PART IV

Unitarity Triangle Formalism

How well we know CKM

The Unitarity Triangle

The CKM is unitary

$$V V^\dagger = V^\dagger V = 1$$

The non-diagonal elements of the matrix products correspond to 6 triangle equations

$$V^*_{ud} V_{us} + V^*_{cd} V_{cs} + V^*_{td} V_{ts} = 0 \qquad \lambda \ \lambda \ \lambda^5$$

$$V^*_{ub} V_{ud} + V^*_{cb} V_{cd} + V^*_{tb} V_{td} = 0 \qquad \lambda^3 \ \lambda^3 \ \lambda^3$$

$$V^*_{us} V_{ub} + V^*_{cs} V_{cb} + V^*_{ts} V_{tb} = 0 \qquad \lambda^4 \ \lambda^2 \ \lambda^2$$

$$V^*_{ud} V_{td} + V^*_{us} V_{ts} + V^*_{ub} V_{tb} = 0 \qquad \lambda^3 \ \lambda^3 \ \lambda^3$$

$$V^*_{td} V_{cd} + V^*_{ts} V_{cs} + V^*_{tb} V_{cb} = 0 \qquad \lambda^4 \ \lambda^2 \ \lambda^2$$

$$V^*_{ud} V_{cd} + V^*_{us} V_{cs} + V^*_{ub} V_{cb} = 0 \qquad \lambda \ \lambda \ \lambda^5$$

Remember that :

$$\begin{pmatrix} 1-\lambda^2/2-\lambda^4/8 & \lambda & A\lambda^3(\rho-i\eta) \\ -\lambda+\frac{A^2\lambda^5}{2}(1-2\rho)-i A^2\lambda^5\eta & 1-\lambda^2/2-\lambda^4(\frac{1}{8}+\frac{A^2}{2}) & A\lambda^2 \\ A\lambda^3(1-(1-\lambda^2/2)(\rho+i\eta)) & -A\lambda^2(1-\lambda^2/2)(1+\lambda^2(\rho+i\eta)) & 1-\frac{A^2\lambda^4}{2} \end{pmatrix}$$

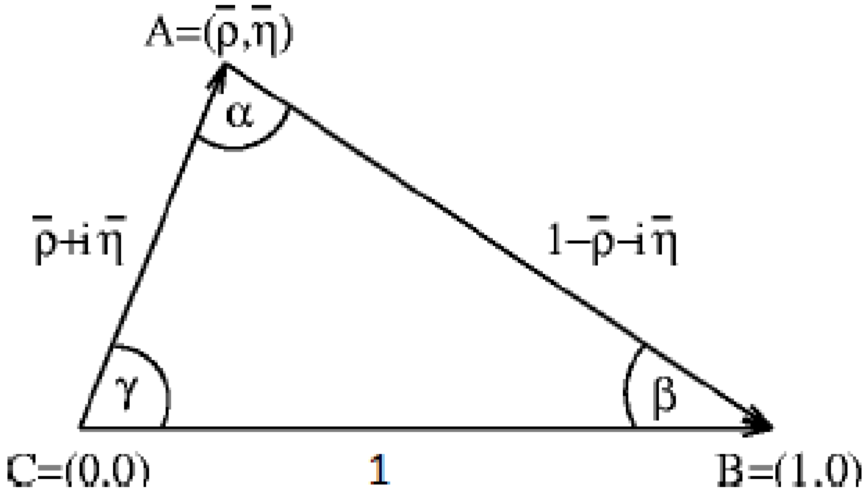
$$\overline{\rho}(\overline{\eta}) = (1-\lambda^2/2)\rho(\eta)$$

$$V_{ub}^{*}V_{ud}+V_{cb}^{*}V_{cd}+V_{tb}^{*}V_{td}=0$$

$$V_{ud}V_{ub}^{*}=A\lambda^{3}(\overline{\rho}+i\overline{\eta})$$

$$V_{cd}V_{cb}^{*}=-A\lambda^{3}$$

$$V_{td}V_{tb}^{*}=A\lambda^{3}(1-\overline{\rho}-i\overline{\eta})$$



$$\overline{AB}=\frac{V_{td}V_{tb}^{*}}{V_{cd}V_{cb}^{*}}=\sqrt{(1-\overline{\rho})^{2}+\overline{\eta}^{2}}=\frac{1}{\lambda}\left|\frac{V_{td}}{V_{cb}}\right|\sim\frac{1}{\lambda}\left|\frac{V_{td}}{V_{ts}}\right|$$

