Topology in particle production: Applications to early universe cosmology

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Work with D.J.H Chung

Introduction: Particle Production

1. Particle production in cosmology is sourced by an expanding space-time geometry. The dynamics of this process is similar to the following notable examples.
2. **Schwinger pair production: (1951)** e^+e^- pairs produced under strong a electric field -'conducting vacuum'.
3. **Hawking radiation: (1974)** particle anti-particle pairs created near the horizon can extract energy from the black hole and radiate real particles outside the horizon -'black holes evaporate'.
4. Particle production during the expansion of the universe may even explain the dark matter abundance today.[rf. D.J.H.Chung -1998]

Preview

Analogous to the anomalous current being sourced by topology i.e.

$$\partial_\mu J^\mu \sim cF\tilde{F}$$

particle production can be understood as the current associated with particle number being sourced by the topology of asymptotic expansions.

Particle Production

1. Interested in particle production due to breaking of time translation invariance.
2. Consider a scalar field in flat space-time coupled to a time dependent background field

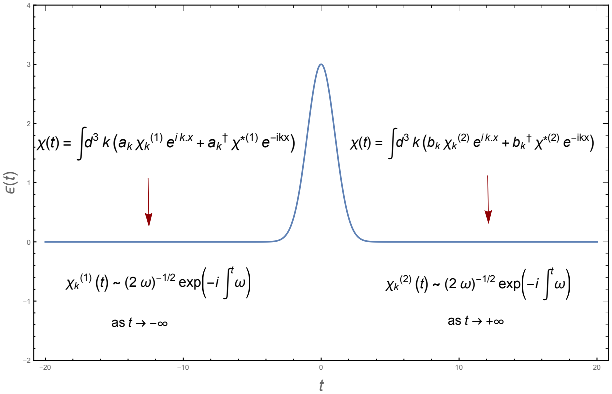
$$S = \int \mathrm{d}^4x \left(\partial_\mu \chi \partial^\mu \chi + g\phi^2(t)\chi^2\right) \quad \text{with} \;\; \mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}\mathbf{x}^2$$

3. Quantizing on the background

$$\hat{\chi} = \int \mathrm{d}^3k \left[a_k \chi_k(t) e^{ik.x} + a_k^\dagger \chi_k^*(t) e^{-ikx}\right] \quad \text{where} \;\; \partial_t^2 \chi_k + \left[k^2 + g\phi^2(t)\right]\chi_k = 0 \;\&\, \text{B.C}$$

   To every $\chi_k(t)$ corresponds to a notion of vacuum defined as $a_k \left|0\right\rangle = 0$.
4. Define a measure for breaking of time translational invariance

$$\epsilon(t) = \frac{\partial_t \omega(t)}{\omega^2(t)} = \frac{\phi(t)\partial_t\phi(t)}{\left(k^2 + g\phi^2(t)\right)^{3/2}} \quad \text{where} \;\; \omega^2(t) = k^2 + g\phi^2(t).$$



1. Since $\chi_k^{(1)}, \chi_k^{(1)*}$ and $\chi_k^{(2)}, \chi_k^{(2)*}$ are two sets of independent solutions of the mode equation

$$\chi_k^{(1)} = \alpha_k \chi_k^{(2)} + \beta_k \chi_k^{*(2)} \implies a_k = \alpha_k^* b_k - \beta_k b_k^\dagger$$
$$N_k = \langle 0| \hat{N}_k |0\rangle = \langle 0| a_k^\dagger a_k |0\rangle = \|\beta_k\|^2$$

Bogoliubov Transformation Method

1. Canonical transformation to 'coefficients of the WKB modes' — Conformal time η

$$\chi_k(\eta) = \alpha_k(\eta) f_-(\eta) + \beta_k(\eta) f_+(\eta); \quad \partial_\eta \chi_k(\eta) = i\,\omega(\eta)\left[\beta_k(\eta) f_+(\eta) - \alpha_k(\eta) f_-(\eta)\right];$$

$$\text{where} \quad f_\pm(\eta) = (2\omega(\eta))^{-1/2} exp\left(\pm i \int_{\eta_0}^{\eta} \mathrm{d}\eta' \omega(\eta')\right)$$

