A tale of tails via QCD insights

Alex Edison

with

Michèle Levi

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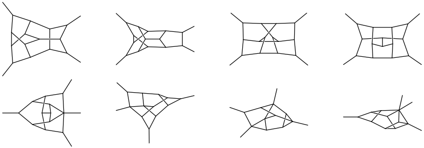
arXiv:2310.20066

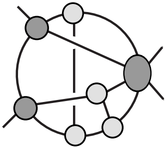
PIKIMO, May 04 2024

Northwestern

Who am I?

- UCLA The Mani L. Bhaumik Institute for Theoretical Physics → UPPSALA UNIVERSITET → Northwestern
- Expertise: multi-loop methods in (supersymmetric) Yang–Mills, (super)gravity





- Current interests: applications in BBH/GW physics, EFTofLSS

TL;DR

New approach to GW tails (scale-crossing interference)

- Bringing together insights from particle physics, dissipative systems, GR EFT
- Discovering hidden patterns and iterations
- Surpassing traditional GR methods.
  Them: T in 1988, T^3 in 2017.
  Us: T-T^3 in 2022, T^4 in 2023.

BBH in post-Newtonian approximation

Blanchet *Living Rev.Rel.* 17 (2014) 2

Assumption: objects are *slow-moving* and *weakly interacting*.
Good approximation for quasi-circular inspiral.

1. Classical effective action – multipoles coupled to gravity:

$$S_{\mathrm{GR+matter}} = \frac{1}{16\pi G}\int d^4x\sqrt{g}\,R + S_{\mathrm{mp}}$$

$$S_{\mathrm{mp}} = \int dt\sqrt{g}\left[E(t) - \sum_{l=2}^{\infty}\frac{1}{l!}I^L(t)\nabla_{L-2}\mathcal{E}_{i_{l-1}i_l} + \ldots\right]$$

2. GR is non-linear ⇒ radiation and potential modes can interfere
3. Goal: "Integrate out" gravitational field
   - Iteratively solve EoM DiffEqs
   - Use Feynman diagrams
   - Use modern unitarity methods

End result: perturbative effective action for evolution of mutlipoles

$$S_{\mathrm{tails}} = \int d\omega f(\omega)\underbrace{I^{ij}(\omega)I_{ij}(-\omega)}_{\kappa(\omega)}$$

Radiated energy via modified action principles (Schwinger-Keldysh)

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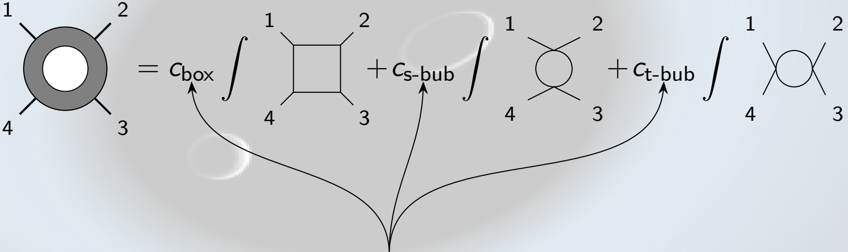
Radiated energy via modified action principles (Schwinger-Keldysh)

Unitarity: loops from trees

Two imporant facts:

- Feynman integrals (including numerators) can be reduced to a basis of *scalar* integrals using integration-by-parts relations (IBPs)
  EX: 4-point massless one-loop

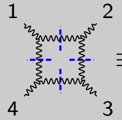
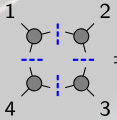


Rational functions of D and external data

Unitarity: loops from trees

- Basis coefficients c_X can be determined by matching *generalized unitarity cuts*, constructed via repeated application of the QFT optical theorem

$$\mathrm{Cut}_G = \sum_{\substack{\text{states}\\ \text{of } E(G)}} \prod_{v \in V(G)} A_{\text{tree}}(v)$$