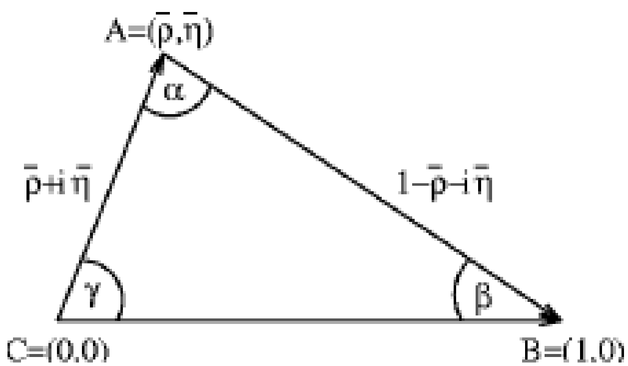
$$\overline{AC}=\frac{V_{ud}V_{ub}^{*}}{V_{cd}V_{cb}^{*}}=\sqrt{\overline{\rho}^{2}+\overline{\eta}^{2}}=\left(1-\frac{\lambda^{2}}{2}\right)\frac{1}{\lambda}\left|\frac{V_{ub}}{V_{cb}}\right|$$

Each of the angles of the unitarity triangle is the relative phase of two adjacent sides (a part for possible extra π and minus sign)

$$\beta = \arg\left(\frac{V_{td}V_{tb}^*}{V_{cd}V_{cb}^*}\right) = \text{atan}\left(\frac{\bar{\eta}}{(1-\bar{\rho})}\right)$$

$$\gamma = \arg\left(\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}\right) = \text{atan}\left(\frac{\bar{\eta}}{\bar{\rho}}\right)$$

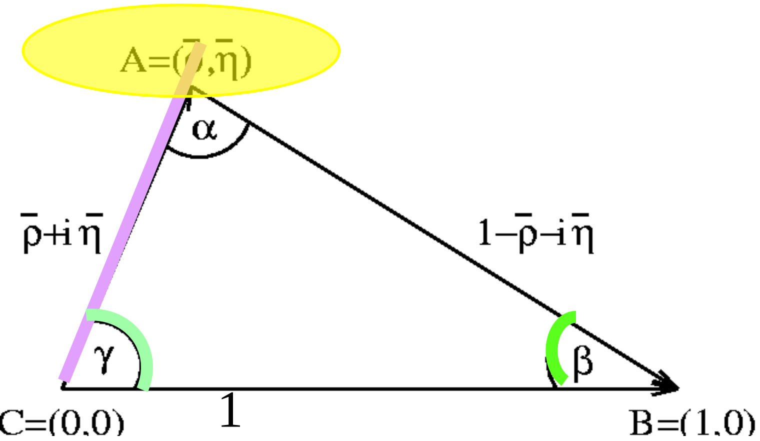
$$\alpha + \beta + \gamma = \pi$$



The reason of making the arg of the ratio of two legs is simple

$$x = |x|e^{i\vartheta};\ y = |y|e^{i\chi} \qquad x/y = (|x|/|y|)e^{i(\vartheta-\chi)}$$

$\rightarrow$ $\quad \arg(x/y) = (\vartheta - \chi)$ So the relative phase



$$\alpha + \beta + \gamma = \pi$$

$$\overline{AB} = \left|\frac{V_{td}V_{tb}^*}{V_{cd}V_{cb}^*}\right| = \sqrt{(1-\bar{\rho})^2 + \bar{\eta}^2} = \frac{1}{\lambda}\left|\frac{V_{td}}{V_{cb}}\right| \sim \frac{1}{\lambda}\left|\frac{V_{td}}{V_{ts}}\right|$$

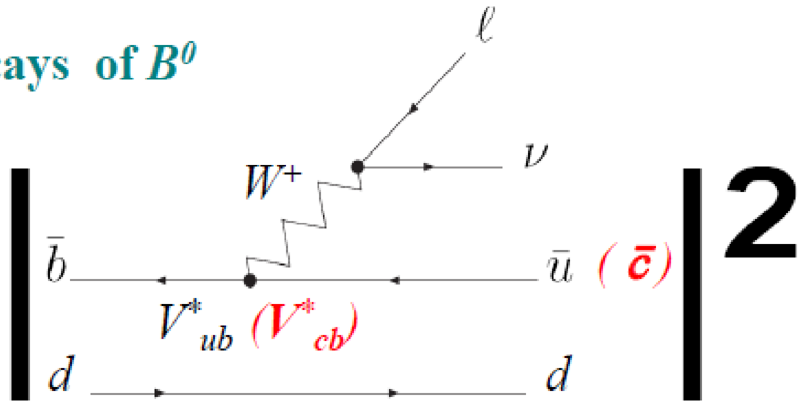
$$\overline{AC} = \left|\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}\right| = \sqrt{\bar{\rho}^2 + \bar{\eta}^2} = \left(1-\frac{\lambda^2}{2}\right)\frac{1}{\lambda}\left|\frac{V_{ub}}{V_{cb}}\right|$$

