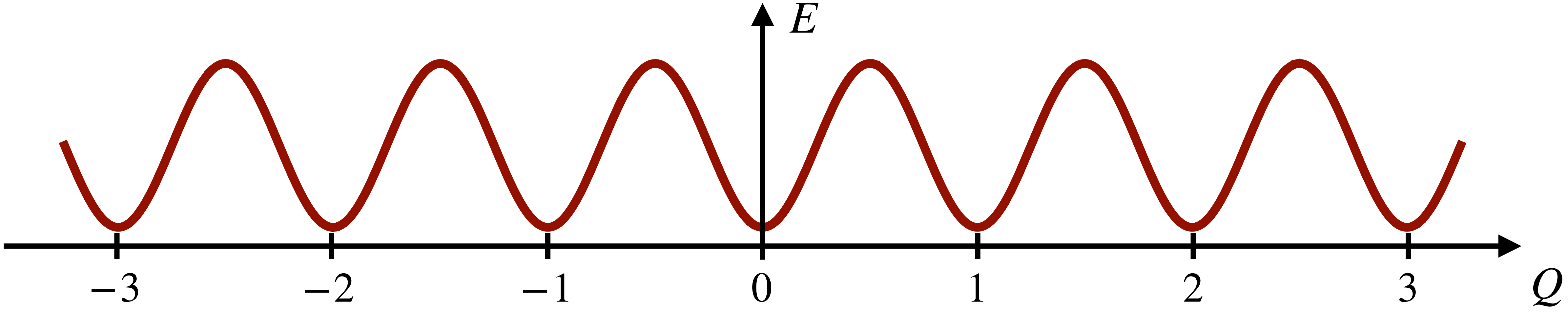
This has dramatic consequences for the vacuum of the gauge theory!

- suppose the vacuum of our system consists only of field configurations with $Q = 0$:
  $|Q = 0\rangle = |0\rangle$ (this is the vacuum state of ordinary perturbation theory)
- do a large gauge trafo: $|0\rangle \xrightarrow{g_Q} |Q\rangle$
- physical vacuum must be superposition of all possible Q-vacua: $|\Omega\rangle = \sum_{Q\in\mathbb{Z}} c_Q |Q\rangle$
- Vacuum must be stable under gauge trafos, including large ones (it can only change by a phase, $|\Omega\rangle \to e^{i\Theta}|\Omega\rangle$)

$$\longrightarrow \boxed{\theta\text{-vacuum: } |\Omega\rangle = \sum_{Q\in\mathbb{Z}} e^{i\theta Q} |Q\rangle}$$

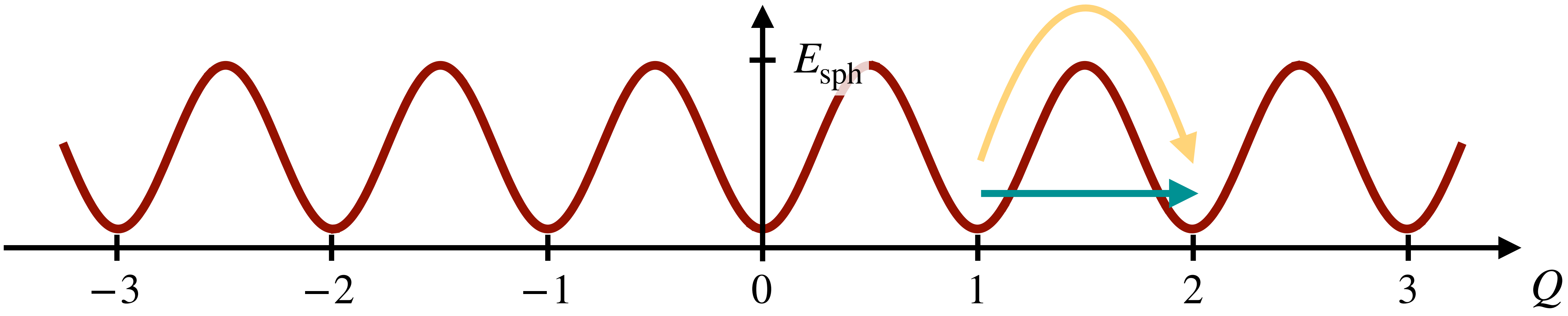
[Callan, Dashen, Gross (1976)]
[Jackiw, Rebbi (1976)]

a true **free parameter** in the SM!



VACUUM TRANSITIONS

Many vacua + topological field configurations = transitions!



- $E \gtrsim E_{\mathrm{sph}}$: transitions through hopping
    - need to overcome energy barrier, e.g. through sufficiently large T
    - transition probability determined by Boltzmann factor $\sim e^{-E_{\mathrm{sph}}/T}$
- $E < E_{\max}$: transitions through tunneling
    - "imaginary time phenomenon" (QM: $\psi(x) \sim \exp\left(it\sqrt{E - E_{\max}}\right)$)
    - transition probability $\sim e^{-8\pi^2|Q|/g^2}$ (as we will see shortly)

SO WHAT?

Non-Abelian gauge theories have topological features, but what are the physical consequences?

→ consider their coupling to matter

Fundamentally, gauge fields couple to fermions. Topology leads to two important effects:

- axial $U(1)_A$ symmetry is anomalous: **axial anomaly**
- fermions acquire zero modes on a topological background

$$\mathscr{M}^{(Q)}\psi^{(Q)} = \lambda_Q \psi^{(Q)} \quad \text{with} \quad \lambda_Q = 0$$

Dirac operator

$$\mathscr{M}^{(Q)} = i\gamma_\mu\left(\partial_\mu + A_\mu^{(Q)}\right)$$

these have net chirality, and their number is fixed by Q (index theorem)

$$\boxed{N_f Q = n_L - n_R}$$

# of left- and right-handed fermion zero modes

[Atiyah, Singer (1963)]
['t Hooft (1976)]

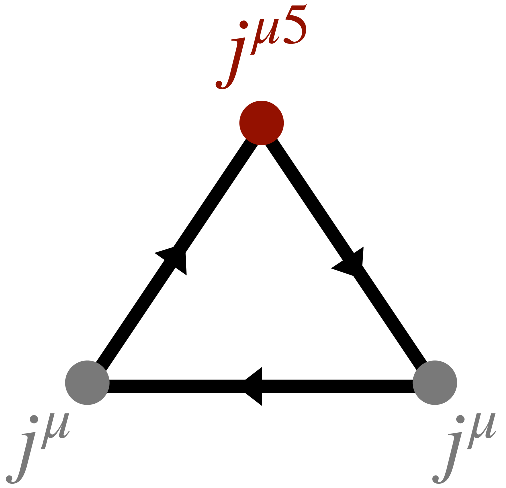
AXIAL ANOMALY

N_f massless (Dirac-) fermions have chiral symmetry

$$U(N_f)_L \times U(N_f)_R \sim SU(N_f)_V \times SU(N_f)_A \times U(1)_V \times U(1)_A$$

axial trafo
$U(N)_A : \; e^{i\gamma_5 \alpha_a T^a}$

Axial current $j^{\mu 5} = \bar{\psi}\gamma^\mu \gamma^5 \psi$ of $U(1)_A$ is classically conserved, but anomalously broken due to quantum effects:



