EXPLORING THE INFRARED STRUCTURE OF MASSLESS GAUGE THEORIES

Lorenzo Magnea

University of Torino - INFN Torino

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Outline

• Infrared factorisation of scattering amplitudes
• The subtraction problem
• A celestial viewpoint
• Outlook
INFRARED FACTORISATION
Singularities arise only when propagators go on shell

\[(p + k)^2 = 2p \cdot k = 2E_p w_k (1 - \cos \theta_{pk}) = 0 \rightarrow w_k = 0 \text{ (soft)}; \quad \cos \theta_{pk} = 1 \text{ (collinear)}\]

- Emission is not suppressed at long distances
- Isolated charged particles are not true asymptotic states of unbroken gauge theories

- A serious problem: the $S$ matrix does not exist in the usual Fock space
- Possible solutions: construct finite transition probabilities (KLN theorem)
  construct better asymptotic states (coherent states)
- Long-distance singularities obey a pattern of exponentiation

\[A = A_0 \left[1 - \kappa \frac{\alpha}{\pi} \frac{1}{\epsilon} + \ldots\right] \quad \Rightarrow \quad A = A_0 \exp \left[-\kappa \frac{\alpha}{\pi} \frac{1}{\epsilon} + \ldots\right]\]
Soft-collinear factorisation
Soft-collinear factorisation

A gauge theory Feynman diagram with soft and collinear enhancements
**Soft-collinear factorisation**

- Divergences arise in scattering amplitudes from leading regions in loop momentum space.
- Potential singularities can be located using Landau equations.
- Actual singularities can be identified using power-counting techniques in the relevant regions.
- For renormalised massless theories only soft and collinear regions give divergences.
- Soft and collinear emissions have universal features, common to all hard processes.
- Ward identities can be used to prove decoupling of soft and collinear factors to all orders.
- A soft-collinear factorisation theorem for multi-particle matrix elements follows.
- Similar factorisation theorems hold for inclusive (soft and collinear safe) cross sections.

A gauge theory Feynman diagram with soft and collinear enhancements
The factorised amplitude
Infrared divergences in fixed-angle multi-particle scattering amplitudes factorise:

\[ \mathcal{A}_n \left( \frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon \right) = \mathcal{Z}_n \left( \frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon \right) \mathcal{F}_n \left( \frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon \right), \]

The infrared factor is a colour operator determined by a finite anomalous dimension matrix:

\[ \mathcal{Z}_n \left( \frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon \right) = \mathcal{P} \exp \left[ \frac{1}{2} \int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2} \Gamma_n \left( \frac{p_i}{\lambda}, \alpha_s(\lambda^2, \epsilon) \right) \right], \]

All infrared poles arise from the scale integration, through the d-dimensional running coupling:

\[ \lambda \frac{\partial \alpha_s}{\partial \lambda} \equiv \beta(\alpha_s, \epsilon) = -2\epsilon \alpha_s - \frac{\alpha_s^2}{2\pi} \sum_{k=0}^{\infty} \left( \frac{\alpha_s}{\pi} \right)^k b_k. \]

For massless theories, the all-order structure of the anomalous dimension in known, up to corrections due to higher-order Casimir operators of the gauge algebra:

\[ \Gamma_n \left( \frac{p_i}{\mu}, \alpha_s(\mu^2) \right) = \Gamma_n^{\text{dip}} \left( \frac{s_{ij}}{\mu^2}, \alpha_s(\mu^2) \right) + \Delta_n \left( \rho_{ijkl}, \alpha_s(\mu^2) \right), \]

\[ \rho_{ijkl} = \frac{p_i \cdot p_j p_k \cdot p_l}{p_i \cdot p_l p_j \cdot p_k} = \frac{s_{ij} s_{kl}}{s_{ij} s_{jk}}. \]
The amplitude can be expressed in a process-dependent orthonormal basis of colour tensors:

\[ A_n^{a_1 \ldots a_n} \left( \frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon \right) = \sum_L A_L^{a_1 \ldots a_n} \left( \frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon \right) c_L^{a_1 \ldots a_n}. \]

\[ \sum_{\{a_i\}} c_L^{a_1 \ldots a_n} (c_M^{a_1 \ldots a_n})^* = \delta_{LM}. \]

A simple example is quark-antiquark scattering, where colour space is two-dimensional.