$$\beta = \arg\left(\frac{V_{td}V_{tb}^*}{V_{cd}V_{cb}^*}\right) = \operatorname{atan}\left(\frac{\bar{\eta}}{(1-\bar{\rho})}\right)$$

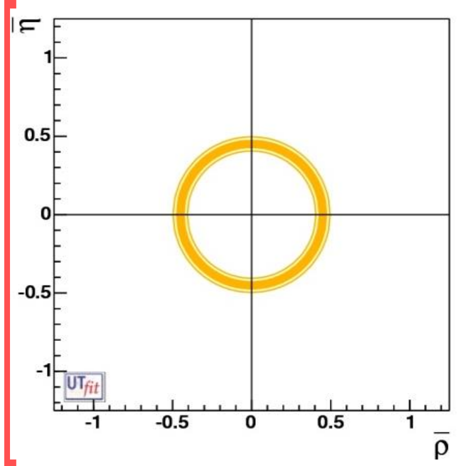
$$\gamma = \arg\left(\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}\right) = \operatorname{atan}\left(\frac{\bar{\eta}}{\bar{\rho}}\right)$$

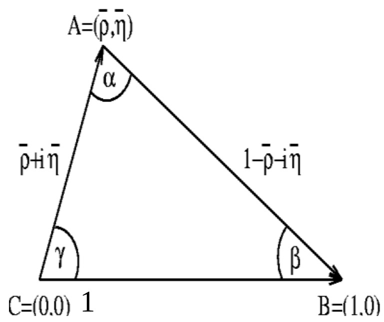
Rates of semileptonic decays of B^0

Provide information on V_{ub} (V_{cb})



$\propto |V_{ub}|^2$ ($|V_{cb}|^2$)

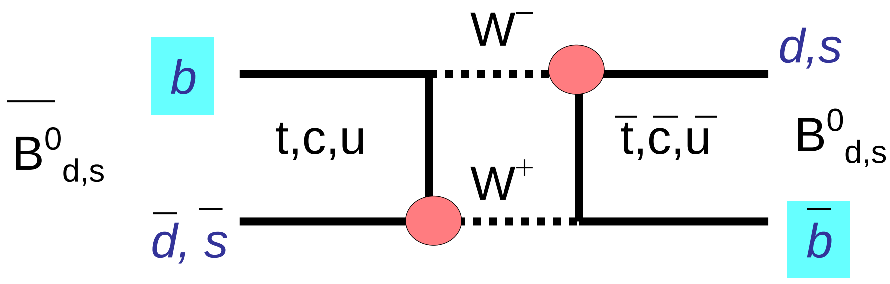




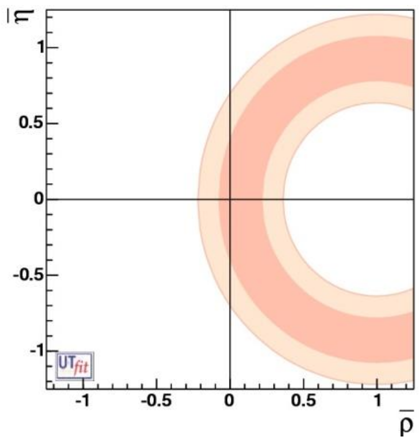
$$\overline{AC} = \frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*} = \sqrt{\bar{\rho}^2 + \bar{\eta}^2} = \left(1 - \frac{\lambda^2}{2}\right)\frac{1}{\lambda}\left|\frac{V_{ub}}{V_{cb}}\right|$$

Circle around (0,0) in the ρ–η plane

You have to measure B decays b→c and b → c transitions



V_{td} V_{ts}



Bs and Bd oscillations give acces to Vts abd Vtd

$$\overline{AB} = \frac{V_{td}V_{tb}^*}{V_{cd}V_{cb}^*} = \sqrt{(1-\overline{\rho})^2 + \overline{\eta}^2} = \frac{1}{\lambda}\left|\frac{V_{td}}{V_{cb}}\right| \sim \frac{1}{\lambda}\left|\frac{V_{td}}{V_{ts}}\right|$$

$$\propto (1-\overline{\rho})^2 + \overline{\eta}^2$$

Circle around (1,0) in the ρ–η plane

You have to measure B meson oscillations
More precisely you have to measure Oscillation frequency