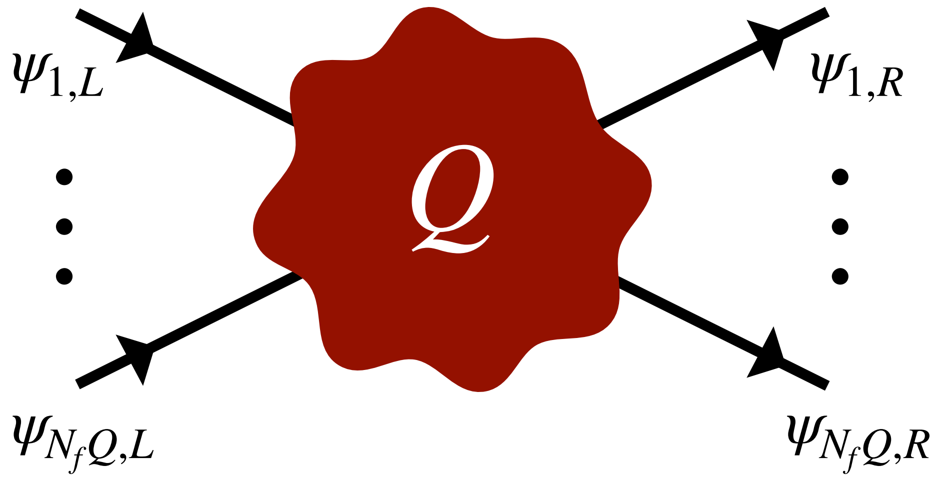
[Adler, Bell & Jackiw (1969)]

$$\boxed{\partial_\mu j^{\mu 5} = \frac{iN_f}{8\pi^2}\,\mathrm{tr}\, F\tilde{F}}$$

→ $U(1)_A$ **anomaly from topological field configurations**

Axial charge $Q_5 = i\int d^3x\, j^{05}$ changes in the presence of topological field configurations:

$$\Delta Q_5 = Q_5(t = +\infty) - Q_5(t = -\infty) = 2N_f\, Q$$



FERMION ZERO MODES

Fermion contribution to the path integral:

$$\langle \Omega | \Omega \rangle \sim \int \mathscr{D}A \mathscr{D}\psi \mathscr{D}\bar{\psi}\, e^{iS[A,\psi,\bar{\psi}]} = \int \mathscr{D}A e^{iS[A]} \det \mathscr{M}$$

⟶ functional determinant $\det \mathscr{M} = \prod_n \lambda_n = 0$ in the presence of zero-modes

So topology always drops out and $\langle \Omega | \Omega \rangle = \langle 0 | 0 \rangle$?

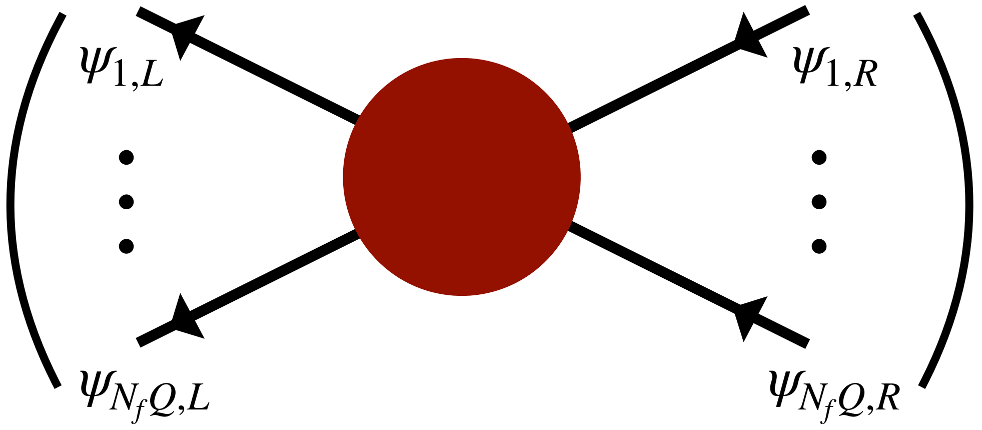
No, because $\langle \Omega | \Omega \rangle \supset \langle Q + \Delta Q | Q \rangle$:

top. charges changes by $\Delta Q \xrightarrow{\text{anomaly}}$ axial charge changes by $\Delta Q_5 = 2N_f \Delta Q$

To account for this change, we have to introduce a source:

annihilate $N_f Q$ R-fermions and create $N_f Q$ L-fermions (and vice versa):

$\langle Q + \Delta Q |$



$| Q \rangle \neq 0$ ⟶ anomalous $2N_f Q$-quark correlation functions

(arise from fermion zero modes)

['t Hooft (1976)]
[Pisarski, FR (2019)], [FR (2020)]

SEMI-CLASSICAL ANALYSIS

SADDLE-POINT APPROXIMATION

Now we want to compute topological effects. Try weak coupling!

Consider the path integral,

$$Z = \int \mathscr{D}\Phi \, e^{iS[\Phi]}$$

Expand about a "background" field $\bar{\Phi}$

$$\Phi = \bar{\Phi} + \delta\Phi \qquad \longrightarrow \qquad Z = \int \mathscr{D}\delta\Phi \, e^{iS[\bar{\Phi}+\delta\Phi]}$$

- if $\bar{\Phi}$ solves classical EoM $S'[\bar{\Phi}] = 0$: classical field
- if corrections are small, one can do a systematic expansion in $\delta\Phi$

$$S[\bar{\Phi} + \delta\Phi] = S[\bar{\Phi}] + \frac{1}{2} S''[\bar{\Phi}] \, \delta\Phi^2 + \ldots$$

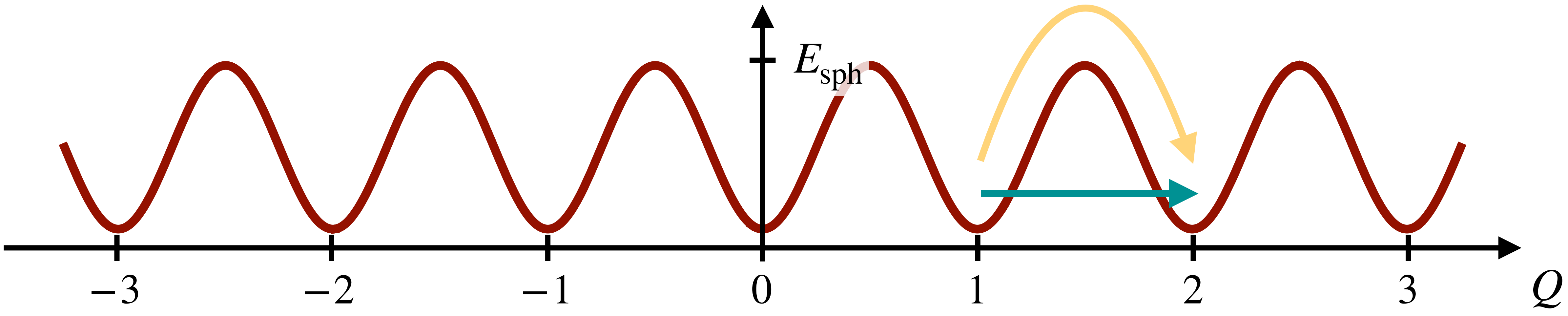
- leading corrections to Z : functional determinants

$\longrightarrow$ semi-classical analysis

For $\bar{\Phi} = 0$, this is essentially what we do in perturbation theory:
a semi-classical analysis is perturbation theory on an arbitrary background!

INSTANTONS AND SPHALERONS

Two ways to change Q: hopping and tunneling



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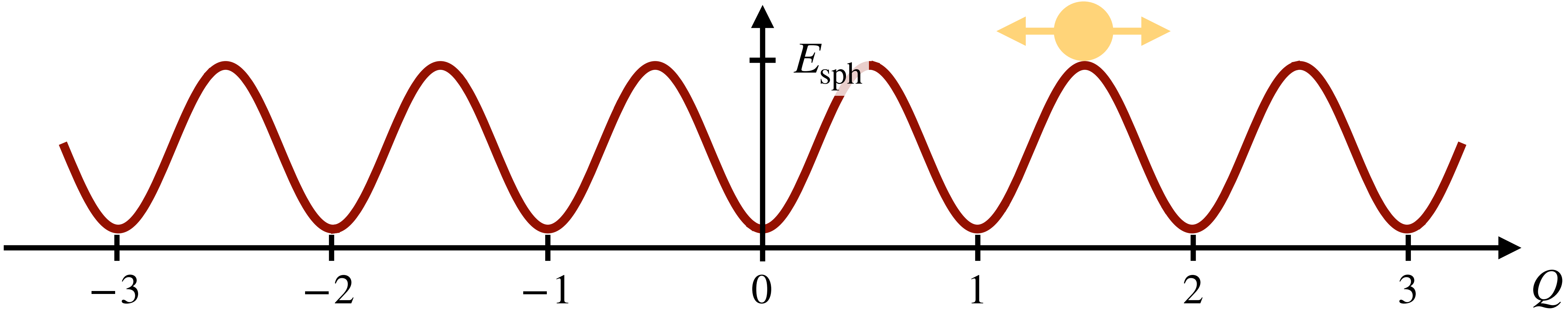


Hopping is described by fluctuations around field configuration that sits on top of barrier

$\longrightarrow$ **sphaleron** Greek: slippery

It's a static, unstable solution of the EoM with finite energy $E = E_{\mathrm{sph}}$ at real time.

