2-loop RGEs from geometry

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Outline

1) RGEs from counterterms

2) 't Hooft's method for counterterms

3) Extension to EFTs

4) Extension to two loops

5) Application: 2-loop RGE of $O(N)$ EFT, SMEFT, χPT

RGEs from UV counterterms

$$\mathcal{L} \supset C_i^b \mathcal{O}_i$$

$\frac{1}{\epsilon^n}$ pole @ ℓ loops

$$C_i^b = \mu^{n_i \epsilon}(C_i + \delta_i) , \qquad \delta_i = \sum_{\ell=1}^{\infty} \sum_{n=0}^{\ell} \frac{a^{(\ell,n)}}{\epsilon^n}$$

Using $\mu \frac{d}{d\mu} C_i^b = 0$ (and topological identities)

$$\dot{C}_i^{(1)} = 2 a_i^{(1,1)}$$

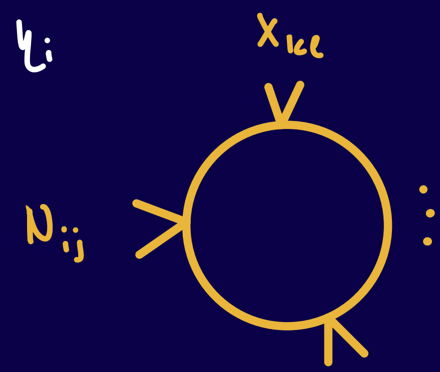
$$\dot{C}_i^{(2)} = 4 a_i^{(2,1)} - 2 a_j^{(1,0)} \frac{\partial a_i^{(1,1)}}{\partial C_j} - 2 a_j^{(1,1)} \frac{\partial a_i^{(1,0)}}{\partial C_j}$$

$$\dot{C}_i^{(\ell)} = 2\ell \, a^{(\ell,1)} + \dots$$

where $\dot{C} = \mu \frac{d}{d\mu} C$.

't Hooft's algorithm

Start with $\mathcal{L}(\phi_i)$. Write $\phi_i \longrightarrow \phi_i + \eta_i$ and expand in η_i



$\mathcal{O}(\eta^2)$ term is

$$\mathcal{L}^{(1)} = \frac{1}{2}\partial_\mu \eta_i \partial_\mu \eta_i + N^\mu_{ij}\partial_\mu \eta_i \eta_j + \frac{1}{2} X_{ij}\eta_i\eta_j$$

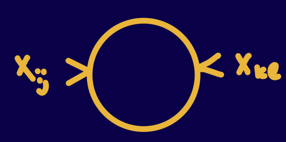
$$= \ldots = \frac{1}{2}(D_\mu \eta)_i (D_\mu \eta)_i + \frac{1}{2}\eta_i X_{ij}\eta_j \quad \text{with} \quad D^\mu = \partial^\mu + N^\mu$$

$\mathcal{L}^{(1)}$ invariant under local $O(N)$ $\longrightarrow$ $\Delta\mathcal{L}^{(1)}$ built from X, D_μ and

$$Y^{\mu\nu} = [D^\mu, D^\nu] = \partial^\mu N^\nu - \partial^\nu N^\mu + N^\mu N^\nu - N^\nu N^\mu$$

Dimensional analysis: $\boxed{\Delta\mathcal{L}^{(1)} = \frac{1}{16\pi^2\epsilon}\left(a\, X_{ij}X_{ji} + b\, Y^{\mu\nu}_{ij}Y^{\mu\nu}_{ji}\right)}$

compute (XX), $(N^\mu N^\nu N^\rho N^\sigma)$ $\longrightarrow$ $a = -\frac{1}{4}$, $b = -\frac{1}{24}$ ['t Hooft, 1973]



Example: Renormalization of ϕ^4 theory

E.g. $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 - \Lambda$

$$X = \frac{\delta^2 \mathcal{L}}{\delta\phi\,\delta\phi} = -m^2 - \frac{\lambda}{2}\phi^2 , \quad Y^{\mu\nu} = 0$$

So the counterterms at one loop are

$$\Delta\mathcal{L}^{(1)} = -\frac{1}{4\epsilon}X^2 = -\frac{1}{4\epsilon}\left(-m^2 - \frac{\lambda}{2}\phi\right)^2$$

$$= \underbrace{-\frac{m^4}{4\epsilon}}_{\delta_\Lambda} \underbrace{-\frac{\lambda m^2}{4\epsilon}\phi^2}_{\delta_m} \underbrace{-\frac{\lambda^2}{16\epsilon}\phi^4}_{\delta_\lambda}$$

and $\delta_\phi = 1$

Instead of calculating 

Inclusion of Effective Operators

Higher-dimensional operators with two derivatives

$$\mathcal{L} = \frac{1}{2} g_{ij}(\phi)\,(D_\mu \phi)^i\,(D^\mu \phi)^j - V(\phi)$$