Introduction to mixing and CP phenomena

Pairs of self-conjugate mesons that can be transformed to each other via flavour changing weak interaction transitions are:

$$|K^0\rangle = |\bar{s}d\rangle \qquad |D^0\rangle = |c\bar{u}\rangle \qquad |B_d^0\rangle = |\bar{b}d\rangle \qquad |B_s^0\rangle = |\bar{b}s\rangle$$

They are **flavour eigenstates** with definite quark content $|B^0\rangle, \ |\overline{B}^0\rangle$

- useful to understand particle production and decay

Apart from the flavour eigenstates there are **mass eigenstates**: $|B_L\rangle, \ |B_H\rangle$

- eigenstates of the Hamiltonian
- states of definite mass and lifetime

$$|B_L\rangle = p|B^0\rangle + q|\overline{B}^0\rangle$$

$$|B_H\rangle = p|B^0\rangle - q|\overline{B}^0\rangle$$

$|B_L\rangle, \ |B_H\rangle$: mass eigenstates

$|B^0\rangle, \ |\overline{B}^0\rangle$: flavour eigenstates

Since flavour eigenstates are not mass eigenstates, the flavour eigenstates are mixed with one another as they propagate through space and time

$|B^0(t)\rangle$ ($|\bar{B}^0(t)\rangle$) : the flavour state of a B meson that was a B^0 ($\bar{B}^0$) at $t=0$.

Schrödinger equation governs time evolution of the B^0-$\bar{B}^0$ System:

$$i\frac{d}{dt}\begin{pmatrix}|B^0(t)\rangle \\ |\bar{B}^0(t)\rangle\end{pmatrix} = \underbrace{\left(M - \tfrac{i}{2}\Gamma\right)}\begin{pmatrix}|B^0(t)\rangle \\ |\bar{B}^0(t)\rangle\end{pmatrix}$$

T conservation ➡ $|H_{21}| = |H_{12}|$

CP conservation ➡ $|H_{21}| = |H_{12}|$, $H_{11} = H_{22}$

CPT conservation ➡ $H_{11} = H_{22}$

=> ***H*** (effective Hamiltonian)

$$H|B_L^0\rangle = (M_L - i/2\Gamma_L)|B_L^0\rangle$$

$$H|B_H^0\rangle = (M_H - i/2\Gamma_H)|B_H^0\rangle$$

eigenvalues

Mass states are eigenvectors of ***H***

$$\Delta m_B \equiv M_H - M_L \approx 2|M_{12}|$$

$$\Delta\Gamma_B \equiv \Gamma_H - \Gamma_L = 2\mathrm{Re}\left(M_{12}\Gamma_{12}^*\right)/|M_{12}|$$

$$m_B \equiv \frac{M_H + M_L}{2}$$

$$\Gamma_B \equiv \frac{\Gamma_H + \Gamma_L}{2}$$

$$\frac{q}{p} \equiv -\sqrt{\frac{H_{21}}{H_{12}}} = \frac{\Delta m_B + i\Delta\Gamma_B/2}{2M_{12} - i\Gamma_{12}}$$

The time evolution of the mass eigenstates is governed by their eigenvalues :

$$|B_{H,L}(t)\rangle = e^{-i\left(M_{H,L} - i\frac{\Gamma_{H,L}}{2}\right)t}|B_{H,L}(t=0)\rangle$$

+

$$|B_L\rangle = p|B^0\rangle + q|\bar{B}^0\rangle$$

$$|B_H\rangle = p|B^0\rangle - q|\bar{B}^0\rangle$$

Time evolution

Time evolution of the physical states $|B^0(t)\rangle$ $(|\bar{B}^0(t)\rangle)$

$$|B^0(t)\rangle = g_+(t)|B^0\rangle + \frac{q}{p}g_-(t)|\bar{B}^0\rangle \quad g_+(t) = e^{-i\left(m_B - i\frac{\Gamma_H}{2}\right)t}\left[\cosh\frac{\Delta\Gamma t}{4}\cos\frac{\Delta m t}{2} - i\sinh\frac{\Delta\Gamma t}{4}\sin\frac{\Delta m t}{2}\right]$$

$$|\bar{B}^0(t)\rangle = \frac{p}{q}g_-(t)|B^0\rangle + g_+(t)|\bar{B}^0\rangle \quad g_-(t) = e^{-i\left(m_B - i\frac{\Gamma_H}{2}\right)t}\left[-\sinh\frac{\Delta\Gamma t}{4}\sin\frac{\Delta m t}{2} + i\cosh\frac{\Delta\Gamma t}{4}\cos\frac{\Delta m t}{2}\right]$$