First solution found in $SU(2)$-Higgs theory [Klinkhamer, Manton (1984)]:

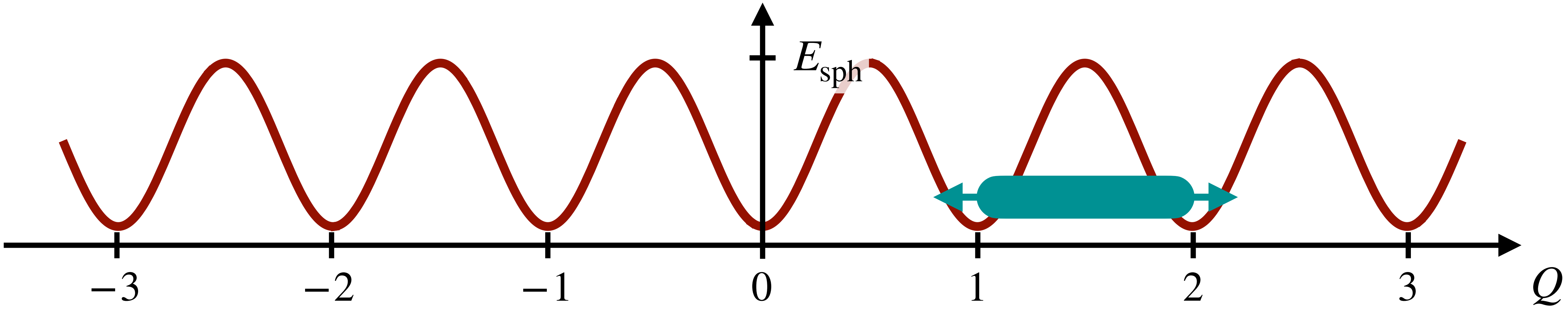
$$\mathscr{L} = -\frac{1}{2g^2}\,\mathrm{tr}\,F^2 + |D_\mu \phi|^2 + V\big(|\phi|^2\big) \longrightarrow \begin{aligned} \vec{A} &= \nu\,\frac{f(\xi)}{\xi}\,\hat{r}\times\vec{\sigma} \\ \phi &= \frac{\nu}{\sqrt{2}} h(\xi)\,\hat{r}\cdot\sigma\,\phi_0 \end{aligned}$$

ν : Higgs-VEV
f, h : numeric functions
$\xi = rg\nu$

The sphaleron has energy $E_{\mathrm{sph}} \sim 4\pi\nu/g$ and topological charge $Q = n + 1/2$

INSTANTONS AND SPHALERONS

Two ways to change Q: hopping and tunneling



Tunneling is described by fluctuations around field configuration that sits between minima

→ **instanton**

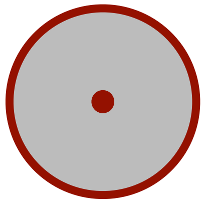
It is a minimum of the Euclidean action of Yang-Mills theory.

First solution for $SU(2)$ with $Q=1$: BPST instanton [Belavin, Polyakov, Schwartz, Tyupkin (1975)]

global $SU(N_c)$ matrix: orientation in the gauge group

positive real number: instanton size

$$A_\mu^{(1)}(x) = U_1\,\bar\sigma^{\mu\nu}\,U_1^\dagger\,\frac{\rho_1^2}{(x-z_1)^2}\,\frac{(x-z_1)_\nu}{(x-z_1)^2+\rho_1^2} =$$



four-vector: instanton location

$$\sigma^\mu = (-i,\vec{\sigma})^\mu$$
$$\bar\sigma^\mu = (i,\vec{\sigma})^\mu$$
$$\bar\sigma^{\mu\nu} = \frac{1}{2}\left(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu\right)$$

This can be generalized to arbitrary topological charges $Q \in \mathbb{Z}$

INSTANTONS

Minimize the classical action of Euclidean Yang-Mills theory,

$$S = -\frac{1}{2g^2}\int d^4x \,\mathrm{tr}\, F^2 \,,$$

using $\mathrm{tr}\, F^2 = \mathrm{tr}\, \tilde{F}^2$, write

$$S = -\frac{1}{4g^2}\int d^4x \left[\underbrace{\mathrm{tr}\left(F_{\mu\nu} \mp \tilde{F}_{\mu\nu}\right)^2}_{\geq 0} \pm 2\,\mathrm{tr}\, F\tilde{F}\right] \overset{\text{assume finite action!}}{\geq} -\frac{1}{2g^2}\left|\int d^4x \,\mathrm{tr}\, F\tilde{F}\right|$$

- action minimized by (anti) selfdual gauge fields with $F^{(Q)} = \pm \tilde{F}^{(Q)}$: **instantons $A_\mu^{(Q)}$**
- solutions of the classical EoM $D^\mu F_{\mu\nu}^{(Q)} = 0$ (Bianchi identity $D^\mu \tilde{F}_{\mu\nu}^{(Q)} = 0$)

→ instantons are classical fields that minimize the YM action

Classical action of these solutions:

$$S^{(Q)} = -\frac{1}{2g^2}\left|\int d^4x \,\mathrm{tr}\, F^{(Q)}\tilde{F}^{(Q)}\right| = -\frac{8\pi^2}{g^2}|Q|$$

→ self-dual solutions are topological

CONSTRUCTION OF INSTANTONS

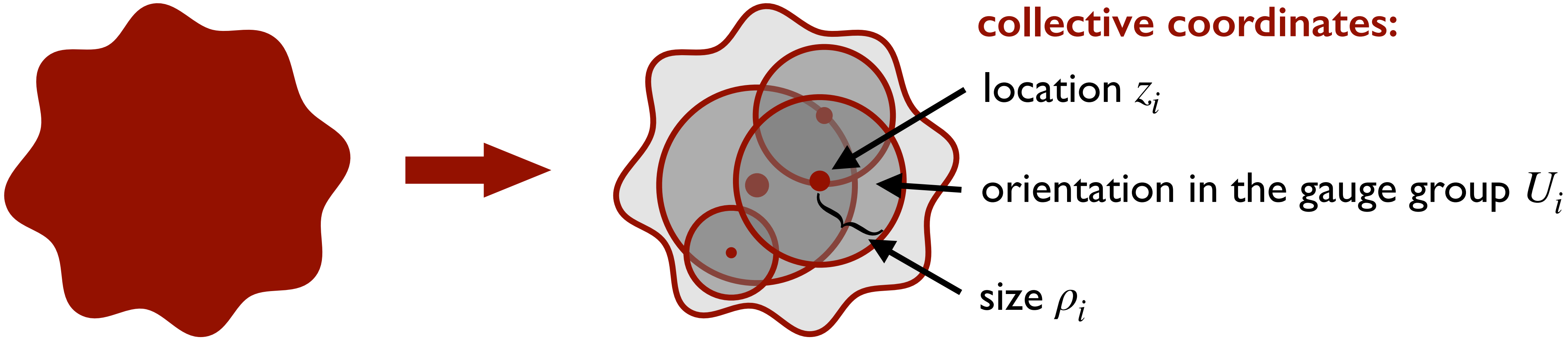
For a semi-classical analysis, we need to know general Q-instantons $A_\mu^{(Q)}$ and the corresponding fermion zero modes $\psi^{(Q)}$.