$\mathrm{Cut}_{\text{box}} =$  $\equiv \sum_{\text{states}}$  $\xrightarrow{IBPs} c_{\text{box}}$

$$\sum_{\text{states}} \varepsilon_k^{\mu\nu} \bar{\varepsilon}_k^{\alpha\beta} = \mathcal{P}^{\mu\nu;\alpha\beta} = \frac{1}{2}\left(P_k^{\mu\alpha} P_k^{\nu\beta} + P_k^{\mu\beta} P_k^{\nu\alpha} - \frac{2}{D-2} P_k^{\mu\nu} P_k^{\alpha\beta}\right)$$

$$P_k^{\mu\nu} = \eta^{\mu\nu} - \frac{k^\mu q^\nu + k^\nu q^\mu}{k \cdot q}$$

Gauge invariance $\Leftrightarrow$ Cut is independent of q

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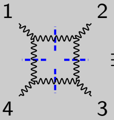
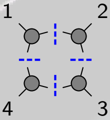


$\equiv \sum_{\text{states}}$



$\xRightarrow{\text{IBPs}} c_{\text{box}}$

$$\sum_{\text{states}} \varepsilon_k^{\mu\nu} \varepsilon_k^{\alpha\beta} \equiv \mathcal{P}_k^{\mu\nu;\alpha\beta} = \frac{1}{2}\left(P_k^{\mu\alpha} P_k^{\nu\beta} + P_k^{\mu\beta} P_k^{\nu\alpha} - \frac{2}{D-2} P_k^{\mu\nu} P_k^{\alpha\beta}\right)$$

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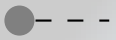
Building tails

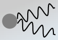
1. Identify basis integrals and corresponding cuts
2. Model quadrupole-gravity coupling as spin-1 particle coupled to gravity

$$\mathcal{M}_{Qg} \equiv \lambda_Q J_m^{\mu\nu} \varepsilon_{\mu\nu} = \quad I^{ij}$$  $$\varepsilon_{\mu\nu}$$

$$\sim (\omega_g k_g^i \varepsilon_g^0 \varepsilon_g^j + \omega_g k_g^j \varepsilon_g^0 \varepsilon_g^i - k_g^i k_g^j \varepsilon_g^0 \varepsilon_g^0 - \omega_g^2 \varepsilon_g^i \varepsilon_g^j)$$

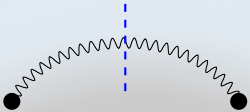
3. Similarly, model potential mode sources as massive spin-0 coupled to gravity

$$\mathcal{M}_{Mg} = \mathcal{M}_{sgs} = \frac{\lambda_E}{m_s^2} p_s^\mu p_s^\nu \varepsilon_{\mu\nu} = \quad M$$  $$\varepsilon_{\mu\nu}$$

$$\mathcal{M}_{Mgg} = \lim_{m_S \to \infty} \mathcal{M}_{sggs} = \quad M$$  $$\varepsilon_1^{\mu\nu} \quad \varepsilon_2^{\rho\sigma}$$

4. Sew sources together with bulk graviton amplitudes
5. Integrate over *spatial* graviton momenta (energy is “external” input)

Radiation reaction: Unitarity



Basis integral: $$F^{(1)}(1;\omega^2) = \int \frac{\mathrm{d}^d \ell_E}{(2\pi)^d} \frac{1}{(-\ell_E^2 + \omega^2)} = -\frac{\Gamma(1-d/2)(-\omega^2)^{d/2-1}}{(4\pi)^{d/2}}$$

Effective Action: $$S_{\mathrm{RR}} = \int \frac{\mathrm{d}\omega}{2\pi} c_{\mathrm{RR}} F^{(1)}(1;\omega^2)$$

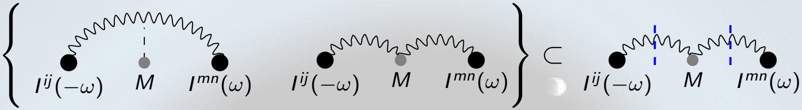
$$\mathrm{Cut}_{\mathrm{RR}} = \lambda_Q^2 J_1^{\mu\nu}\ \mathcal{P}^{\mu\nu;\alpha\beta} J_2^{\alpha\beta} \delta(\ell_E^2 - \omega^2) = \delta(P_\ell)\lambda_Q^2 \frac{(d+1)(d-2)}{(d+2)(d-1)} \omega^4 \underbrace{\kappa_{ab}(\omega)}_{I_a^{ij}(-\omega) I_{ij,b}(\omega)}$$