2. Mode equation re-written

$$\partial_\eta \begin{bmatrix} \alpha_k(\eta) \\ \beta_k(\eta) \end{bmatrix} = \underbrace{\frac{\epsilon(\eta)\omega(\eta)}{2} \begin{bmatrix} 0 & e^{+2i\int_{\eta_0}^{\eta_1} \omega} \\ e^{-2i\int_{\eta_0}^{\eta_1} \omega} & 0 \end{bmatrix}}_{\mathbf{M}(\eta)} \begin{bmatrix} \alpha_k(\eta) \\ \beta_k(\eta) \end{bmatrix} ; \qquad \epsilon(\eta) = \frac{\partial_\eta \omega(\eta)}{\omega^2(\eta)}$$

Standard approximation scheme

1. For $(\alpha_{-\infty}, \beta_{-\infty}) = (1, 0)$

$$\alpha(\eta) \approx 1,\ |\beta(\eta)| \ll 1 \implies \beta_{+\infty} \approx \int_{-\infty}^{+\infty} \mathrm{d}\bar{\eta}\, \frac{\omega'}{2\omega^2} e^{-2i\int_{\eta_0}^{\bar{\eta}} \omega}$$

Integral is estimated after contour deformation along steepest descent curves in $\mathbb{C}$. These pass through stationary points η_s

$$\partial_\eta \left[-2i \int_{\eta_0}^{\eta} \omega\right]_{|\eta_s} = 0 \implies \omega^2(\eta_s) = 0$$

Zeroes of $\omega^2(\eta)$

2. Steepest descent approximation only valid for well separated zeroes.
3. S. Enomoto, T. Matsuda (2020) use **Stokes phenomenon** to compute $\|\beta_{+\infty}\|^2$ from global analytic properties of $\omega^2(\eta)$ for well separated zeroes.
   (*rf. N.Froman, O.Fromann - 1965; E.W.Kolb, A.J.Long -2023; S.Hashiba, Y. Yamada -2021*)
4. We extend this work using Stokes phenomenon combined with symmetries [rf. N.Froman and O.Froman] to expose topological nature of the $\|\beta\|^2$ in the limit $k \to 0$.

Stokes Phenomenon

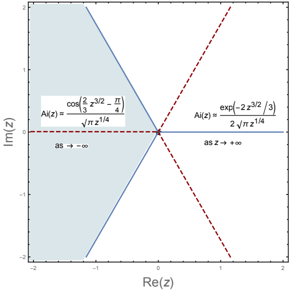
1. Given a Schroedinger-like differential equation

$$\psi''(z) + \omega^2(z)\psi(z) = 0$$

express solutions in terms of the WKB modes

$$f_{\pm}(z) = \frac{exp\left[\pm i \int_0^z \mathrm{d}\bar{z}\, \omega(\bar{z})\right]}{\sqrt{2\omega(z)}}$$

$\omega^2(z) = -z$ Airy Functions!



$$\lim_{z\to+\infty} Ai(z) = 0; \lim_{z\to+\infty} Ai'(z) = 0$$

"Given an exact solution to a complex Schrodinger-like differential equation, it's WKB series jumps discretely over boundaries in the complex plane called Stokes lines."