$\longrightarrow$ ADHM construction

[Atiyah, Drinfeld, Hitchin, Manin],
[Corrigan, Fairlie, Templeton, Goddard],
[Osborn] (1987)

- reduces classical self-dual YM equations to a set of nonlinear algebraic equations
- still, exact solutions for $Q > 2$ are unknown

The gist: Q-instanton can be viewed as composition of constituent-instantons with $Q = 1$:



collective coordinates:

location z_i

orientation in the gauge group U_i

size ρ_i

- $4N_c|Q|$ collective coordinates describe a Q-instanton
- arise from symmetries that yield inequivalent instanton solutions
- ADHM can be done systematically for any Q in the 'small constituent-instanton limit', $|z_i - z_j| \gg \rho_i$

[Christ, Weinberg, Stanton (1978)]
[Pisarski, FR (2019)], [FR (2020)]

PARTITION FUNCTION

Partition function in a Q-instanton background

$$Z_Q[J] = \int \mathscr{D}\delta\Phi \exp\left\{ -S[\bar{\Phi}^{(Q)} + \delta\Phi] + \int_x \bar{\psi} J \psi \right\}$$

$\Phi = (A, c, \bar{c}, \psi, \bar{\psi})$

$\bar{\Phi}^{(Q)} = (A^{(Q)}, 0, 0, 0, 0)$

source for quark-antiquark pairs (to account for ΔQ_5)

- consider small fluctuations around topological background $A_\mu^{(Q)}$
- collective coordinates correspond to symmetries: resulting gauge field zero modes need to be treated exactly. The same is true for fermion zero modes.
- replace integral over zero modes by integral over collective coordinates:

$$Z_Q[J] = \int \left[N \prod_{i=1}^{Q} d^4 z_i \, d\rho_i \, dU_i \right] n_Q\big(\{z_i, \rho_i, U_i\}\big) \det{}_0 (J)$$

Q-instanton density

- gluon and ghost fluctuations
- fermion determinant over nonzero modes
- Jacobian of coordinate change from zero modes to collective coordinates

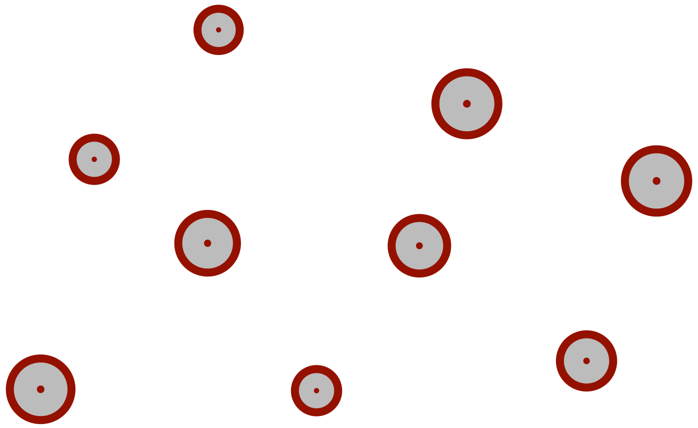
fermion zero mode determinant

$$\det \int d^4x \, \bar{\psi}^{(Q)\dagger}(x) \, J(x) \, \psi^{(Q)}(x)$$

DILUTE GAS & CORRECTIONS

Z_Q is partition function in the background of one instanton, but all possible field configurations contribute to the path integral

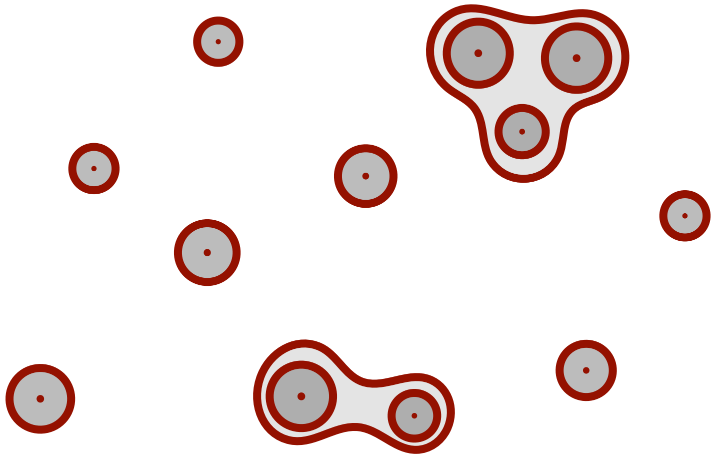
- If separation large against instanton size: dilute gas



$$Z[J] = \sum_{n_1} \frac{1}{n_1!} Z_1[J]^{n_1} = e^{Z_1[J]}$$

['t Hooft (1976)]

- multi-instanton corrections for smaller separation



$$Z[J] = \prod_Q \sum_{n_Q} \frac{1}{n_Q!} Z_Q[J]^{n_Q} = e^{\sum_Q Z_Q[J]}$$

[FR (2020)]

Dilute approximation reasonable at large T due to thermal screening of the instanton density

$$\bar{\rho} \ll \frac{1}{\pi T}$$

[Pisarski, Yaffe (1980)]
[Gross, Pisarski, Yaffe (1981)]

average size of (constituent) instanton

THETA VACUUM FROM THE DILUTE GAS

Sum over all Q-sectors should also include the phase from the θ-vacuum

$$Z_Q \to Z_Q \, e^{iQ\theta}$$

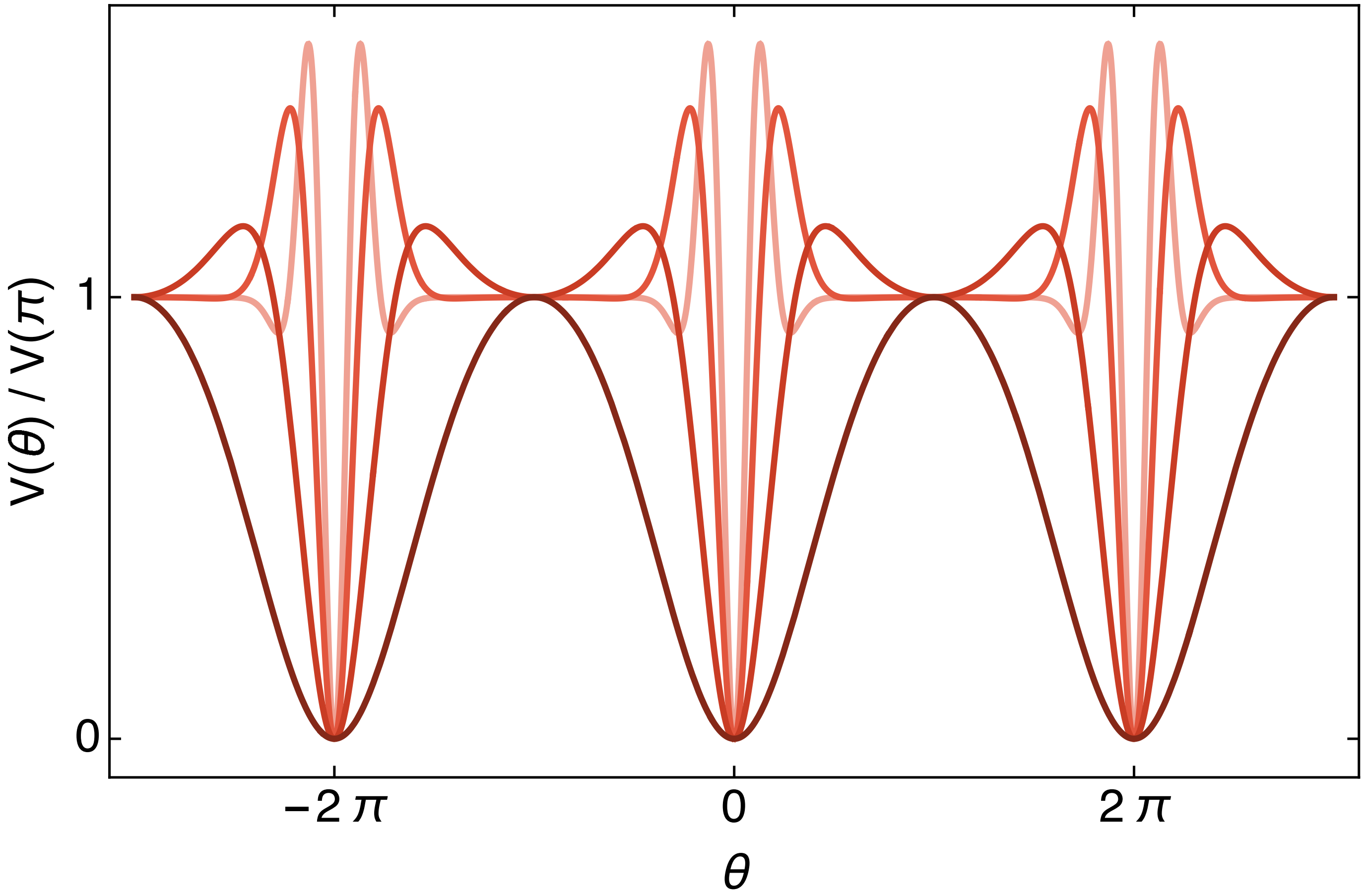
The resulting free energy density in the dilute gas then is