Effective action (after CTP sum):

Reproduces Einstein quadrupole $E \sim I^{(3)} I^{(3)}$

$$S_{\mathrm{RR}} = -i \frac{G_N}{5} \int \frac{\mathrm{d}\omega}{2\pi} \omega^5 \kappa_{-+}(\omega)$$

Leading tail via unitarity



Basis integral:

$$F^{(2)}(1,1,0) \equiv \int \frac{\mathrm{d}^d\ell_1\,\mathrm{d}^d\ell_2}{(2\pi)^{2d}} \frac{1}{(-\ell_1^2+\omega^2)(-\ell_2^2+\omega^2)} = F^{(1)}(1;\omega^2)^2$$

Cut that we need to calculate

$$\begin{aligned}\mathrm{Cut}_{\mathrm{tail}} &= \sum_{\mathrm{states}} \mathcal{M}_{Qg(-\omega)}\mathcal{M}_{sggs}\mathcal{M}_{Qg(\omega)}\Big|_{\substack{P_{\ell_1}=0,P_{\ell_2}=0\\ m_s\to\infty}} \\ &= \lambda_Q^2\,\delta(P_{\ell_1})\delta(P_{\ell_2}) J^{\mu\nu}_{I(-\omega)} P^{\mu\nu;\alpha\beta}\mathcal{M}_{sggs}^{\alpha\beta;\gamma\sigma} P^{\gamma\sigma;\rho\tau} J^{\rho\tau}_{I(\omega)}\Big|_{m_s\to\infty}\end{aligned}$$

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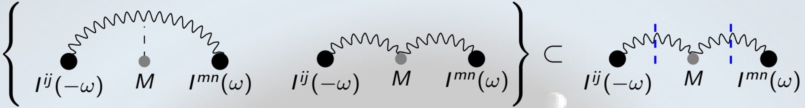

After integral reduction, CTP sum, and DimReg ($d \to 3 + \epsilon_d$)

$$S_{\mathrm{T}} = \frac{2}{5} G_N^2 E \int \frac{\mathrm{d}\omega}{2\pi}\, \omega^6 \kappa_{-+}(\omega) \left[\frac{1}{\epsilon_d} + \log\left(\frac{\omega^2}{\mu_T^2}\right) - i\pi\, \mathrm{sgn}(\omega) \right],$$

Goldberger & Ross 2009, Galley et.al. 2016

Keep this in mind for later:

$$S_{\mathrm{T}} = \int \frac{\mathrm{d}\omega}{2\pi} \kappa_{ab}(\omega)\, \omega^4 \frac{(2\pi G_N)(d+1)(d-2)}{(d+2)(d-1)}$$

$$\left(-(16\pi G_N E) \frac{12 - 2d + 5d^2 - 4d^3 + d^4}{2(d-3)(d-1)d(d+1)} \right) F^{(1)}(1; \omega^2$$

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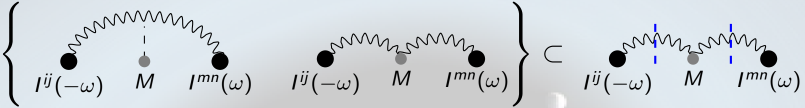

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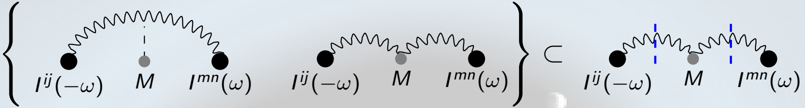
← Repeated from RR

Leading tail via unitarity



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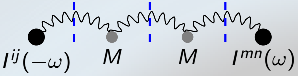
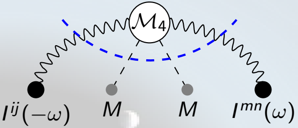
New in T $\longrightarrow$ $$\boxed{\left(-(16\pi G_N E) \frac{12 - 2d + 5d^2 - 4d^3 + d^4}{2(d-3)(d-1)d(d+1)} \right)} F^{(1)}(1; \omega_{ab}^2)^2$$