Now ϕ^i are coordinates on $\mathcal{M}^{scalar}$. Quantum corrections to $\mathcal{L}$ from expansion in geodesics $\phi^i(\lambda)$:

$$\frac{d^2\phi^i}{d\lambda^2} + \Gamma^i_{jk}(\phi)\,\frac{d\phi^j}{d\lambda}\frac{d\phi^k}{d\lambda} = 0\,, \qquad \frac{d\phi^i}{d\lambda} = \eta^i$$

ϕ_i: coordinates

η: tangent vectors of geodesics

$$\phi^i \longrightarrow \phi^i + \frac{d\phi^i}{d\lambda} + \frac{1}{2}\frac{d^2\phi^i}{d\lambda^2} + \dots$$

$$= \phi^i + \eta^i - \frac{1}{2}\Gamma^i_{jk}\,\eta^j\eta^k + \dots$$

Inclusion of Effective Operators

In coordinate - free notation:

$$D g = 0, \quad D_\lambda \eta = 0, \quad T(X,Y) = D_X Y - D_Y X - [X,Y] = 0$$

$$\delta \mathcal{L} = g(D_\mu \eta, D_\mu \phi) - D_\lambda V \qquad \text{EOM}$$

$$\delta^2 \mathcal{L} = \frac{1}{2} g(D_\mu \eta, D_\mu \eta) + \frac{1}{2} g(R(\eta, D_\mu \phi)_\eta, D_\mu \phi) - \frac{1}{2} D_\lambda D_\lambda V \qquad \text{1-loop}$$

Comparison with the 't Hooft formula

$$\Delta \mathcal{L}^{(1)} = -\frac{1}{4\epsilon} X^i{}_j X^j{}_i - \frac{1}{24\epsilon} (Y_{\mu\nu})^i{}_j (Y^{\mu\nu})^j{}_i$$

gives:

$$X_{;ij} = -R_{ikjl} (D_\mu \phi)^k (D_\mu \phi)^l - D_i D_j V$$

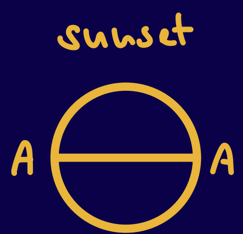
$$Y^{\mu\nu}_{;ij} = R_{ijkl} (D^\mu \phi)^k (D^\nu \phi)^l + D_j t_{\alpha;i} F^\alpha_{\mu\nu}$$

[Alonso, Jenkins, Manohar, 2016]

[Helset, Jenkins, Manohar, 2023]

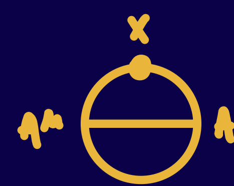
The Background Field Method at two loops

Generic two-loop diagram:



or

e.g.



Expand up to $\mathcal{O}(\eta^4)$

$$\mathcal{L}^{(2)}[\phi] = A_{ijk}\,\eta^i\eta^j\eta^k + A^{\mu}_{ijk}\,D_\mu\eta^i\eta^j\eta^k + A^{\mu\nu}_{ijk}\,D_\mu\eta^i D_\nu\eta^j\,\eta^k$$

$$+ B_{ijkl}\,\eta^i\eta^j\eta^k\eta^l + B^{\mu}_{ijkl}\,D_\mu\eta^i\eta^j\eta^k\eta^l + B^{\mu\nu}_{ijkl}\,D_\mu\eta^i D_\nu\eta^j\,\eta^k\eta^l$$

The two-loop counterterms $\Delta\mathcal{L}^{(2)}$

Dimensional analysis

$$[D_\mu] = 1 \quad [X] = [Y] = 2 \quad [A] = 1 \quad [A^\mu] = 0 \quad [A^{\mu\nu}] = -1$$

$$[B] = 0 \quad [B^\mu] = -1 \quad [B^{\mu\nu}] = -2$$

Ansatz for $\Delta\mathcal{L}^{(2)}$

$$\Delta\mathcal{L}^{(2)} = AAD^2 + AAX + AAY + A^\mu AD^3 + A^\mu ADX + \dots$$
$$+ BD^4 + BXD^2 + BYD^2 + BXX + BXY + BYY + B^\mu D^5 + \dots$$