1. $\psi(z)$ is given by the asymptotic series

$$Ai(z) \approx \frac{exp\left(-2z^{3/2}/3\right)}{2\sqrt{\pi}z^{1/4}}\left(1 - \frac{u_1}{z^{3/2}} + \frac{u_2}{z^3} + ...\right) + Exp.\ \ supp.\ \ trms \quad \text{(Region I)}$$

$$Ai(z) \approx \frac{\cos\left(z^{3/2} - \frac{\pi}{4}\right)}{\sqrt{\pi}z^{1/4}}\left(1 - \frac{u_2}{z^3} + ...\right) + \frac{\sin\left(z^{3/2} - \frac{\pi}{4}\right)}{\sqrt{\pi}z^{1/4}}\left(\frac{u_1}{z^{3/2}} + ...\right) + Exp.\ \ supp.\ \ trms \quad \text{(Region II)}$$

2. The exponentially suppressed terms in Region I(II) grow to become significant in Region II(I). Transitions 'almost discontinuously' - hence 'jumps'.

Parametrising these jumps

1. **How 'n' When?** Move across contours along which

$$Im\left[i\int_0^z \omega\right] = 0 \quad \text{'Stokes lines'}$$

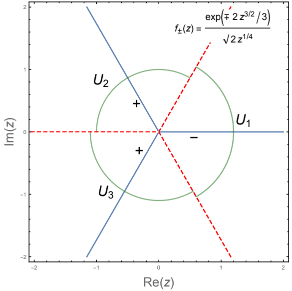
Transformation

$$f_+ \to f_+ + S f_-, \quad f_- \to f_- \quad (+\text{ SL})$$
$$f_- \to f_- + S f_+, \quad f_+ \to f_+ \quad (\text{ - SL})$$

2. **Into matrices:** Transformations of vector $(\alpha, \beta)^T$

$$U_1 \approx \begin{bmatrix} 1 & 0 \\ S & 1 \end{bmatrix};\ U_2 \approx \begin{bmatrix} 1 & S' \\ 0 & 1 \end{bmatrix};\ U_3 \approx \begin{bmatrix} 1 & S'' \\ 0 & 1 \end{bmatrix}$$

Q: What are the red dashed lines?



$$\lim_{|z|\to\infty} \exp\left[i\int_0^z \omega\right] \to \infty \quad \text{'(+) Stokes line'}$$

$$\lim_{|z|\to\infty} \exp\left[-i\int_0^z \omega\right] \to \infty \quad \text{'(-) Stokes line'}$$

Relation to topology?

1. **Symmetries:**
   For $\omega^2(z) = Az^n$, mode equation is symmetric under $z \to \exp\left\{\frac{2\pi i}{n+2}\right\} z$

   $\mathbb{Z}_{n+2}$ group

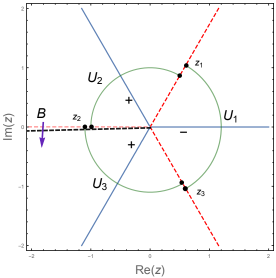
$$\implies U_+^T = U_- \approx \begin{bmatrix} 1 & 0 \\ S_n & 1 \end{bmatrix}$$

U_- across all (-) SL, U_+ across all (+) SL !!

2. **Single valuedness:**
   $\psi(z)$ single valued on $\mathbb{C} \implies (\alpha, \beta)^T$ transforms non-trivially across branch cut

$$\implies U_1 \cdot U_2 \ldots U_{n+2} \cdot B_n = \mathbb{I}_{2\times 2}$$

Fixes $\mathrm{S}_n = 2i \cos\left(\frac{\pi}{n+2}\right)$



$$f_\pm(z) = \frac{\exp\left\{\pm i \frac{2A}{n+2} z^{\frac{n+2}{2}}\right\}}{\sqrt{2}A^{1/4} z^{n/4}}$$

$$F(z) = \left(f_+(z), f_-(z)\right)$$

$$F(z \exp\{2\pi i\}) = F(z) \cdot B_n^{-1}$$

Topological!