Tail-of-tail





vs $\sim$ 5 Feynman diagrams

Related to Goldberger & Ross 2009

$$S_{\text{TT}} = \frac{107}{175}\, G_N^3 E^2 \int \frac{\mathrm{d}\omega}{2\pi}\, \omega^7 \kappa_{-+}(\omega) \left[\pi\, \text{sgn}(\omega) + i \left[\frac{2}{3\,\epsilon_d} + \log\left(\frac{\omega^2}{\mu_{TT}^2} \right) \right] \right]$$

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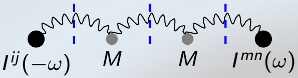

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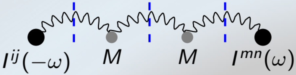
Repeated from RR

Tail-of-tail



vs $\sim$ 5 Feynman diagrams

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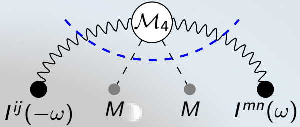
Iteration of T! $\longrightarrow$

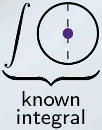
$$\boxed{\left(-(16\pi G_N E) \frac{12 - 2d + 5d^2 - 4d^3 + d^4}{2(d-3)(d-1)d(d+1)} \right)^2} F^{(1)}(1; \omega_{ab}^2)^3$$

Tail-of-tail



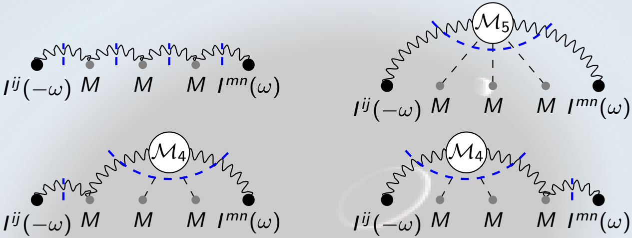
vs $\sim$ 5 Feynman diagrams

$$S_{\mathrm{TT}}^{\mathrm{right}} \sim \int \frac{\mathrm{d}\omega}{2\pi} \kappa_{ab}(\omega) \frac{\text{deg 9 poly in } d}{(d-3)\ldots} \int$$



Tail-of-tail-of-tail

Never approached via EFT – GR via Blanchet 2017

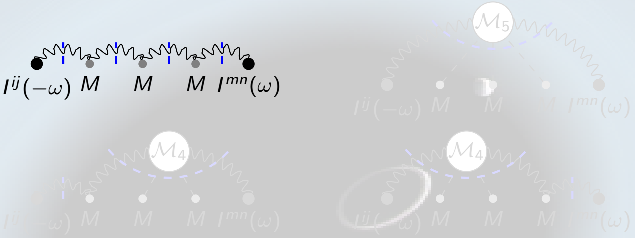


vs many Feynman diagrams

$$S_{\rm TTT} = -\frac{4}{525} G_N^4 E^3 \int \frac{\mathrm{d}\omega}{2\pi}\,\omega^8 \kappa_{-+}(\omega)\left[\frac{107}{2\epsilon_d^2} + \frac{107}{\epsilon_d}\log\left(\frac{\omega^2}{\mu_{TTT}^2}\right) + 107\log^2\left(\frac{\omega^2}{\mu_{TTT}^2}\right)\right.$$
$$\left. + \frac{20707426967}{60399360} - \frac{3103}{4}\zeta_2 - 420\,\zeta_3 - i\pi\,\mathrm{sgn}(\omega)\left[\frac{107}{\epsilon_d} + 214\log\left(\frac{\omega^2}{\mu_{TTT}^2}\right)\right]\right],$$