$$V(\theta) = -\frac{1}{\mathscr{V}} \ln Z(\theta) = -\frac{2}{\mathscr{V}} \sum_Q Z_Q \cos(Q\theta)$$

[FR (2020)]

→ describes the distribution of topological charge

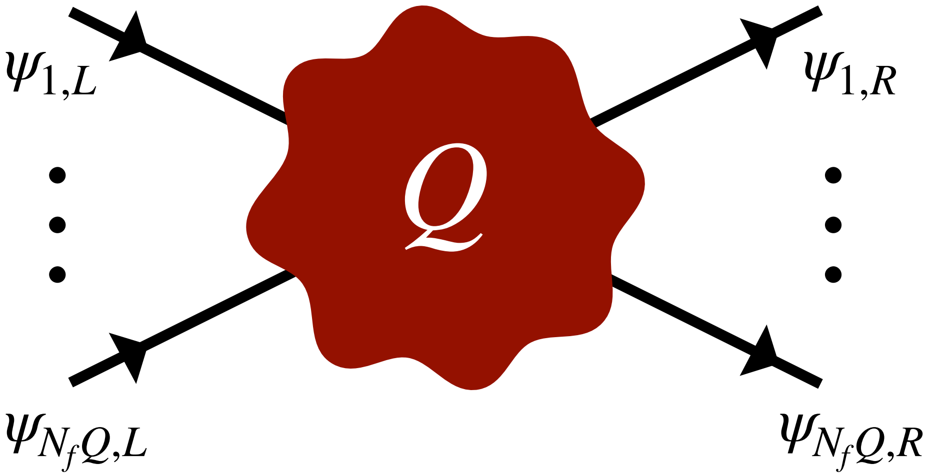
Approximation based on small-instanton limit:



from darkest to lightest red: increasing instanton density

PHENOMENOLOGY

PHENOMENOLOGY



Most effects arise from the coupling to fermions and the axial anomaly

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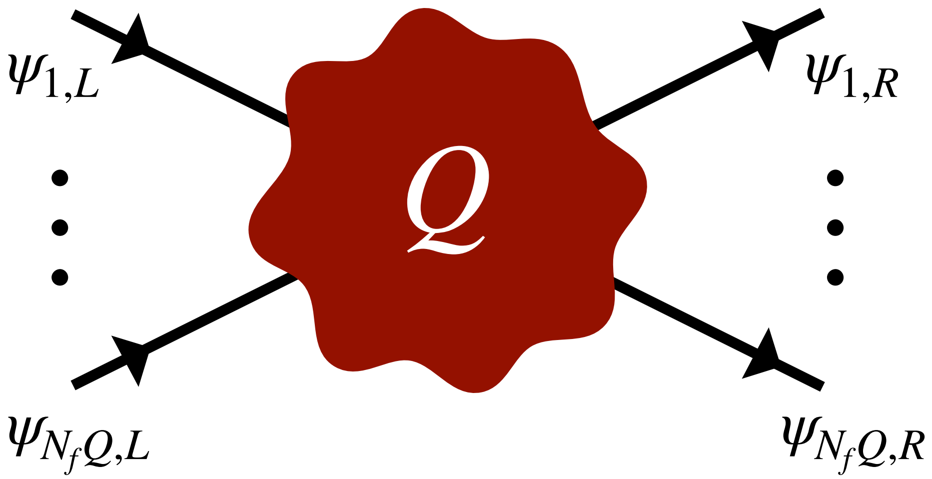
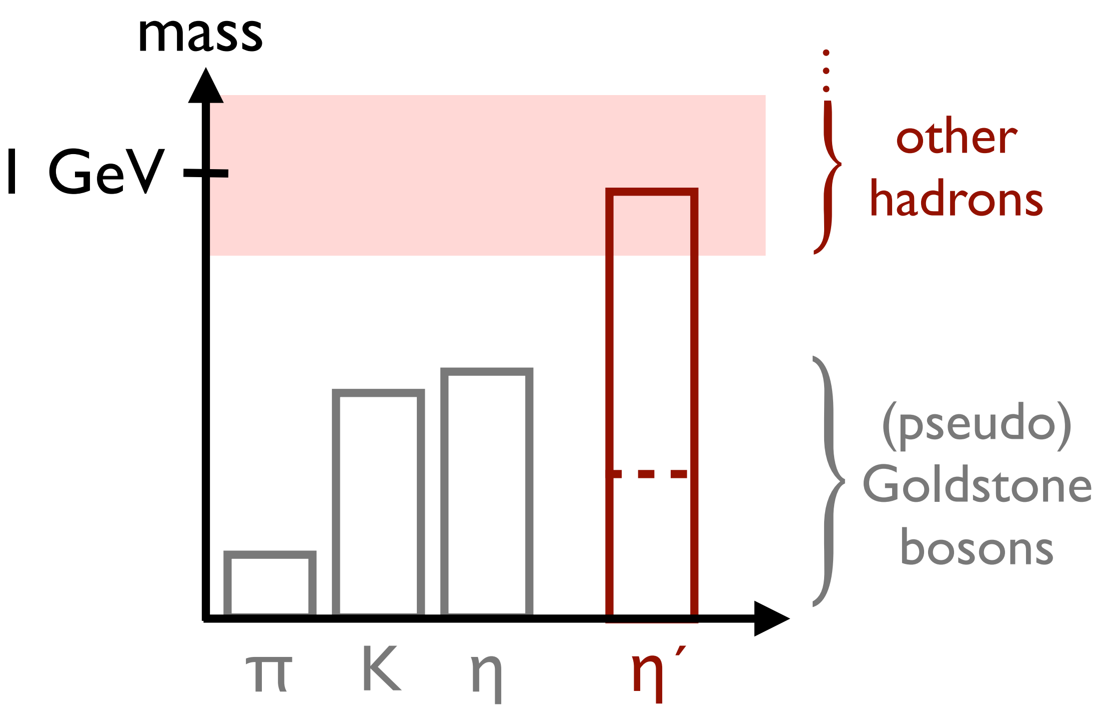


Chiral symmetry in QCD:

$$U(3)_L \times U(3)_R \sim SU(3)_V \times SU(3)_A \times U(1)_V \times U(1)_A$$



- 'classical' chiral symmetry breaking should lead to 9 Goldstone bosons: π, K, η, η'
- but η' too heavy to be a Goldstone