Any term can have multiple independent contractions

The two-loop counterterms $\Delta\mathcal{L}^{(2)}$

AA	$D^2 \quad X \quad Y$
$A^\mu A$	$D^3 \quad XD \quad YD$
$A^\mu A^\mu$	$D^4 \quad XD^2 \quad YD^2 \quad X^2 \quad XY \quad YY$
$A^{\mu\nu} A$	$D^4 \quad XD^2 \quad YD^2 \quad X^2 \quad XY \quad YY$
$A^{\mu\nu} A^\mu$	$D^5 \quad XD^3 \quad YD^3 \quad X^2D \quad XYD \quad YYD$
$A^{\mu\nu} A^{\mu\nu}$	$D^6 \quad XD^4 \quad YD^4 \quad X^2D^2 \quad XYD^2 \quad YYD^2 \quad XXX \quad XXY \quad XYY \quad YYY$
B	$D^4 \quad XD^2 \quad YD^2 \quad XX \quad XY \quad YY$
B^μ	$D^5 \quad XD^3 \quad YD^3 \quad XXD \quad XYD \quad YYD$
$B^{\mu\nu}$	$D^6 \quad XX \quad XXD^2 \quad XYD^2 \quad YYD^2 \quad XXX \quad XXY \quad XYY \quad YYY$

Each corresponds to a Green's function

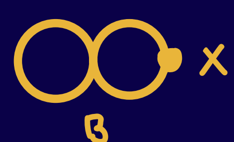
The two-loop counterterms $\Delta\mathcal{L}^{(2)}$

AA	$D^2 \quad X \quad \cancel{Y}$
$A^\mu A$	$\cancel{D^3} \quad XD \quad YD$
$A^\mu A^\mu$	$D^4 \quad XD^2 \quad YD^2 \quad X^2 \quad XY \quad YY$
$\cancel{A^{\mu\nu} A}$	$\cancel{D^4 \quad XD^2 \quad YD^2 \quad X^2 \quad XY \quad YY}$
$\cancel{A^{\mu\nu} A^\mu}$	$\cancel{D^5 \quad XD^3 \quad YD^3 \quad X^2D \quad XYD \quad YYD}$
$\cancel{A^{\mu\nu} A^{\mu\nu}}$	$\cancel{D^6 \quad XD^4 \quad YD^4 \quad X^2D^2 \quad XYD^2 \quad YYD^2 \quad XXX \quad XXY \quad XYY \quad YYY}$

Geometry magic

B	$\cancel{D^4} \quad \cancel{XD^2} \quad \cancel{YD^2} \quad XX \quad \cancel{XY} \quad \cancel{YY}$
B^μ	$\cancel{D^5} \quad \cancel{XD^3} \quad \cancel{YD^3} \quad XXD \quad XYD \quad \cancel{YY}D$
$B^{\mu\nu}$	$\cancel{D^6} \quad XX \quad XXD^2 \quad XYD^2 \quad YYD^2 \quad XXX \quad XXY \quad XYY \quad YYY$

E.g. $A_{ijk} A_{ijk} Y^{\mu\mu} = 0$, $\quad A^\mu_{ijk} A_{ijk} D^3 = 0$ (D_μ preserves symmetry)

E.g. $\langle BXD^2 \rangle$: 

scaleless & power divergent

Result

$$\boxed{\Delta\mathcal{L}^{(2)} = -\frac{3}{4\epsilon} D_\mu A_{ijk} D_\mu A_{ijk} + \left(\frac{5}{2\epsilon^2} - \frac{9}{2\epsilon}\right) A_{ijk} A_{ijl} X_{kl} + \frac{3}{\epsilon^2} B_{ijkl} X_{ij} X_{kl} + \text{terms with } A^\mu,\ B^\mu,\ B^{\mu\nu}}$$

the 2-loop counterterm for any scalar theory

(with no more than 2 derivatives)

assumes $\quad \mathcal{L} = \frac{1}{2} g_{ij}(\phi)\,(D_\mu\phi)^i\,(D^\mu\phi)^j - V(\phi) \quad$ with $\quad g_{ij} = \delta_{ij}$.

Generalization to $g_{ij} \neq \delta_{ij}$ again from geodesic expansion.