Tail-of-tail-of-tail

Never approached via EFT – GR via Blanchet 2017



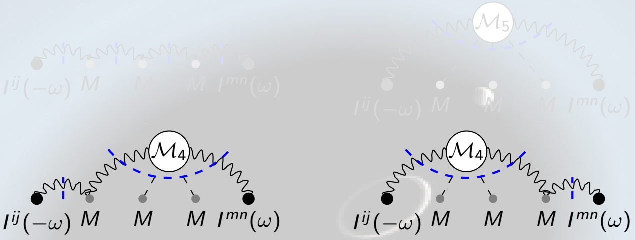
vs many Feynman diagrams

$$S_{\text{TTT}}^{\text{top-left}} = \int \frac{\mathrm{d}\omega}{2\pi} \frac{(2\pi G_N)(d+1)(d-2)}{(d+2)(d-1)} \omega^4 \kappa_{ab}(\omega) \longleftarrow \text{Repeated from RR}$$

$$\text{Iteration of T!} \longrightarrow \left(-(16\pi G_N E) \frac{12 - 2d + 5d^2 - 4d^3 + d^4}{2(d-3)(d-1)d(d+1)} \right)^3 F^{(1)}(1;\omega_{ab}^2)^4$$

Tail-of-tail-of-tail

Never approached via EFT – GR via Blanchet 2017



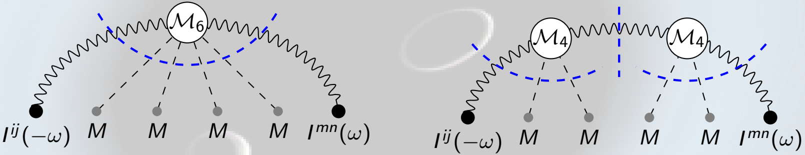
vs many Feynman diagrams

$$S_{\mathrm{TTT}}^{\text{bottom}} = \int \frac{\mathrm{d}\omega}{2\pi} \,(\text{term from } \mathcal{M}_4 \text{ including integral})$$

$$\times \left(-(16\pi G_N E) \frac{12 - 2d + 5d^2 - 4d^3 + d^4}{2(d-3)(d-1)d(d+1)} \right) F^{(1)}(1; \omega_{ab}^2)$$

Tail-of-tail-of-tail-of-tail

Never before computed in generality!
Mostly predicted by iteration, need two new diagrams.



Tail-of-tail-of-tail-of-tail

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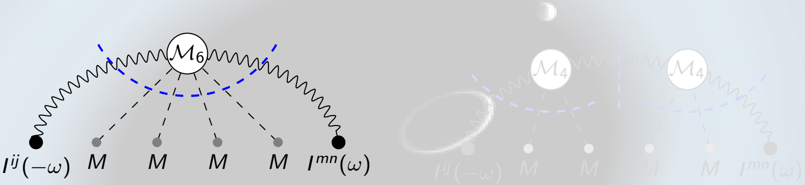

$$S^{\text{left}}_{\mathrm{T}^4} \sim \int \frac{\mathrm{d}\omega}{2\pi} \kappa_{ab}(\omega) \text{ (ratio of deg 22 polys in } d\text{)} \int$$ 

known integral

Tail-of-tail-of-tail-of-tail

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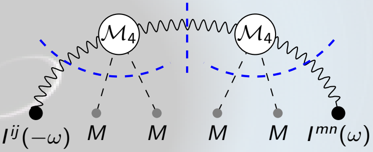


$$S^{\text{right}}_{T^4} \sim \int \frac{d\omega}{2\pi} \kappa_{ab}(\omega)\ (\mathcal{M}_4 \text{ term})^2 \int$$

numerical integral

Conclusions

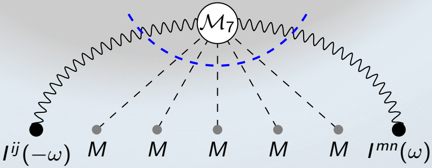
New approach to GW tails!

- Established new state of the art
- Exploited efficient methods from particle physics in new context
- Found interesting structure, hints of more

Future directions:

- Search for more structure
- Translate energy losses to orbit shifts
- Explore BSM contributions?

Thanks for your attention!