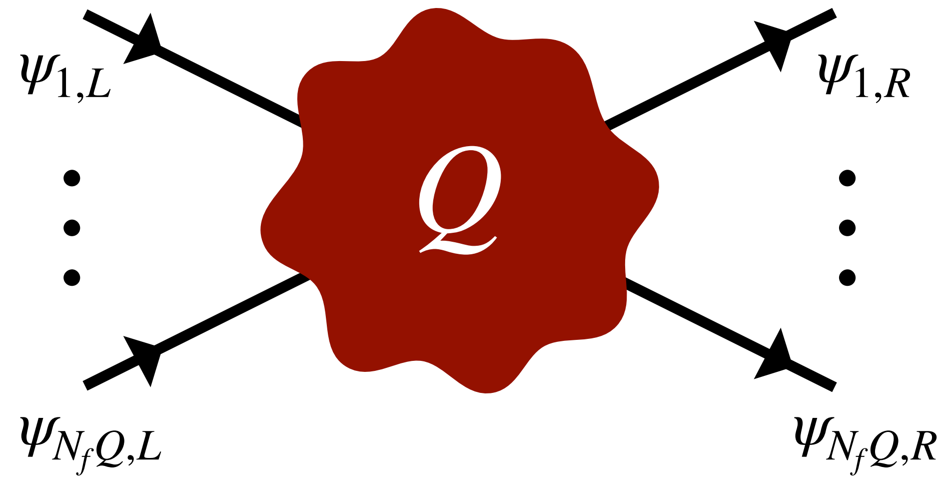
→ anomaly makes η' heavy

(adds explicit mass contribution)

['t Hooft (1976)]

PHENOMENOLOGY

Most effects arise from the coupling to fermions and the axial anomaly



Anomalous effects in QCD semi-classically from instanton-induced interactions:

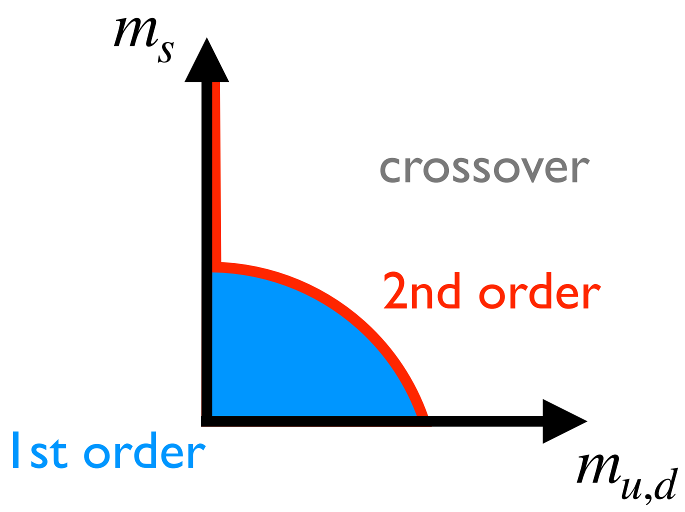
- sources to account for $\Delta Q_5 \neq 0$ in Q-instanton background give rise to nonzero determinant over quark zero modes
- turns out to be a $2N_f Q$-quark correlation function that explicitly breaks $U(1)_A$

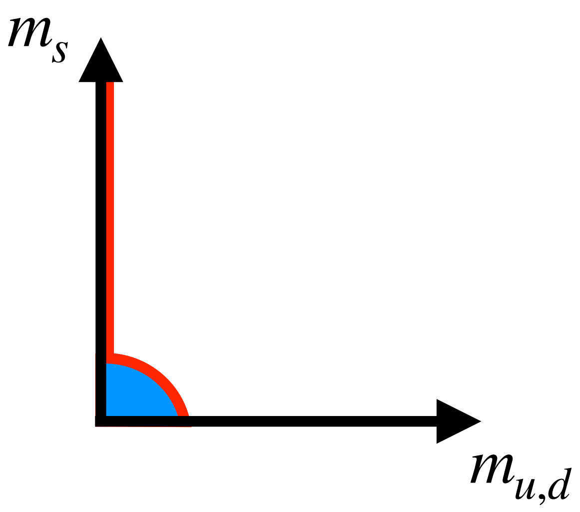
fluctuations around $A_\mu^{(Q)}$ $\longrightarrow$ $\det_f\left(\bar\psi_R\psi_L\right)^{|Q|} + \det_f\left(\bar\psi_L\psi_R\right)^{|Q|}$

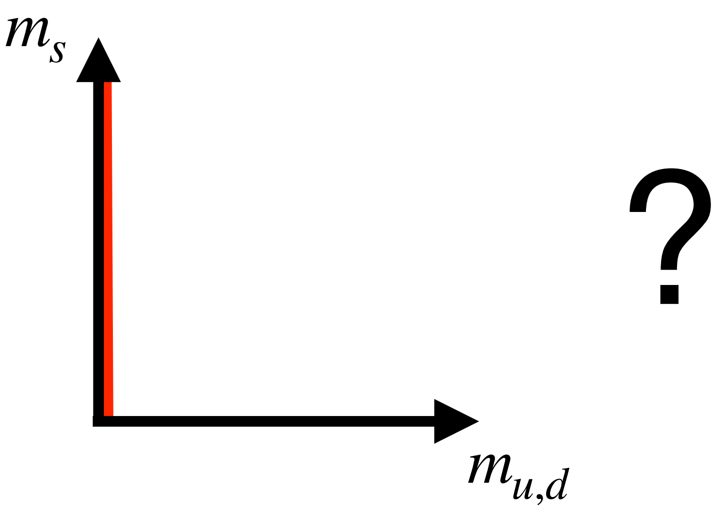
|Q|=1: ['t Hooft (1976)]
|Q|=2: [Pisarski, FR (2019)]
|Q|>2: [FR (2020)]

Give mass to η', affect the order of the chiral phase transition, ...

[Pisarski, Wilczek (1983)]
[Pisarski, FR (2024)]

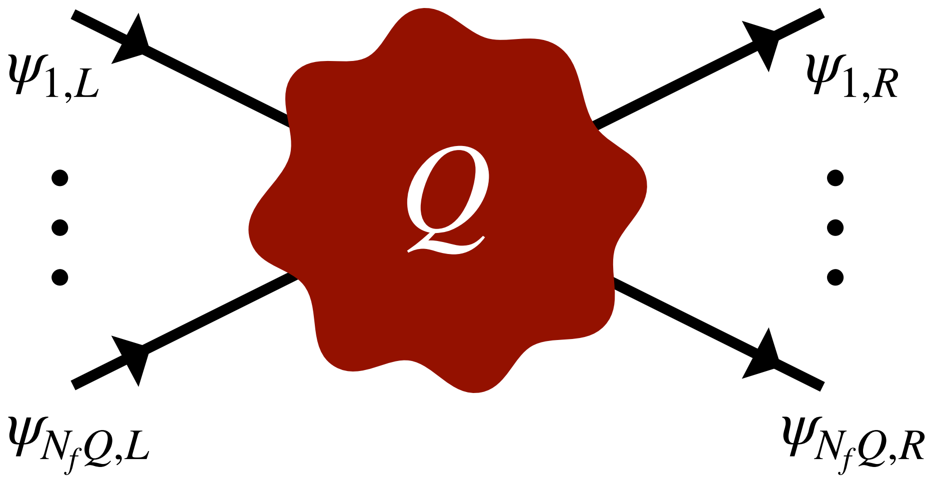






PHENOMENOLOGY

Most effects arise from the coupling to fermions and the axial anomaly



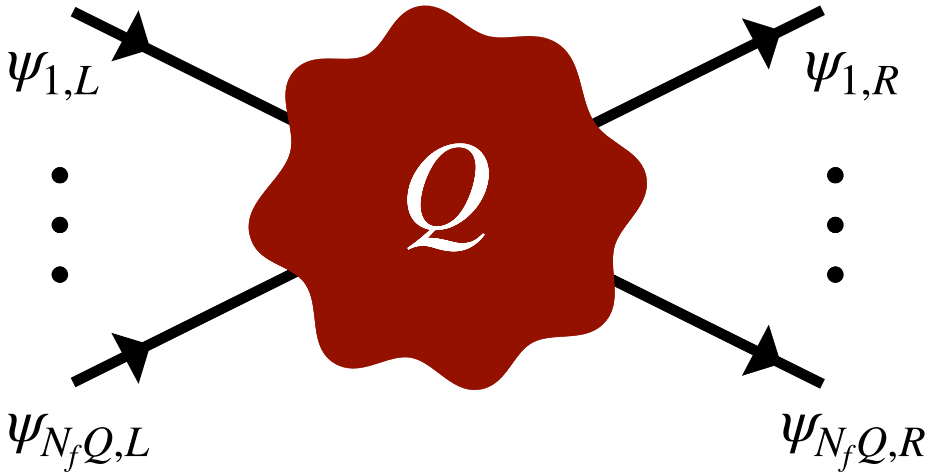
Chiral Magnetic Effect

(and its cousins)