Geometry at two loops

$$\delta\mathcal{L} = g(D_\mu\eta, D_\mu\phi) - D_\lambda V$$

$$\delta^2\mathcal{L} = \frac{1}{2} g(D_\mu\eta, D_\mu\eta) + \frac{1}{2} g(R(\eta, D_\mu\phi)\eta, D_\mu\phi) - \frac{1}{2} D_\lambda D_\lambda V$$

$$\delta^3\mathcal{L} = \frac{1}{6} g(D_\lambda R(\eta, D_\mu\phi)\eta, D_\mu\phi) + \frac{2}{3} g(R(\eta, D_\mu\phi)\eta, D_\mu\eta) - \frac{1}{6} D_\lambda D_\lambda D_\lambda V$$

$$\delta^4\mathcal{L} = \frac{1}{24} g(D_\lambda D_\lambda R(\eta, D_\mu\phi)\eta, D_\mu\phi) + \frac{1}{4} g(D_\lambda R(\eta, D_\mu\phi)\eta, D_\mu\eta) + \frac{1}{6} g(R(\eta, D_\mu\eta)\eta, D_\mu\eta)$$

$$+ \frac{1}{6} g(R(\eta, D_\mu\phi)\eta, R(\eta, D_\mu\phi)\eta) - \frac{1}{24} D_\lambda D_\lambda D_\lambda D_\lambda V$$

Can read off

$$A_{abc} = -\frac{1}{6} D_{(a} D_b D_{c)} V - \frac{1}{18}(D_a R_{bdce} + \dots)(D_\mu\phi)^d (D_\mu\phi)^e$$

$$A^\mu_{abc} = \frac{1}{3}(R_{abcd} + R_{acbd})(D^\mu\phi)^d$$

$$A^{\mu\nu}_{abc} = 0$$

$$\vdots$$

Application: Procedure

1) Read off g_{ij}, compute Γ^i_{jk} and R_{ijkl}
2) Use our formulas to calculate X, A, A^μ, B, B^μ, $B^{\mu\nu}$
3) Plug into $\Delta\mathcal{L}^{(2)}$, raise and lower indices using g_{ij}
4) Perform field redefinition and derive RGE

Application: The O(N) EFT

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\cdot\partial_\mu\phi - \frac{m^2}{2}\phi\cdot\phi - \frac{\lambda}{4}(\phi\cdot\phi)^4 - \Lambda$$
$$+ C_1(\phi\cdot\phi)^3 + C_E(\phi\cdot\phi)(\partial_\mu\phi\cdot\partial_\mu\phi)$$

$$g_{ij} = \delta_{ij} + 2C_E(\phi\cdot\phi), \quad g^{ij} = \delta^{ij} - 2C_E(\phi\cdot\phi)$$

$$\Gamma^i_{jk} = 2C_E(\delta_{ik}\phi_j + \delta_{ij}\phi_k + \delta_{jk}\phi_i)$$

$$R_{ijkl} = 4C_E(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl})$$

$$\dot{C}_i = \{\ldots\}_1 + \{\ldots\}_2$$

Application: χPT

$SU(N) \times SU(N) \longrightarrow SU(N)$

$u(x) = e^{\frac{i}{f_\pi}\pi(x)}$, $\pi(x) = \pi^a(x)\, T^a$, $u_\mu = i(u^\dagger \partial_\mu u - u \partial_\mu u^\dagger)$

$\mathcal{L} = \frac{f_\pi^2}{4} \langle u_\mu u^\mu \rangle + \mathcal{L}_{p^4} + \mathcal{L}_{p^6}$

$R_{abcd} = \frac{1}{f^2} f_{abg} f_{cdg}$, $D_e R_{abcd} = 0$

Compute X, Y, A, A^μ, B, B^μ, $B^{\mu\nu}$

$\rightarrow$ full agreement on 2-loop counterterms

[Bijnens, Colangelo, Ecker]

Application: SMEFT

We include $\mathcal{L}_{SM,\,Higgs}$ and insertions of

H^6: $\mathcal{O}_H$,

H^2D^2: $\mathcal{O}_{H\Box}$, $\mathcal{O}_{HD}$

X^2H^2: $\mathcal{O}_{HG}$, $\mathcal{O}_{HW}$, $\mathcal{O}_{HB}$, $\mathcal{O}_{HWB}$

$\mathcal{O}_{H\tilde{G}}$, $\mathcal{O}_{H\tilde{W}}$, $\mathcal{O}_{H\tilde{B}}$, $\mathcal{O}_{H\tilde{W}B}$

- 1-loop RGE ✓
- 2-loop SM RGE ✓
- Consistency relations ✓
- 2-loop SMEFT RGE $\longrightarrow$ new

?