The amplitude is a vector in colour space, to all perturbative orders:

\[ A_{abcd} = A_1 c_{abcd}^{(1)} + A_2 c_{abcd}^{(2)}, \quad c_{abcd}^{(1)} = \delta_{ac} \delta_{bd}, \quad c_{abcd}^{(2)} = \delta_{ab} \delta_{cd}. \]

The exchange of a virtual gluon will shuffle the colour components, even if the gluon is soft:

QED: \[ A_{\text{div}} = Z A_{\text{Born}}; \]

QCD: \[ [A_{\text{div}}]_J = [Z]_{JK} [A_{\text{Born}}]_K. \]
A powerful \textit{basis-independent} notation uses \textit{colour operators} `inserting' soft gluons

\[
\mathcal{A}_{n+1}^{a_1 \ldots a_n} \big|_{\text{soft}} \propto \sum_{i=1}^{n} \left[T_i^a\right]^{b_i}_{c_i} \mathcal{A}_n^{b_1 \ldots b_{i-1} c_i \ldots b_n},
\]

\textbf{Soft gluon operators} are \textit{generators} of the algebra in the \textit{representation} of the emitter

At \textit{leading power} in $k$:

For different \textit{emitters}:

\textbf{Colour operators} \textit{obey identities} inherited by the \textit{algebra} and dictated by \textit{gauge invariance}
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At leading power in \( k \):

\[ g\mu \epsilon \vec{u}_s(p_i) \gamma_\alpha \frac{\not{p}_i + \not{k}}{2p_i \cdot k} (T^c)_{c_i d_i} \hat{A}_{s_1 \ldots s_n}^{c_1 \ldots d_i \ldots c_n} \{p_j\}, k) \epsilon^*_\chi (k), \]

For different emitters:

\[ T_i \bigg|_{\text{q, out}} \rightarrow T^a_{cd}, \quad T_i \bigg|_{\text{q, out}} \rightarrow -T^a_{dc}, \quad T_i \bigg|_{\text{g, out}} \rightarrow -i f^a_{cd}, \]

Colour operators obey identities inherited by the algebra and dictated by gauge invariance.

\[ [T_i^a, T_i^b] = i f^a_{c e} T_i^c, \quad T_i \cdot T_i = T_i^a T_i^b \delta_{ab} = C_i^{(2)}, \quad \sum_{i=1}^{n} T_i = 0, \]
The dipole formula

Let's take a closer look at the structure of the infrared anomalous dimension matrix.

The dipole term:
\[
\Gamma_{n}^{\text{dip}} \left( \frac{s_{ij}}{\mu^2}, \alpha_s(\mu^2) \right) = \frac{1}{2} \hat{\gamma}_K \left( \alpha_s(\mu^2) \right) \sum_{i=1}^{n} \sum_{j=i+1}^{n} \log \left( \frac{s_{ij} e^{i\pi\lambda_{ij}}}{\mu^2} \right) T_i \cdot T_j + \sum_{i=1}^{n} \gamma_i \left( \alpha_s(\mu^2) \right),
\]

The cusp anomalous dimension in the `Casimir scaling' limit:
\[
\gamma_{K,r}(\alpha_s) = C_r^{(2)} \hat{\gamma}_K(\alpha_s),
\]

Corrections start at three loops, with quadrupoles:
\[
F_{ijkl}(\{\rho\}) f_{abe} f_{cd} e^{T^a_{i} T^b_{j} T^c_{k} T^d_{l}},
\]

- The colour dipole is the natural structure arising at one loop from gluon exchange.
- The fact that it survives at two loops is a non-trivial consequence of symmetries.
- Field anomalous dimensions in color-uncorrelated terms govern collinear singularities.
- Unitarity phases contain crucial analytic information. For final-state pairs: \( \lambda_{ij} = 1 \).
- The cusp anomalous dimension plays a very special role: a universal infrared coupling.
- The structure emerges from the constraints of scale invariance in the soft limit.
Infrared factorisation: pictorial

A pictorial representation of soft-collinear factorisation for fixed-angle scattering amplitudes

\[ A = S \times H \times (J \times n) \]
Here we introduced dimensionless four-velocities $\beta_i = \frac{p_i}{Q}$, and factorisation vectors $n_i^\mu$, $n_i^2 \neq 0$ to define the jets in a gauge-invariant way. For outgoing quarks

$$A_n\left(\frac{p_i}{\mu}\right) = \prod_{i=1}^{n} \left[ \frac{\mathcal{J}_i\left((p_i \cdot n_i)^2/(n_i^2 \mu^2)\right)}{\mathcal{J}_{E,i}\left((\beta_i \cdot n_i)^2/n_i^2\right)} \right] S_n(\beta_i \cdot \beta_j) \ H_n\left(\frac{p_i \cdot p_j}{\mu^2}, \frac{(p_i \cdot n_i)^2}{n_i^2 \mu^2}\right)$$