→ talks by Adrien & Zhiwan

PHENOMENOLOGY

Most effects arise from the coupling to fermions and the axial anomaly

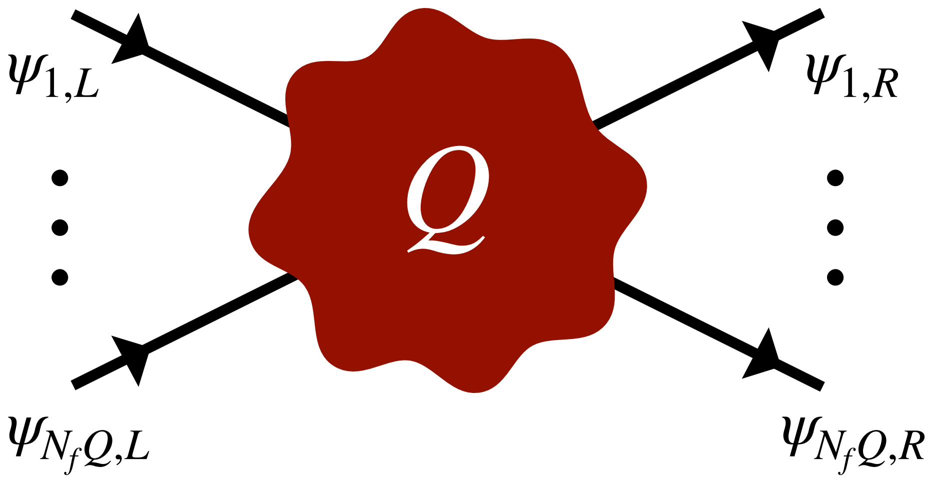


instanton/sphaleron processes at colliders

→ talks by Valentin & Ynyr

PHENOMENOLOGY

Most effects arise from the coupling to fermions and the axial anomaly



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strong CP problem

Phase of the θ-vacuum enters the QCD Lagrangian:

$$\mathscr{L} = \bar{\psi}\,\gamma_\mu D_\mu \psi + \frac{1}{2}\mathrm{tr}\,FF + \frac{i\theta}{16\pi^2}\,\mathrm{tr}\,F\tilde{F}$$

neutron electric dipole moment

strong CP violation for $\theta \neq 0, \pi$ → $d_n \sim e\theta \dfrac{m_u m_d}{f_\pi^2(m_u + m_d)}$

[Crewther, Di Vecchia, Veneziano, Witten (1979)]

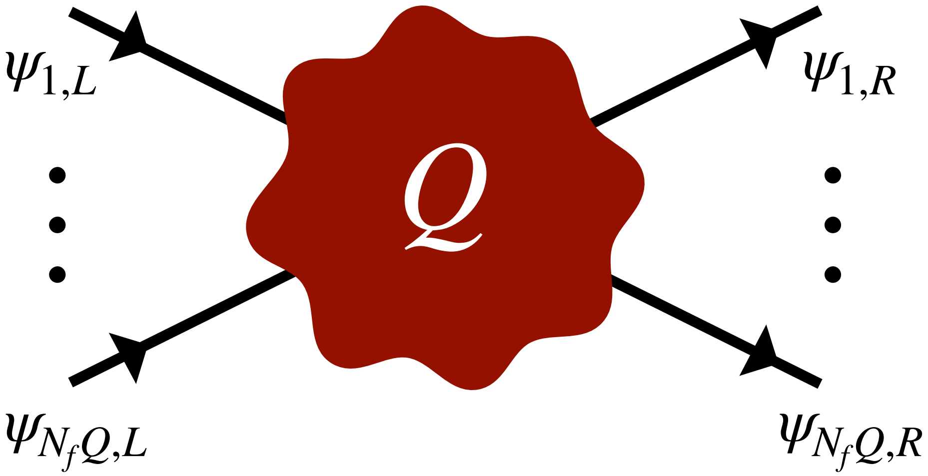
Most recent measurements yield $|d_n| < 1.8 \times 10^{-26}\,e$cm → $\theta \lesssim 10^{-10}$

[Abel et al. (nEDM) (2020)]

why is $\theta \approx 0$?

PHENOMENOLOGY

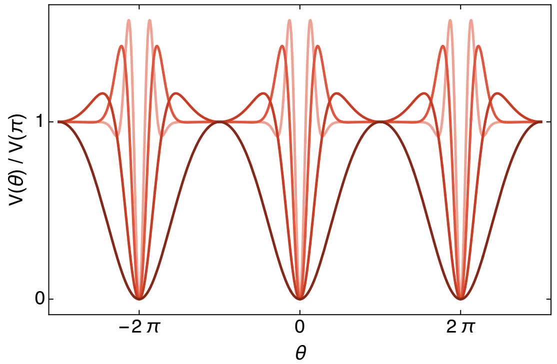
Most effects arise from the coupling to fermions and the axial anomaly



- augment SM with global axial $U(1)_{\mathrm{PQ}}$ + charged scalar that couples to quarks
- $U(1)_{\mathrm{PQ}}$ is spontaneously broken at scale f_a, resulting Goldstone boson: **axion**
- the axial anomaly in $U(1)_{\mathrm{PQ}}$ dictates non-derivative couplings of the axion:

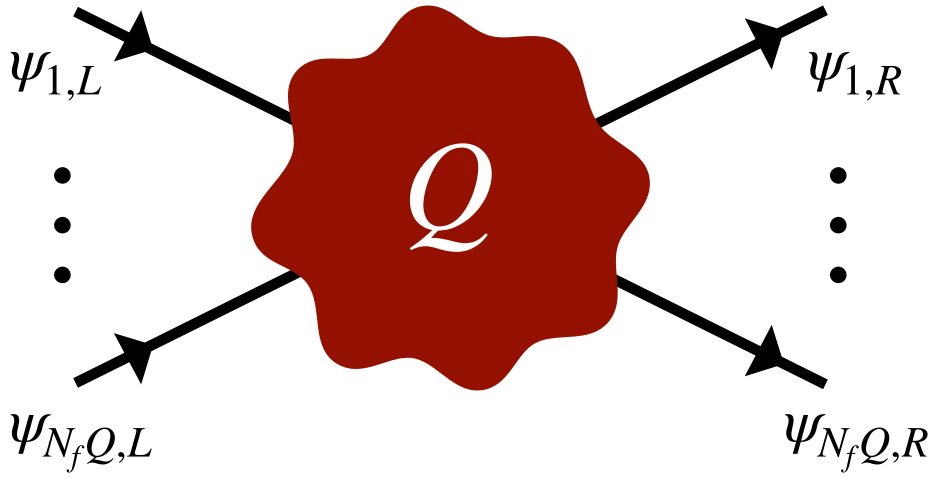
$$\mathscr{L} = \bar{\psi}\,\gamma_\mu D_\mu \psi + \frac{1}{2}\mathrm{tr}\,FF + \frac{i\theta}{16\pi^2}\,\mathrm{tr}\,F\tilde{F} + \frac{a(x)}{f_a}\frac{i}{16\pi^2}\,\mathrm{tr}\,F\tilde{F} + \ldots$$

→ 'dynamical' θ-angle $\bar{\theta}(x) = f_a\theta + a(x)$
solves the strong CP problem



PHENOMENOLOGY

Most effects arise from the coupling to fermions and the axial anomaly



Baryogenesis

Matter-antimatter imbalance requires baryon-number violation (+ C and CP violation and non-equilibrium: Sakharov conditions)

- electroweak SM inherently chiral (only L-fermions couple to $SU(2)$)
- baryon and lepton number conservation related to a $U(1)_A$ symmetry

$$\Delta(B+L) \sim \Delta Q$$

[Kuzmin, Rubakov, Shaposhnikov (1985)]

Must occur through sphaleron transitions $\sim e^{-E_{\mathrm{sph}}/T}$, since instanton process $\sim e^{-\frac{8\pi^2}{g^2}|Q|}$ are highly unlikely for the weak interactions!