Backup: Steps in 2-loop calculation

- Start with first nonvanishing Green's function
- Generate diagrams with qgraf
- Calculate UV divergences using

$$\frac{1}{(q+p)^2-M^2} = \frac{1}{q^2-m^2} + \frac{M^2-p^2-2p\cdot q-m^2}{q^2-m^2}\,\frac{1}{(q+p)^2-M^2}$$

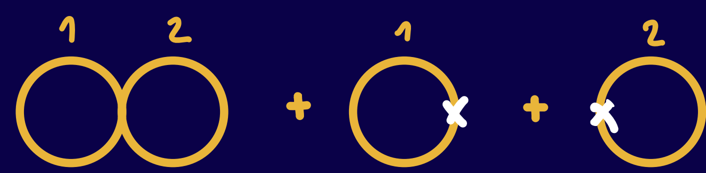
- Identify divergent subgraphs $\rightarrow$ add terms to $\Delta\mathcal{L}_Q^{(1)}$
- Evaluate 1-loop integrals with package-X
- Make sure non-local terms cancel, using FORM
- Map local divergences to $\Delta\mathcal{L}^{(2)}$

[Nogueira, 1993] [Chetyrkin, Misiak, Muenz, 1997]

[Patel, 2015] [Vermaseren, 2000]

Factorizable graphs

Factorizable graphs only give $1/\epsilon^2$ poles

$\underline{I}^{\,sub} =$ 

$$\stackrel{MS}{=} \left(\frac{I_{1\infty}}{\epsilon} + I_{1f}\right)\left(\frac{I_{2\infty}}{\epsilon} + I_{2f}\right) + \left(\frac{I_{1\infty}}{\epsilon} + I_{1f}\right)\left(-\frac{I_{2\infty}}{\epsilon}\right) + \left(-\frac{I_{1\infty}}{\epsilon}\right)\left(\frac{I_{2\infty}}{\epsilon} + I_{2f}\right)$$

$$= -\frac{I_{1\infty} I_{2\infty}}{\epsilon^2} + I_{1f} I_{2f}$$

→ predicts divergence

→ $\frac{1}{\epsilon}$ pole cancels, two-loop CT is purely $\frac{1}{\epsilon^2}$

→ does not affect RGE

But what if additional factor of ϵ in numerator ?

Enumerating Contractions

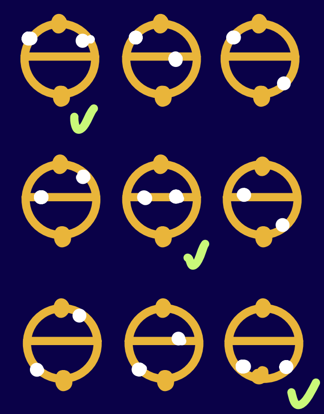
Possible flavor contractions in $(A^\mu A X D)$

but

$A^\mu_{ijk} A_{jk\ell} X_{i\ell}$ $\quad$ $A^\mu_{ijk} A_{ij\ell} X_{k\ell}$ $\quad$ $(A^\mu_{ijk} + A^\mu_{jki} + A^\mu_{kij}) A_{ij\ell} X_{k\ell} = 0$

Flavor contractions in $(A^\mu A^\mu X Y)$

2 independent contractions



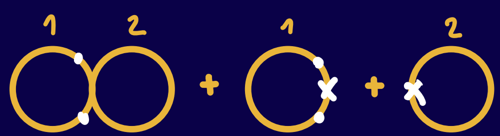
3 independent contractions

Factorizable graphs

Case 1: ϵ generated by an individual loop. $\eta_{\alpha\alpha} = d = 4-2$


$$= \eta^{\alpha}_{\alpha}\left(\frac{I_\infty}{\epsilon} + I_f\right) = \frac{4}{\epsilon} I_\infty + d I_f - 2 I_\infty$$

from $(D_\mu \eta D_\nu \eta)$



$$= \underline{d}\left(\frac{I_{1\infty}}{\epsilon} + I_{1f}\right)\left(\frac{I_{2\infty}}{\epsilon} + I_{2f}\right)$$