The precise functional form of this graphical factorisation is

$$\bar{u}_s(p) \mathcal{J}_q\left(\frac{(p \cdot n)^2}{n^2 \mu^2}\right) = \langle p, s | T \left[ \bar{\psi}(0) \Phi_n(0, \infty) \right] | 0 \rangle,$$

where $\Phi_n$ is the Wilson line operator along the direction $n$. For outgoing gluons

$$g_s \varepsilon^{*(\lambda)}_\mu (k) \mathcal{J}\nu^\lambda\mu\nu\left(\frac{(k \cdot n)^2}{n^2 \mu^2}\right) \equiv \langle k, \lambda | T\left[ \Phi_n(\infty, 0) iD^\nu \Phi_n(x, \infty) \right]_{x=0} | 0 \rangle.$$
The soft function $S$ is a color operator, mixing the available color tensors. It is defined by a correlator of Wilson lines.

$$S_n (\beta_i \cdot \beta_j) = \langle 0 | T \left[ \prod_{k=1}^{n} \Phi_{\beta_k}(\infty, 0) \right] |0\rangle,$$

The eikonal jet function $J_E$ contains soft-collinear poles: it is defined by replacing the field in the ordinary jet $J$ with a Wilson line in the appropriate color representation.

$$J_E \left( \frac{(\beta \cdot n)^2}{n^2} \right) = \langle 0 | T [ \Phi_{\beta}(\infty, 0) \Phi_n(0, \infty) ] |0\rangle.$$

Wilson-line matrix elements exponentiate non-trivially and have tightly constrained functional dependence on their arguments. They are known to three loops.
On functional dependences

Straight semi-infinite Wilson lines are scale-invariant

\[ \Phi_\beta(\infty, 0) \equiv P \exp \left[ ig \int_0^\infty d\lambda \beta \cdot A(\lambda \beta) \right]. \]

Correlators involving light-like Wilson lines break scale invariance due to collinear poles: a quantum `anomaly' proportional to the cusp anomalous dimension.

The anomaly must cancel in combination that are free from collinear poles

The reduced function depends only on scale-invariant combinations

At the level of anomalous dimensions the cancellation is particularly striking

Singular terms in \( \Gamma_s \) must be diagonal.

Finite diagonal terms in \( \Gamma_s \) must form \( \rho_{ij} \)'s.

Off-diagonal terms in \( \Gamma_s \) must be finite, and must depend only on cross-ratios \( \rho_{ijkl} \).

An exact equation for the soft anomalous dimension
THE SUBTRACTION PROBLEM
THE SUBTRACTION PROBLEM
A diagram contributing a double-virtual NNLO correction to t-tbar-jet production
A diagram contributing a double-virtual NNLO correction to t-tbar-jet production

Pictorial infrared

\[ \frac{1}{\epsilon^4} \]
Pictorial infrared

A diagram contributing a real-virtual NNLO correction to $t\bar{t}$-jet production
A diagram contributing a real-virtual NNLO correction to $t$-$t\bar{t}$-jet production

Pictorial infrared

$\frac{1}{\epsilon^2}$
A diagram contributing a real-virtual NNLO correction to t-tbar-jet production
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A diagram contributing a double-real NNLO correction to t-tbar-jet production
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The subtraction problem

- Infrared divergences (soft and collinear) cancel between configurations with different numbers of particles.
- Collider observables are algorithmically complex and need elaborate phase-space constraints.
- Divergences must be canceled analytically before performing numerical integrations.
- Existing subtraction algorithms beyond NLO are computationally very intensive.
- LHC is now a precision machine: we are interested in subtraction for complicated process at very high orders.
- The factorisation of virtual corrections contains all-order information, not fully exploited.
- The structure of virtual singularities can be used as an organising principle for subtraction.
NLO Subtraction

The computation of a generic IRC-safe observable at NLO requires the combination

\[
\frac{d\sigma_{\text{NLO}}}{dX} = \lim_{d \to 4} \left\{ \int d\Phi_n V_n \delta_n(X) + \int d\Phi_{n+1} R_{n+1} \delta_{n+1}(X) \right\},
\]

The necessary numerical integrations require finite ingredients in \(d=4\). Define counterterms

\[
K_{n+1}^{(1)} = L^{(1)} R_{n+1},
\]

\[
I_n^{(1)} \equiv \int d\Phi_{n+1}^{r,1} K_{n+1}^{(1)},
\]

Add and subtract the same quantity to the observable: each contribution is now finite.

\[
\frac{d\sigma_{\text{NLO}}}{dX} = \int d\Phi_n \left( V_n + I_n^{(1)} \right) \delta_n(X) + \int d\Phi_{n+1} \left( R_{n+1} \delta_{n+1}(X) - K_{n+1}^{(1)} \delta_n(X) \right),
\]

Search for the simplest fully local integrand \(K_{n+1}\) with the correct singular limits.
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Minimize complexity: split phase space in sectors with sector functions in order to have at most one soft (i) and one collinear (ij) singularity in each sector.