Backup slides

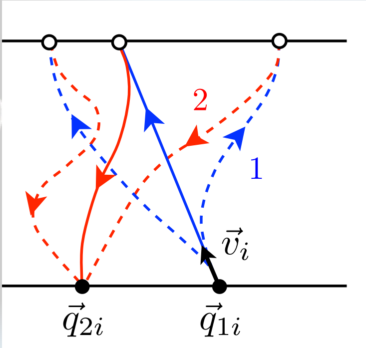
Dissipative effects via Closed Time Path

Galley et al. 1210.2745 1412.3082

Closed Time Path: action/variation principles for systematically handling non-conservative systems (classical implementation of Schwinger/Keldysh)

1. Double all degrees of freedom $(-,+)$, "causal" $(- \to +)$ and "anti-causal" $(+ \to -)$ branches
2. "Integrate out" inaccessible DoF
3. Conservative piece, L: $(-,+)$ symmetric; Non-conservative, K: asymmetric
4. Calculus of variations with "$-$" variables

Image from Galley 1210.2745



NB: Enters momentum-space calculations by changing $i0$ prescription, analytic continuations

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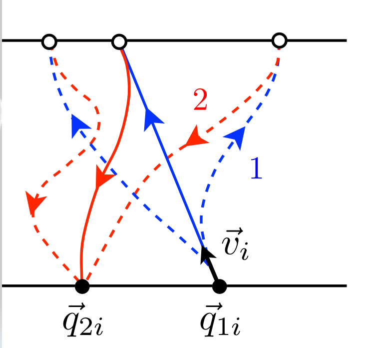

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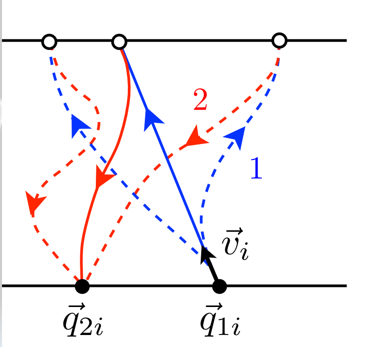

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Extracting energy spectra from CTP actions

CTP extension to Noether theorem:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{\partial L}{\partial t} + \dot{q}^J \left[\frac{\partial K}{\partial q^J_-}\right]_{\mathsf{PL}} + \ddot{q}^J \left[\frac{\partial K}{\partial \dot{q}^J_-}\right]_{\mathsf{PL}} + \frac{\partial}{\partial \begin{smallmatrix}\text{higher}\\ \text{time}\\ \text{derivs}\end{smallmatrix}}$$

$$\int \frac{\mathrm{d}E}{\mathrm{d}t}\,\mathrm{d}t = \Delta E = \int \frac{\mathrm{d}E}{\mathrm{d}\omega}\,\mathrm{d}\omega = \int \mathrm{d}\omega\,(i\omega) q^J(\omega) \left[\frac{\partial K}{\partial q^J_-(\omega)}\right]_{\mathsf{PL}}$$

Remove symmetric in $(-,+)$

Generic tail actions:

$$S_{\mathsf{T}^x} = \int \mathrm{d}\omega f(\omega) I^{ij}_-(-\omega) I_{ij,+}(\omega)$$

$$\Rightarrow \Delta E = \int \mathrm{d}\omega \kappa(\omega)\omega \,\mathrm{Im}\, f_{\mathrm{odd}}(\omega)$$

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Raw energy spectra

$$(\Delta E)_{\mathrm{RR}} = -\frac{G_N}{5\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega \kappa(\omega)\omega^6 \left\{1 - \frac{\epsilon_d}{2}\left[\frac{9}{10} - \log\left(\frac{\omega^2 e^{\gamma_E}}{\mu^2\pi}\right)\right] + \dots\right\}$$

$$(\Delta E)_{\mathrm{T}} = -\frac{2}{5} G_N^2 E \int_{-\infty}^{\infty} \mathrm{d}\omega \kappa(\omega)\omega^7 \left\{1 + \epsilon_d\left[\log\left(\frac{\omega^2 e^{\gamma_E}}{\mu^2\pi}\right) - \frac{41}{30}\right] + \dots\right\}$$

$$(\Delta E)_{\mathrm{TT}} = \frac{214 G_N^3 E^2}{525\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega \kappa(\omega)\omega^8 \left\{\frac{1}{\epsilon_d} + \left[\frac{3}{2}\log\left(\frac{\omega^2 e^{\gamma_E}}{\mu^2\pi}\right) - \frac{420\zeta_2}{107} - \frac{675359}{89880}\right]\right\}$$

$$(\Delta E)_{\mathrm{TTT}} = \frac{428}{525} G_N^4 E^3 \int_{-\infty}^{\infty} \mathrm{d}\omega \kappa(\omega)\omega^9 \left\{\frac{1}{\epsilon_d} + \left[2\log\left(\frac{\omega^2 e^{\gamma_E}}{\mu^2\pi}\right) - \frac{252583}{29960}\right]\right\}$$