SUMMARY

$A_\mu \sim$

$=$

Disclaimer:

- there are many more semi-classical objects I didn't mention; they can be periodic (finite T), have non-trivial holonomy (confinement), fractional topological charge, magnetic charge, ...
- they can also interact, which I didn't cover either
- interactions/correlations become strong, the semi-classical picture breaks down

BACKUP

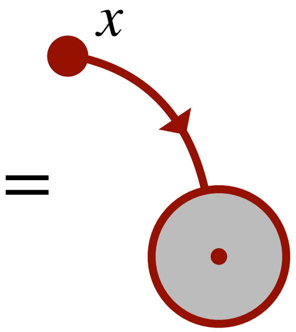
Q=1 QUARK ZERO MODE

Zero mode of BPST instanton with $Q = 1$: ['t Hooft (1976)]

$$\psi^{(1)}(x) = \nu \frac{U_i \, \rho_i}{\left[(x-z_i)^2 + \rho_i^2\right]^{3/2}} \frac{\gamma_\mu \, (x-z_i)_\mu}{|\, x - z_i \,|} \underset{\uparrow}{\varphi_R} \xrightarrow{|x-z_i| \gg \rho_i} \sim \rho_i \, U_i \, \underset{\uparrow}{\Delta}(x-z_i) \, \varphi_R =$$

RH spinor

free quark propagator!

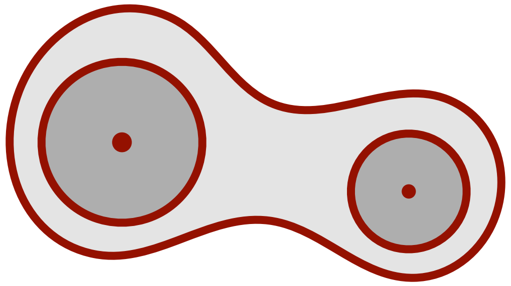


Q-INSTANTONS & QUARK ZERO MODES

Results to order $\rho^4/|R|^4$ (R: separation of constituent-instantons):

- Q-instanton:

$Q = 2$ illustration:



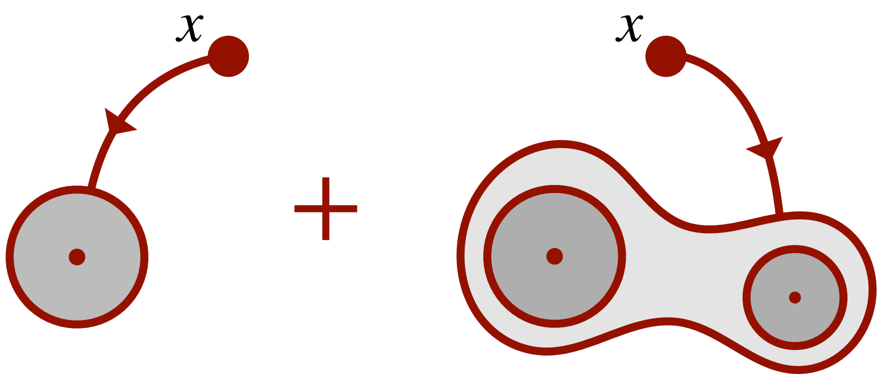
$$A_\mu^{(Q)}(x) = \frac{1}{\xi_0(x, \{z_i, \rho_i\})} \sum_{i=1}^{Q} A_\mu^{(1)}(x; z_i, \rho_i, U_i) + \mathcal{O}\left(\frac{\rho^4}{|R|^4}\right)$$

[Christ, Weinberg, Stanton (1978)]

- $N_f\, Q$ quark zero modes: ($f = 1, \ldots, N_f$, $i = 1, \ldots, Q$)

$$\psi_{fi}^{(Q)}(x) = \psi_{fi}^{(1)}(x, z_i, \rho_i, U_i) - \sum_{j \neq i} \mathbb{X}_{ij}(x, z_i, \rho_i, \rho_j)\, \psi_{fj}^{(1)}(x, z_j, \rho_j, U_i) + \mathcal{O}\left(\frac{\rho^4}{|R|^4}\right)$$

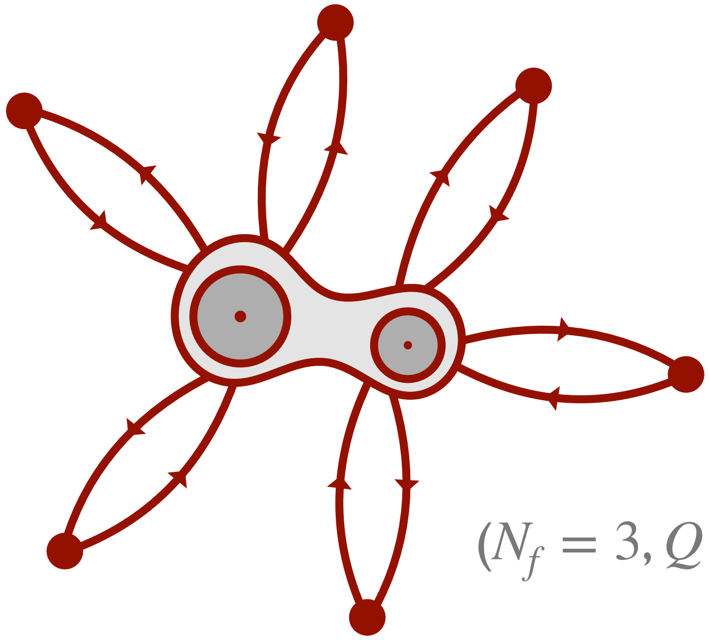
[Pisarski, FR; 1910.14052]
[FR; 2003.13876]



CORRELATION FROM QUARK ZERO MODES

Use the large-distance form of the quark zero modes:

$$\det{}_0(J) \sim \prod_i \prod_f \int d^4x_{fi}\, \Delta(z_i - x_{fi})\, J(x_{fi})\, \Delta(x_{fi} - z_i)$$



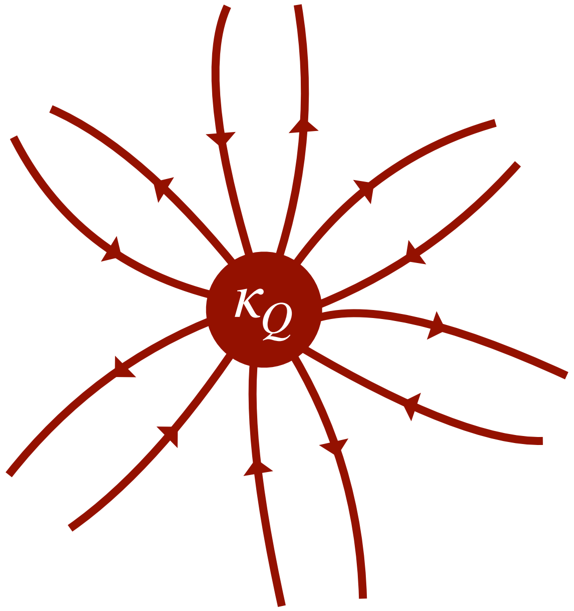
$Z[J]$ is identical to an effective partition function
(details in [Pisarski, FR, 1910.14052])

no instanton background!

$$Z^{\text{eff}}[\bar{J}] = \int \mathscr{D}\Phi\, e^{-S[\Phi] + \sum_Q \Delta S_Q^{\text{eff}} + \int_x \bar{\psi}\bar{J}\psi}$$

store "instanton stuff" in coupling κ_Q

$$\Delta S_Q^{\text{eff}}\Big|_{\text{singlet}} \sim \int d^4z\, \kappa_Q \det_{fg} \left[\bar{\psi}_f(z)\, \mathbb{P}_R\, \psi_g(z)\right]^Q$$



- local $2N_f\,Q$-quark correlation function
- for anti-instantons ($Q < 0$): $\mathbb{P}_R \to \mathbb{P}_L$