Sector functions $\mathcal{W}_{ij}$ must form a partition of unity.

In order not to appear in analytic integrations, sector functions must obey sum rules. Denoting with $S_i$ the soft limit for parton $i$ and $C_{ij}$ the collinear limit for the $ij$ pair,

$$S_i \sum_{k \neq i} \mathcal{W}_{ik} = 1,$$

$$C_{ij} \left[ \mathcal{W}_{ij} + \mathcal{W}_{ji} \right] = 1.$$

Sector functions are defined in terms of Lorentz invariants before choosing an explicit parametrisation of phase space. A possible choice is

$$e_i \equiv \frac{s_{qi}}{s}, \quad w_{ij} \equiv \frac{ss_{ij}}{s_{qi}s_{qj}},$$

$$\sigma_{ij} \equiv \frac{1}{e_i w_{ij}}, \quad \mathcal{W}_{ij} \equiv \frac{\sigma_{ij}}{\sum_{k \neq l} \sigma_{kl}},$$

With the help of sector functions, one can now define a candidate counterterm

$$L^{(1)} R_{n+1} = \sum_i \sum_{j \neq i} \left( S_i + C_{ij} - S_i C_{ij} \right) R_{n+1} \mathcal{W}_{ij}.$$
Kinematic complexity

In order to factorise a Born matrix element $B_n$ with $n$ on-shell particles conserving momentum, we need a mapping from the $(n+1)$-particle to the Born phase spaces. We use

\[ k_i^{(abc)} = k_i, \quad \text{if } i \neq a, b, c, \]
\[ k_i^{(abc)} = k_a + k_b - \frac{s_{ab}}{s_{ac} + s_{bc}} k_c, \]
\[ k_i^{(abc)} = \frac{s_{abc}}{s_{ac} + s_{bc}} k_c, \]

We can now redefine soft and collinear limits to include the re-parametrisation. Explicitly

\[ \bar{S}_i R(\{k\}) = -N_1 \sum_{l, m} \delta_{f_{i} g} \frac{s_{lm}}{s_{il} s_{im}} B_{lm}(\{k\}^{ilm}), \]
\[ \bar{C}_{ij} R(k) = \frac{N_1}{s_{ij}} \left[ P_{ij} B(\{k\}^{ijr}) + Q_{ij}^{\mu \nu} B_{\mu \nu}(\{k\}^{ijr}) \right], \]
\[ \bar{S}_i \bar{C}_{ij} R(\{k\}) = 2N_1 C_{f_{ij}} \delta_{f_{i} g} \frac{s_{jr}}{s_{ij} s_{ir}} B(\{k\}^{ijr}), \]

Note that we have assigned parametrisation triplets differently in different terms. Then

\[ K = \sum_{i, j \neq i} K_{ij}, \quad K_{ij} \equiv (\bar{S}_i + \bar{C}_{ij} - \bar{S}_i \bar{C}_{ij}) R \mathcal{W}_{ij}, \]
The pattern of cancellations is more intricate at higher orders.

\[
\frac{d\sigma_{\text{NNLO}}}{dX} = \lim_{d \to 4} \left\{ \int d\Phi_n \, VV_n \, \delta_n(X) + \int d\Phi_{n+1} \, RV_{n+1} \, \delta_{n+1}(X) \right. \\
+ \left. \int d\Phi_{n+2} \, RR_{n+2} \, \delta_{n+2}(X) \right\},
\]

More counterterm functions need to be defined:

\[
K^{(1)}_{n+2} = L^{(1)} \, RR_{n+2}, \quad K^{(2)}_{n+2} = L^{(2)} \, RR_{n+2}, \quad K^{(12)}_{n+2} = L^{(1)} \, L^{(2)} \, RR_{n+2}, \quad K^{(\text{RV})}_{n+1} = \tilde{L}^{(1)} \, RV_{n+1}.
\]