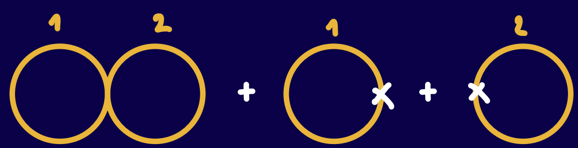
$$+ \underline{d}\left(\frac{I_{1\infty}}{\epsilon} + I_{1f}\right)\left(-\frac{I_{2\infty}}{\epsilon}\right) + \underline{4}\left(-\frac{I_{1\infty}}{\epsilon}\right)\left(\frac{I_{2\infty}}{\epsilon} + I_{2f}\right) = -\frac{4 I_{1\infty} I_{2\infty}}{\epsilon^2} + \text{finite}$$

$\longrightarrow$ minimal subtraction gives no $\frac{1}{\epsilon}$ pole. Nice.

Factorizable graphs

Case 2: ϵ generated after combining both loops

E.g. if $I_1^{\alpha\beta} = \eta^{\alpha\beta} I_1$, $I_2^{\alpha\beta} = \eta^{\alpha\beta} I_2$



$$= -\frac{1}{\epsilon^2} I_1^{\alpha\beta} I_2^{\alpha\beta} = -\frac{\overbrace{\eta^{\alpha}_{\alpha}}^{4-2\epsilon}}{\epsilon^2} I_1 I_2$$

→ factorizable topologies generate $\frac{1}{\epsilon}$ poles.

But suppose we split

$$\mathcal{L}_{eff} = \bar{\mathcal{L}}_{eff} + \hat{\mathcal{L}}_{eff}$$

Now

$\bar{\mathcal{L}}_{eff}$ generates $\bar{\eta}^{\alpha}_{\alpha} = 4$ → no effect on RGE

$\hat{\mathcal{L}}_{eff}$ generates $\hat{\eta}^{\alpha}_{\alpha} = -2\epsilon$ → no effect on RGE when we deviate from MS

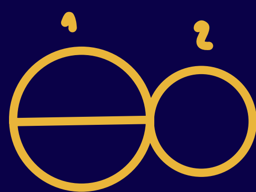
[Dugan, Grinstein, 1991]

Factorizable graphs

Argument generalizes to arbitrary loop order

E.g.  $+ \ldots CT \ldots = \dfrac{I_{1\infty} I_{2\infty} I_{3\infty}}{\epsilon^3} +$ finite

E.g.  $+ \ldots CT \ldots = \dfrac{(-1)^{n+1}}{\epsilon^n} \prod_i I_{i\infty}$

E.g.  $+ \ldots CT \ldots = -\left(\dfrac{I^2_{1\infty}}{\epsilon^2} + \dfrac{I^1_{1\infty}}{\epsilon}\right)\left(\dfrac{I^1_{2\infty}}{\epsilon}\right) +$ finite

With $I^i_{1\infty}$ the subdivergence subtracted (local) divergences of diag 1.

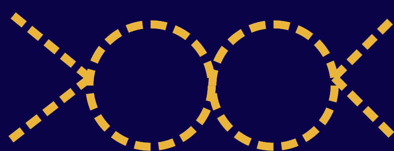
$\longrightarrow$ predict subtracted divergences of factorizable graphs

$\boxed{\text{Factorizable Graphs do not contribute to RGEs}}$

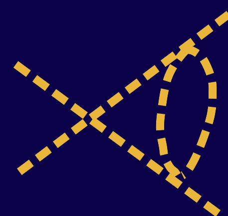
Factorizable graphs

Therefore factorizable 2-loop diagrams do not affect RGEs.

Example 1:



no effect

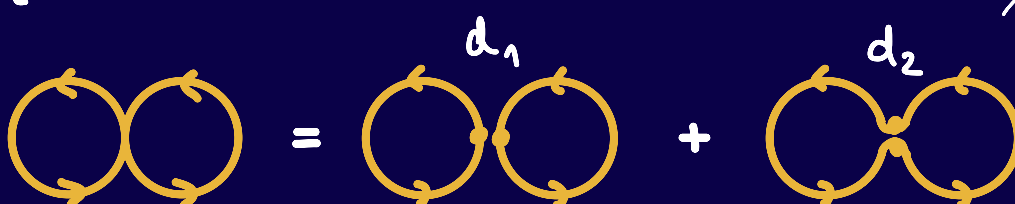


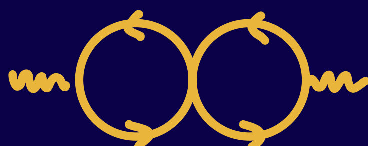
affects RGE

[Machacek, Vaughn, 1985]