1. For $n \in$ even, $\omega^2(z) \geq 0$ for $z \in \mathbb{R}$. Combining connection matrices from $\mathbb{R}_-$ to $\mathbb{R}_+$

$$\beta\left(z_{+\infty}\right) = \cot\left[\frac{\pi}{n+2}\right]$$

for boundary condition $\left(\alpha\left(z_{-\infty}\right), \beta\left(z_{-\infty}\right)\right) = (1, 0)$. **Topological! Counts the no.of Stokes lines**

2. Suprising? May be re-derived in terms of Wronskian identities of Bessel functions - the topological nature may be attributed to scale invariance of the Wronskian.

Extending to realistic dispersion relations

1. More realistic dispersion relation

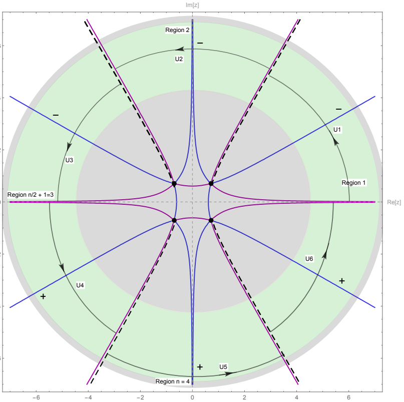
$$\psi''(z) + \left(\bar{k}^2 + z^n\right)\psi(z) = 0$$
$$\omega^2(z) = \bar{k}^2 + z^n$$

2. **Symmetry:** $z \to \gamma z, \bar{k} \to \gamma^{-1}\bar{k}$ with $\gamma = \exp\left\{\frac{2\pi i}{n+2}\right\}$ ($\mathbb{Z}_{n+2}$ symmetry)

$$\implies S_j(\bar{k}) = S_1(\gamma^j \bar{k})$$

Analyticity: Stokes constants are analytic functions for $\bar{k}^{\frac{n+2}{n}}$

$$\implies S_1(\bar{k}) = \sum_{i=0}^{\infty} c_i \left(\bar{k}^{\frac{n+2}{n}}\right)^i$$



$n = 4$

1. **Single valuedness** fixes first few coefficients

Q: Why aren't all coefficients determinable?

$$\textbf{For } n=4: \quad S_1\left(\bar{k}\right) = i\sqrt{3} + \frac{2\Gamma^2\left(\frac{1}{4}\right)}{3\sqrt{3\pi}}\bar{k}^{3/2} + \ldots \qquad c_{i\geq 2} \text{ not determinable}$$

$$\textbf{For } n=6: \quad S_1\left(\bar{k}\right) = 2i\cos\left(\frac{\pi}{8}\right) - \frac{4\left(1+\sqrt{2}\right)\sqrt{\pi}\Gamma\left[\frac{7}{6}\right]\sec\left(\frac{\pi}{8}\right)}{\left(3i(1+\sqrt{2})+\sqrt{3}\left(3+\sqrt{2}\right)\right)\Gamma\left[\frac{5}{3}\right]}\bar{k}^{4/3}$$
$$+ \frac{2\left(8038-5233\sqrt{2}+16374\sqrt{3}i-4909\sqrt{6}i\right)\pi\Gamma^2\left[\frac{7}{6}\right]}{147\Gamma^2\left[\frac{5}{3}\right]}\bar{k}^{8/3} + \ldots$$

$c_{i\geq 3}$ not determinable

and so on...

2. Corrections define the scale k_{topo} such that

$$\|\beta_k\|^2 \approx \cot^2\left[\frac{\pi}{n+2}\right] \qquad \forall\; k \lesssim k_{topo}$$