A finite expression for the observable in \( d=4 \) must combine several ingredients:

\[
\frac{d\sigma_{\text{NNLO}}}{dX} = \int d\Phi_n \left[ VV_n + I^{(2)}_n + I^{(\text{RV})}_n \right] \delta_n(X) \\
+ \int d\Phi_{n+1} \left[ \left( RV_{n+1} + I^{(1)}_{n+1} \right) \delta_{n+1}(X) - \left( K^{(\text{RV})}_{n+1} + I^{(12)}_{n+1} \right) \delta_n(X) \right] \\
+ \int d\Phi_{n+2} \left[ RR_{n+2} \, \delta_{n+2}(X) - K^{(1)}_{n+2} \, \delta_{n+1}(X) - \left( K^{(2)}_{n+2} - K^{(12)}_{n+2} \right) \delta_n(X) \right]
\]
A hard problem
A hard problem

A wishlist for an optimal subtraction algorithm at $N^k$LO

- Complete generality across all IR-safe observables with any number of particles.
- Exact locality of the IR and collinear counterterms.
- Exact independence on external slicing parameters.
- Complete analytical results for all integrated counterterms.
- Overall computational efficiency, including interfacing with MC codes.
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Several algorithms exist to perform IR subtraction at NNLO for a range of key processes. For the simplest processes and observables prediction are available at $N^3$LO.
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However

An extreme degree of optimisation will be necessary, and possibly completely new tools.
THE CELESTIAL SPHERE
THE CELESTIAL SPHERE
A new viewpoint on infrared/long-distance phenomena in quantum field theory.

A lesson from gravity: do not trivialise the behaviour and symmetries `at infinity'.

Does this idea lead to new calculational techniques for non-abelian theories?
Many directions

- Electromagnetic, colour and gravitational memory effects
- Full conformal symmetry on the celestial sphere
- Soft, next-to soft, next-to-next-to soft
- Asymptotically flat spacetimes and holography
- Black hole soft hair and the information paradox
- Celestial amplitudes

\[ \mathcal{A}(\Delta_j, z_j) = \left( \prod_{i=1}^{n} \int_0^\infty \frac{d\omega_i}{\omega_i} \omega_i^{\Delta_i} \right) \mathcal{A}(\omega_j, z_j). \]
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See also: H.A. González, F. Rojas, 2104.12979
On dipole correlations

Let us begin by **disentangling** **collinear** poles (which are **colour-singlets**) from **soft** poles (which are **colour-correlated**). We replace the **running** scale $\lambda$ with the **fixed** scale $\mu$ in the logarithmic term, and perform the colour sum using **colour conservation**.

$$
\Gamma_n^{\text{dipole}} \left( \frac{s_{ij}}{\lambda^2}, \alpha_s(\lambda, \epsilon) \right) = \frac{1}{2} \hat{\gamma}_K(\alpha_s(\lambda, \epsilon)) \sum_{i=1}^{n} \sum_{j=i+1}^{n} \ln \left( \frac{-s_{ij} + i\eta}{\mu^2} \right) T_i \cdot T_j \\
- \sum_{i=1}^{n} \gamma_i(\alpha_s(\lambda, \epsilon)) - \frac{1}{4} \hat{\gamma}_K(\alpha_s(\lambda, \epsilon)) \ln \left( \frac{\mu^2}{\lambda^2} \right) \sum_{i=1}^{n} C_i^{(2)}
$$

$$
\equiv \Gamma_n^{\text{corr.}} \left( \frac{s_{ij}}{\mu^2}, \alpha_s(\lambda, \epsilon) \right) + \Gamma_n^{\text{singl.}} \left( \frac{\mu^2}{\lambda^2}, \alpha_s(\lambda, \epsilon) \right)
$$

At **one loop**, integrating the **colour-correlated** term yields **single** **soft** poles, while the **singlet** term yields **single** **collinear** and **double** **soft-collinear** poles.

$$
\alpha_s(\lambda, \epsilon) = \alpha_s(\mu) \left( \frac{\lambda^2}{\mu^2} \right)^{-\epsilon},
$$

$$
\int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2} \alpha_s(\lambda, \epsilon) = -\frac{1}{\epsilon} \alpha_s(\mu), \quad \int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2} \ln \left( \frac{\lambda^2}{\mu^2} \right) \alpha_s(\lambda, \epsilon) = -\frac{1}{\epsilon^2} \alpha_s(\mu), \quad (\epsilon < 0).
$$

At **$h$ loops**, **multiple** poles (up to order $h+1$) are generated by the $\beta$ function. For **conformal gauge theories** the logarithm of the infrared factor has only **single** and **double** poles.
Celestial dipoles