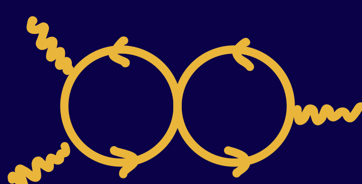
Example 2:

d_1 + Fierz - evanescent





$\longrightarrow$ no φ^4 coefficients in $\gamma^{(2)}_{g_i}$, $\gamma^{(2)}_{\theta_{QCD}}$



$\longrightarrow$ no φ^4 coefficients in $\gamma^{(2)}_{G}, \gamma^{(2)}_{\tilde{G}}, \gamma^{(2)}_{W}, \gamma^{(2)}_{\tilde{W}}$

⋮

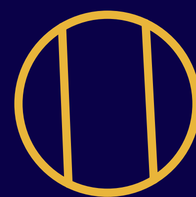
[Bern, Parra-Martinez, Sawyer, 2020]

Factorizable graphs

At 3 loops:

1

2



3

4

5

6

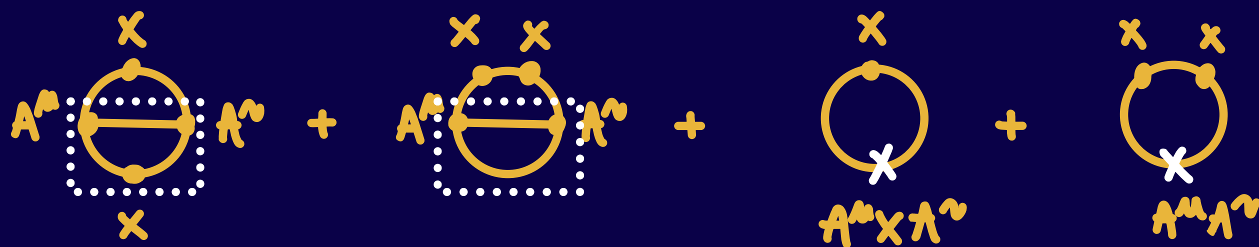
7

8

do not affect RGE

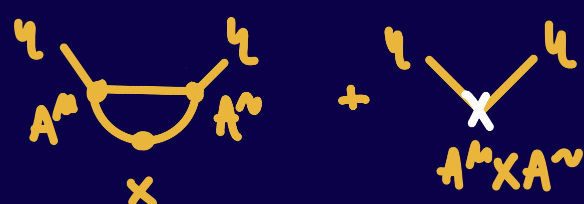
Non-factorizable graphs (AA)

AA graphs produce the $\frac{1}{\epsilon}$ poles which give RGEs. E.g. $A^\mu A^\nu X X$:



Requires additional 1-loop counterterms

$$\Delta R^{(1)}_Q \supset A^\mu X A^\nu \eta\eta + A^\mu A^\nu \eta\eta D^2$$



= finite

More derivatives

We extend $\Delta\mathcal{L}^{(1)}$ to include operators with any number of derivatives, as long as is one per field.

$$\mathcal{L} \supset C^{\mu\nu}_{ab} (D_\mu \eta)^a (D_\nu \eta)^b$$

Perturbative treatment gives

$$\Delta\mathcal{L}^{(1)} \supset \frac{1}{\epsilon}\left(-\frac{1}{12} C^{\mu\mu}_{ab} D^2 X_{ba} - \frac{1}{12} C^{\mu\nu}_{ab} \{D_\mu, D_\nu\} X_{ba} + \dots \right)$$

Non-factorizable graphs (AA)

Can find $\Delta\mathcal{L}_Q^{(1)}$ algebraically

$$\mathcal{L}^{(2)} = A_{ijk}\,\eta^i\eta^j\eta^k\,, \quad \text{shift again}: \quad \eta_i \to \eta_i + x_i$$

$$\mathcal{L}^{(2)} = A_{ijk}(\eta^i + x^i)(\eta^j + x^j)(\eta^k + x^k)$$

$$= \mathcal{O}(x^0) + \mathcal{O}(x^1) + 3A_{ijk}\,x^i x^j \eta^k + \dots$$

Apply the 't Hooft formula again:

$$X_{ij}[\phi,\eta] = 6A_{ijk}[\phi]\,\eta^k$$

$$\Delta\mathcal{L}_Q^{(1)} \supset -\frac{1}{4\epsilon}X_{ij}X_{ji} = -\frac{9}{\epsilon}A_{ijk}A_{ijl}\,\eta^k\eta^l$$