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$$\left. + \left[\frac{5}{2}\log\left(\frac{\omega^2 e^{\gamma_E}}{\mu^2\pi}\right)\left(\frac{5}{4}\log\left(\frac{\omega^2 e^{\gamma_E}}{\mu^2\pi}\right) - \frac{840}{107}\zeta_2 - \frac{5249287}{343470}\right) + \zeta_2, \zeta_3, \zeta_4, \text{rational}\right]\right\}$$

Need to deal with the DimReg divergences: renormalize quadrupole coupling

$$\kappa(\omega) \to \kappa(\omega,\mu)\left(1 + 2\left(\frac{107}{105}\frac{(G_N E\omega)^2}{\epsilon_d} + \frac{107^2}{105^2}\frac{(G_N E\omega)^4}{\epsilon_d^2}\right) + \frac{1695233}{105^3}\frac{(G_N E \dots}{\dots} + \dots\right)$$

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The renormalized energy spectrum and RG flow

Einstein Term

Blanchet & Damour Tail

$$(\Delta E)^{\rm inc}_{\rm T^4} = \int {\rm d}\omega \kappa(\omega,\mu)\Bigg[-\frac{\omega^6 G_N}{5\pi} - \frac{2}{5}\omega^7 G_N^2 E$$

$$+\frac{1}{\pi}G_N^3 E^2 \omega^8\left(\frac{214}{525}\log\left(\frac{\omega^2 e^{\gamma_E}}{\mu^2\pi}\right) - \frac{634913}{220500} - \frac{8\zeta_2}{5}\right)$$

Goldberger & Ross 2009, Bini & Geralico 2021 (FT of Blanchet 1997)

$$+G_N^4 E^3 \omega^9\left(\frac{428}{525}\log\left(\frac{\omega^2 e^{\gamma_E}}{\mu^2\pi}\right) - \frac{634913}{110250}\right)$$

New for generic quadrupole!

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$$+\frac{G_N^5 E^4}{5\pi}\omega^{10}\Bigg\{32\zeta_4 + 2^4\frac{107}{105}\zeta_3 - \frac{1132438}{105^2}\zeta_2 - \frac{275977249}{1944810}$$

$$+\log\left(\frac{\omega^2 e^{\gamma_E}}{\mu^2\pi}\right)\left[\frac{8301847}{257250} + 2^4\frac{107}{105}\zeta_2 - 2\frac{107^2}{105^2}\log\left(\frac{\omega^2 e^{\gamma_E}}{\mu^2\pi}\right)\right]\Bigg\}\Bigg]$$

Logs skip orders!

Renormalized coupling scale dependence to balance log scaling

$$\frac{\rm d}{{\rm d}\mu}(\Delta E)^{\rm inc}_{\rm T^4} = 0 \Rightarrow \frac{{\rm d}\ \kappa(\omega,\mu)}{{\rm d}\log\mu} = -(2G_N E\omega)^2\kappa(\omega,\mu)\left(\frac{107}{105} + \frac{1695233}{105^3}(G_N E\omega)^2\right)$$

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Renormalized coupling includes a scale dependence to balance log scaling

$$\frac{\mathrm{d}}{\mathrm{d}\mu}(\Delta E)^{\text{inc}}_{\text{T}^4} = 0 \Rightarrow \frac{\mathrm{d}\,\kappa(\omega,\mu)}{\mathrm{d}\log\mu} = -(2G_N E\omega)^2\kappa(\omega,\mu)\left(\frac{107}{105} + \frac{1695233}{105^3}(G_N E\omega)^2 + \mathcal{O}(G_N^4)\right)$$