Crucially, we now parametrise the light-cone momenta in celestial coordinates

\[ p_i^\mu = \omega_i \left\{ 1 + z_i \bar{z}_i, z_i + \bar{z}_i, -i(z_i - \bar{z}_i), 1 - z_i \bar{z}_i \right\}, \]

where the energy \( \omega_i \) and the sphere coordinates \( z_i \) have simple transformation properties under the Lorentz group acting as \( \text{SL}(2, \mathbb{C}) \):

\[ \omega' = |cz + d|^2 \omega, \quad z' = \frac{az + b}{cz + d}, \]

Mandelstam invariants are distances on the sphere

\[ s_{ij} = 2p_i \cdot p_j = 4\omega_i \omega_j |z_i - z_j|^2, \]

which unpacks the logarithms

\[ \log (-s_{ij} + i\eta) = \log \left( |z_i - z_j|^2 \right) + \log \omega_i + \log \omega_j + 2 \log 2 + i\pi, \]

Energies give new singlet terms

\[ \hat{\Gamma}^{\text{dipole}} \left( \frac{s_{ij}}{\lambda^2}, \alpha_s(\lambda, \epsilon) \right) \equiv \hat{\Gamma}^{\text{corr.}}_n \left( z_{ij}, \alpha_s(\lambda, \epsilon) \right) + \hat{\Gamma}^{\text{singl.}}_n \left( \frac{\omega_i}{\lambda}, \alpha_s(\lambda, \epsilon) \right), \]

which take the form

\[ \hat{\Gamma}^{\text{singl.}}_n \left( \frac{\omega_i}{\lambda}, \alpha_s(\lambda, \epsilon) \right) = -\sum_{i=1}^n \gamma_i(\alpha_s(\lambda, \epsilon)) - \frac{1}{4} \hat{\gamma}_K(\alpha_s(\lambda, \epsilon)) \sum_{i=1}^n \ln \left( \frac{-4\omega_i^2 + i\eta}{\lambda^2} \right) C_i^{(2)}, \]
Celestial dipoles

The colour-correlated term, responsible for all soft poles, is remarkably simple

$$\hat{\Gamma}_n^{\text{corr.}}(z_{ij}, \alpha_s(\lambda, \epsilon)) = \frac{1}{2} \hat{\gamma}_K(\alpha_s(\lambda, \epsilon)) \sum_{i=1}^{n} \sum_{j=i+1}^{n} \ln \left(|z_{ij}|^2\right) T_i \cdot T_j.$$  

Scale and coupling dependence are completely factored from colour and kinematics, and equal for all dipoles. The scale integral can this be performed in full generality, yielding

$$Z_n^{\text{corr.}}(z_{ij}, \alpha_s(\mu, \epsilon)) \equiv \exp \left[ \int_0^\mu \frac{d\lambda}{\lambda} \hat{\Gamma}_n^{\text{corr.}}(z_{ij}, \alpha_s(\lambda, \epsilon)) \right]$$

$$= \exp \left[ -K(\alpha_s(\mu, \epsilon)) \sum_{i=1}^{n} \sum_{j=i+1}^{n} \ln \left(|z_{ij}|^2\right) T_i \cdot T_j \right].$$

The scale factor $K$ is well-known in QCD from form-factor calculations, and gives the perturbative Regge trajectory in the high-energy limit of four-point amplitudes. It is

$$K(\alpha_s(\mu, \epsilon)) = -\frac{1}{2} \int_0^\mu \frac{d\lambda}{\lambda} \hat{\gamma}_K(\alpha_s(\lambda, \epsilon)).$$

The function $K$ can be computed order by order in terms of the cusp and the $\beta$ function

$$K(\alpha_s, \epsilon) = \frac{\alpha_s}{\pi} \frac{\hat{\gamma}_K^{(1)}}{4\epsilon} + \left(\frac{\alpha_s}{\pi}\right)^2 \left( \frac{\hat{\gamma}_K^{(2)}}{8\epsilon} + \frac{b_0\hat{\gamma}_K^{(1)}}{32\epsilon^2} \right)$$

$$+ \left(\frac{\alpha_s}{\pi}\right)^3 \left( \frac{\hat{\gamma}_K^{(3)}}{12\epsilon} + \frac{b_1\hat{\gamma}_K^{(2)}}{48\epsilon^2} + \frac{b_2\hat{\gamma}_K^{(1)}}{192\epsilon^3} \right) + \mathcal{O}(\alpha_s^4),$$

$$\beta \to 0$$

$$K(\alpha_s, \epsilon) = \sum_{n=1}^{\infty} \frac{\left(\frac{\alpha_s}{\pi}\right)^n \hat{\gamma}_K^{(n)}}{4n\epsilon},$$
A celestial conformal theory

It is natural to mimic the bosonic string, considering free bosons spanning the gauge algebra.

\[ S(\phi) = \frac{1}{2\pi} \int d^2 z \partial_z \phi^a(z, \bar{z}) \partial_{\bar{z}} \phi_a(z, \bar{z}), \]

The free bosons could be organised in a matrix field: gauge generators at different points must then be taken to commute.