Determined all $\Delta\mathcal{L}_Q^{(1)}$ this way, checked cancellation

Application to O(N) model

$$\mathcal{L} = \frac{1}{2}(D_\mu \phi_i)^2 - \frac{m^2}{2}\phi_i^2 - \frac{\lambda}{4}(\phi_i^2)^2$$

$$\frac{\delta \mathcal{L}}{\delta \phi_i} = -m^2 \phi_i - \lambda \phi\cdot\phi\, \phi_i \,, \quad X_{ij} = \frac{\delta^2 \mathcal{L}}{\delta\phi_i \delta\phi_j} = -m^2 \delta_{ij} - \lambda\left(2\phi_i\phi_j + \phi\cdot\phi\,\delta_{ij}\right)$$

$$\underline{\Delta\mathcal{L}^{(1)}} = -\frac{1}{4\epsilon} X_{ij}X_{ji} = -\frac{1}{4\epsilon} N m^2 - \frac{1}{2}(N+2)\lambda m^2 (\phi\cdot\phi) - \frac{1}{4}(N+8)\lambda^2 (\phi\cdot\phi)^2$$

$$A_{ijk} = \frac{1}{3!}\frac{\delta^3 \mathcal{L}}{\delta\phi_i\delta\phi_j\delta\phi_k} = -\frac{1}{3}\lambda\left(\delta_{ij}\phi_k + \delta_{ik}\phi_j + \delta_{jk}\phi_i\right)$$

$$B_{ijkl} = \frac{1}{4!}\frac{\delta^4 \mathcal{L}}{\delta\phi_i\delta\phi_j\delta\phi_k\delta\phi_l} = -\frac{1}{12}\lambda\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$$

$$\underline{\Delta\mathcal{L}^{(2)}} = -\frac{3}{4\epsilon} D_\mu A_{ijk} D_\mu A_{ijk} + \left(\frac{9}{2\epsilon^2} - \frac{9}{2\epsilon}\right) A_{ijk}A_{ijl}X_{kl} + \frac{3}{\epsilon^2} B_{ijkl} X_{ij} X_{kl}$$

$$= -\frac{1}{4\epsilon}(2+N)\lambda^2 (D_\mu\phi)^2 - \frac{3}{2\epsilon}(2+N)\lambda^2 m^2 \phi\cdot\phi$$

$$- \frac{1}{2\epsilon}(22+5N)\lambda^3 (\phi\cdot\phi)^2 + \dots \frac{1}{\epsilon^2} \text{ poles}$$

$\longrightarrow$ extract Z_ϕ, Z_m, Z_λ. Agrees with 2-loop SM RGE.

Backup: Extraction of UV divergences

$$\int d^D k_1\, d^D k_2 \, \frac{k_1^{\mu_1} \dots k_1^{\mu_{n_1}} k_2^{\nu_1} \dots k_2^{\nu_{n_2}}}{(k_1^2)^{b_1} (k_2^2)^{b_2} ((k_1+k_2)^2)^{b_3}}$$

$D_1 = 4 + n_1 - 2b_1 - 2b_3$

$D_2 = 4 + n_2 - 2b_2 - 2b_3$

$D_3 = 4 + n_1 + n_2 - 2b_1 - 2b_2$

$D_{OA} = 8 + n_1 + n_2 - 2b_1 - 2b_2 - 2b_3$

towards tadpoles

$1^0 2^0 3^0 OA^0 + 1^{--} 2^0 3^{--} OA^{--}$

$$\frac{1}{(q+p)^2 - M^2} = \frac{1}{q^2 - m^2} + \frac{M^2 - p^2 - 2qp - m^2}{q^2 - m^2} \, \frac{1}{(q+p)^2 - M^2}$$

towards $(1\text{-loop})^2$

$1^0 2^{++} 3^{--} OA^0 + 1^- 2^{++} 3^0 OA^0$

$$\frac{1}{(k_1+q_2)^2 - M^2} = \frac{1}{k_1^2 - m^2} + \frac{M^2 - q_2^2 - 2k_1 \cdot q_2 - m^2}{k_1^2 - m^2} \, \frac{1}{(k_1+q_2)^2 - M^2}$$

tadpole integrals:

$$J^{(2)}_{n_1 n_2 n_3} := \hat{\mu}^{4\epsilon} \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \, \frac{1}{[k_1^2 - M^2]^{n_1} [k_2^2 - M^2]^{n_2} [(k_1+k_2)^2 - M^2]^{n_3}}$$