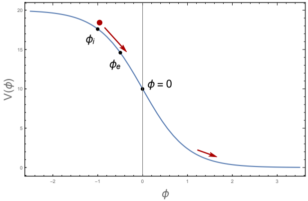
Model

1. Consider a spectator scalar field ϕ rolling on

$$V(\phi) = \rho_0 \left[1 - \tanh\left(\phi/M\right)\right]$$

coupled to dark matter field χ as

$$\mathcal{L} \supset \frac{g}{2\Lambda^2}\phi^4\chi^2$$



2. Field starts rolling from ϕ_i. At the end of inflation $\phi = \phi_e$ lies in linear region of potential.
3. Dispersion relation of χ-modes near non-adiabatic point $\phi(\eta_0) = 0$

$$\omega^2(\eta) = k^2 + \frac{g}{\Lambda^2}a^2(\eta)\phi^4(\eta) \approx k^2 + \frac{g}{\Lambda^2}a^2(\eta_0)\left(\phi'(\eta_0)\right)^4(\eta - \eta_0)^4$$

4. Parameters in the model

$$\bar{\rho} = \frac{\rho_0}{H_I^2 M^2} \approx 10^{-6}, \qquad \bar{g} = \frac{gM^4}{\Lambda^2 H_I^2} \approx 10^{-2}$$

1. Number density of dark matter particles n_χ may be estimated as

$$n_\chi = \int \frac{d^3k}{(2\pi)^3} |\beta_k|^2 \sim f k_{cutoff}^3, \qquad \text{with} \quad k_{cutoff} \approx a_e H_I \left(\bar{g}^{1/4} |c_2|\right)^{\frac{2}{3}} ; \; f \sim O(1)$$

2. The topological contribution is similarly

$$n_{\chi,topo} = \int^{k_{cut,topo}} \frac{d^3k}{(2\pi)^3} |\beta_k|^2$$

where $k_{cut,topo}$ is the smaller of
i) scale of validity of approximation

$$k_{\text{approx}} \sim O(10^{-4})\, k_{cutoff}$$

ii) topological scale inherent to $\bar{\omega}^2 = \bar{k}^2 + z^4$

$$k_{\text{topo}} \sim O(10^{-1})\, k_{cuttoff}$$

$$\implies n_{\chi,topo} \ll n_\chi$$

Summary

1. Analogous to $\partial_\mu J^\mu \sim cF\tilde{F}$, particle production may be understood as the current associated with particle number being sourced by the topology of the asymptotic expansion.
2. The topological nature of β also a consequence of the scale invariance of the Bessel function Wronskian.
3. Topology determines the first few corrections to the β coefficient in small k.
4. Physical models where the above analysis can be applied involve dispersion relations which pass through a zero on the real line.

Thank you!

Appendix 1:

Asymptotic Series

Consider two functions $f(x)$ and $g(x) = f(x) + e^{-\frac{1}{x^2}}$. Since the Taylor series as $x \to +\infty$ of the exponential term is

$$\lim_{x\to\infty} e^{-\frac{1}{x^2}} = 0 + 0.x^{-1} + 0.\frac{x^{-2}}{2} + ...$$

the functions $f(x)$ and $g(x)$ has the exact same asymptotic series.

Wronskian scale invariance

$$\nu = \frac{1}{n+2} \qquad \text{and} \qquad \xi = \frac{n+2}{2}$$

$$\beta = zW\left[J_{-\nu}\left(\frac{z^{\xi}}{\xi}\right), J_{\nu}\left(\frac{z^{\xi}}{\xi}\right)\right] = \frac{2\xi\sin\left(\pi\nu\right)}{\pi}$$

$$W\left[J_{-\nu}\left(\frac{z^{\xi}}{\xi}\right), J_{\nu}\left(\frac{z^{\xi}}{\xi}\right)\right] = J_{-\nu}\left(\frac{z^{\xi}}{\xi}\right)\partial_z J_{+\nu}\left(\frac{z^{\xi}}{\xi}\right) - J_{+\nu}\left(\frac{z^{\xi}}{\xi}\right)\partial_z J_{-\nu}\left(\frac{z^{\xi}}{\xi}\right) = \frac{2\xi\sin\left(\pi\nu\right)}{\pi z}$$

$$\frac{dW}{dz} = \frac{-1}{z}W \qquad (1)$$