The well-known results for free bosons in $d=2$ can be directly transcribed.

The equations of motions are:

\[ \partial_z \partial_{\bar{z}} \phi^a(z, \bar{z}) = 0, \]

implying that the derivatives of the fields are (anti)holomorphic.

A normal-ordered product can be defined, obeying the classical equation of motion

\[ :\phi^a(z, \bar{z}) \phi^b(w, \bar{w}):\ = \phi^a(z, \bar{z}) \phi^b(w, \bar{w}) + \frac{1}{2} \delta^{ab} \log |z - w|^2, \]

There is a traceless conserved energy-momentum tensor, and a conserved Noether current

\[ T(z) = -: \partial_z \phi^a(z, \bar{z}) \partial_{\bar{z}} \phi_a(z, \bar{z}) :, \quad j^a(z) = \partial_z \phi^a(z, \bar{z}). \]
Matrix vertex operators

Guided by the QED example, we can tentatively define a matrix-valued vertex operator

\[ V(z, \bar{z}) \equiv e^{i \kappa \mathbf{T}_z \cdot \phi(z, \bar{z})} = e^{i \kappa \Phi(z, \bar{z})}, \]

In colour space, this is a matrix in the representation of \( T_z \), defined on the boundary sphere and acting on the bulk colour degrees of freedom. But is it a conformal primary field?

For conventional vertex operators (as for example for bosonic strings)

\[ V_{c.s.}(z, \bar{z}) \equiv e^{i k^\mu X_\mu(z, \bar{z})} \quad \rightarrow \quad h = \frac{1}{4} \mathbf{k}^\mu k^\nu \eta_{\mu \nu} = \frac{k^2}{4}, \]

\[ V(z, \bar{z}) \equiv e^{i \kappa \mathbf{T}_z \cdot \phi(z, \bar{z})} \quad \rightarrow \quad h = \frac{k^2}{4} T_z \cdot T_z = \frac{k^2}{4} C_r^{(2)}, \]

Crucially, this is a positive real number and not a matrix. For consistency, two-point functions must evaluate to a power of the distance given by the conformal weight \( \Delta = h + \bar{h} \). Indeed

\[ \langle V(z_1, \bar{z}_1)V(z_2, \bar{z}_2) \rangle \sim |z_{12}|^{-2 \Delta}, \]

by colour conservation \( T_1 + T_2 = 0 \)

Note analogies with other constructions.

Vertex operator construction of Kac-Moody algebras:

Reggeon fields for high-energy scattering:

\[ \begin{align*}
  U^a(z) &= z^{3/2} e^{i g_s Q^a(z)}, \\
  U(z) &= e^{i g_s T^a W^a(z)}. 
\end{align*} \]

(Caron-Huot 2013)
A conformal correlator

Our construction from the beginning targeted the n-point correlator

\[ \mathcal{C}_n(\{z_i\}, \kappa) \equiv \left\langle \prod_{i=1}^{n} V(z_i, \bar{z}_i) \right\rangle. \]

The calculation is a textbook exercise: it can be done with oscillators, after expanding the free fields in modes on the sphere, or computing the path integral (Polchinski). The result is

\[ \mathcal{C}_n(\{z_i\}, \kappa) = C(N_c) \exp \left[ \frac{\kappa^2}{2} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \ln \left( |z_{ij}|^2 \right) T_i \cdot T_j \right], \]

reproducing the structure of the gauge theory infrared operator. Note that

- The correlator has support only on colour conserving configurations
- The field normalisation \( \kappa \) maps to the integral \( K \), carrying scale and regulator dependence.
- In a path integral evaluation on a curved surface (say, a finite sphere with radius \( R \)) the correlator acquires a scale-dependent ‘Weyl’ factor, which in this setting maps to an (undetermined) colour-singlet collinear contribution.

\[ W_n(\{z_i\}, \kappa) = \exp \left[ -\frac{1}{2} \sum_{i=1}^{n} C_i^{(2)} g(z_i, \bar{z}_i) \right], \]
MANY QUESTIONS
The choice of the gauge coupling.
Our construction lends support to the idea the cusp anomalous dimension should be taken as the definition of the strong coupling in the infrared. How far can one take this definition?

Scale and regulator dependence.
It is remarkable, and necessary, that infrared singularities be hidden in the matching condition between the gauge theory and the conformal theory. How can one make this correspondence more precise?

Beyond the free theory.
The celestial conformal theory certainly has corrections involving structure constants (as confirmed by the structure of $\Delta$). The deformed theory is still scale invariant. What drives the deformation?

Constraints from vast field theory data.
Soft and collinear factorisation kernels are known to three loops, and in the massive case to two loops. In most cases their remarkable simplicity is only partly explained. How can we harness these data to constrain the celestial theory?

What is the celestial theory beyond dipoles?
Non-trivial gauge data

Quadrupole corrections to the correlator at three loops and beyond

Ø. Almelid, C. Duhr, E. Gardi; with A. McLeod and C. White; J. Henn, B. Mistlberger.

\[
\Delta_n^{(3)}(\rho_{ijkl}) = 16 f_{abc} f_{cde} \left\{ -C \sum_{i=1}^{n} \sum_{1 \leq j < k \leq n, j \neq k} \left[ T_i^a T_j^b T_k^c \right] \right. \\
+ \sum_{1 \leq i < j < k < l \leq n} \left[ T_i^a T_j^b T_k^c T_l^d F(\rho_{ijkl}, \rho_{ilkj}) + T_i^a T_k^b T_j^c T_l^d F(\rho_{ijkl}, \rho_{ijlk}) \right. \\
+ \left. T_i^a T_j^b T_k^c T_l^d F(\rho_{ijlk}, \rho_{iklj}) \right\} 
\]

\[
F(\rho_{ijkl}, \rho_{ilkj}) = \rho_{ijkl}, \quad (1 - z_{ijkl})(1 - \bar{z}_{ijkl}) = \rho_{ilkj}
\]

\[
F(z) = \mathcal{L}_{10101}(z) + 2\zeta_2[\mathcal{L}_{001}(z) + \mathcal{L}_{100}(z)].
\]

Quantum corrections to the tree-level soft-gluon current


\[
J_{(1)}(k) = -\frac{i}{16\pi^2} \left[ \frac{C_A}{\epsilon^2} J_{(0)}(k) + \frac{1}{\epsilon} f_{abc} \sum_{i \neq j} T_i^b T_j^c \left( \frac{\beta_i^\mu}{\beta_i \cdot k} - \frac{\beta_j^\mu}{\beta_j \cdot k} \right) \log \left( \frac{\mu^2(-2\beta_i \cdot \beta_j)}{(-2\beta_i \cdot k)(-2\beta_j \cdot k)} \right) \right]
\]
Towards an interacting theory?

Interactions in the $d=2$ theory are constrained by gauge and euclidean invariance, whether the theory is conformal or not. With up to four fields one finds

$$
\mathcal{L}(\phi^a) = \frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi_a + i \frac{\lambda_1}{6} \varepsilon^{\mu\nu} f^{abc} \phi_a \partial_\mu \phi_b \partial_\nu \phi_c - \frac{\lambda_2}{24} f^{abc} f^{e\alpha\beta\gamma} \phi_d \partial_\alpha \phi_e \partial_\beta \phi_f \partial_\gamma \phi_g + \ldots
$$

In fact, $\lambda_1 = \lambda_2 = 1$ yields the leading terms of the WZNW action, while $\lambda_1 = 0$, $\lambda_2 = 1$ yields the principal chiral model, which is not conformally invariant.

Correlators in the WZNW model must obey the Knizhnik-Zamolodchikov equation, but this fails for the gauge-theory correlator, and cannot be compensated by quadrupoles.

$$
\mathcal{Z}_n(z_i) \equiv \exp \left[ \mathcal{E}_n(z_i) \right] \rightarrow \frac{\partial \mathcal{Z}_n}{\partial z_i} = K T_i \cdot \sum_{k \neq i} \frac{T_k}{z_i - z_k} + \frac{1}{2} \left[ \mathcal{E}_n(z_i), \frac{\partial \mathcal{E}_n}{\partial z_i} \right] + \ldots
$$

OUTLOOK
The infrared structure of gauge theory scattering amplitudes is theoretically interesting and phenomenologically relevant.

Factorisation of physics at different length scales is the key to progress: it leads to universality, evolution equations, and predictive exponentiation.

The problem of subtraction of IR-singular configurations beyond NLO is intricate both theoretically and computationally.

Infrared factorisation provides general tools to understand subtraction to all orders in perturbation theory. Much technical work however remains to be done.

A new theoretical viewpoint on infrared dynamics emerges from asymptotic symmetries of the $S$-matrix and expresses infrared properties of $d=4$ amplitudes in terms of a $d=2$ conformal field theory, to all orders. Powerful new calculation tools may be at hand.
THANK YOU