More general formulae

$$\Delta m_B \equiv M_H - M_L$$
$$\Delta\Gamma_B \equiv \Gamma_H - \Gamma_L$$
$$m_B \equiv \frac{M_H + M_L}{2}$$
$$\Gamma_B \equiv \frac{\Gamma_H + \Gamma_L}{2}$$
$$\frac{q}{p} = \frac{\Delta m + i\Delta\Gamma/2}{2M_{12} - i\Gamma_{12}}$$

When ΔΓ is small they simplify to :

$$|B^0(t)\rangle = e^{-im_B t}e^{-\Gamma_B t/2}\left(\cos\frac{\Delta m_B t}{2}|B^0\rangle + i\frac{q}{p}\sin\frac{\Delta m_B t}{2}|\bar{B}^0\rangle\right)$$

$$|\bar{B}^0(t)\rangle = e^{-im_B t}e^{-\Gamma_B t/2}\left(\cos\frac{\Delta m_B t}{2}|\bar{B}^0\rangle + i\frac{p}{q}\sin\frac{\Delta m_B t}{2}|B^0\rangle\right)$$

Probability to observe in the state *f* a B^0 produced at time t=0:

$$P\left(B^0(0) \to f\right) = \left|\langle f|H|B^0(t)\rangle\right|^2$$

Probability to observe in the state $\bar{f}$ a B^0 produced at time t=0:

$$P\left(\bar{B}^0(0) \to f\right) = \left|\langle f|H|\bar{B}^0(t)\rangle\right|^2$$

The two master formulae (having however neglected $\Delta\Gamma$:

$$P\left(B^0(0) \to f\right) = \frac{e^{-\Gamma t}}{2}\{(1+\cos\Delta m t)\left|\langle f|H|B^0\rangle\right|^2 + (1-\cos\Delta m t)\left|\frac{q}{p}\right|^2\left|\langle f|H|\overline{B}^0\rangle\right|^2$$
$$-2\sin\Delta m t \times \mathrm{Im}\left(\frac{q}{p}\langle f|H|\overline{B}^0\rangle\langle f|H|B^0\rangle^*\right)\}$$

$$P\left(\overline{B}^0(0) \to f\right) = \frac{e^{-\Gamma t}}{2}\{(1+\cos\Delta m t)\left|\langle f|H|\overline{B}^0\rangle\right|^2 + (1-\cos\Delta m t)\left|\frac{p}{q}\right|^2\left|\langle f|H|B^0\rangle\right|^2$$
$$-2\sin\Delta m t \times \mathrm{Im}\left(\frac{p}{q}\times\langle f|H|B^0\rangle\langle f|H|\overline{B}^0\rangle^*\right)\}$$

Considering only the mixing :

Starting from a B^0

$$\left| \langle B^0 | H | B^0(t) \rangle \right|^2 = \frac{e^{-\Gamma t}}{2} (1 + \cos \Delta m t)$$

CP violation is neglected : q/p=1

Starting from a $\overline{B}^0$

$$\left| \langle \overline{B}^0 | H | B^0(t) \rangle \right|^2 = \frac{e^{-\Gamma t}}{2} (1 - \cos \Delta m t)$$

If one does not neglect $\Delta \wp$ (useful for charm or B_s) the previous formulae become

$$\frac{e^{-\Gamma t}}{4} (\underbrace{e^{\frac{\Delta\Gamma}{2} t} + e^{-\frac{\Delta\Gamma}{2} t}}_{\cosh\left(\frac{\Delta\Gamma}{2} t\right)} \pm 2 \cos \Delta m t)$$

So that one finds for the time dependent mixing asymmetry:

$$A_{\text{mix}}(t) \equiv \frac{N(\text{unmixed}) - N(\text{mixed})}{N(\text{unmixed}) + N(\text{mixed})}(t) = \frac{\cos(\Delta m t)}{\cosh(\Delta \Gamma t / 2)}$$

Mixed : $\overline{B}^0 \rightarrow B^0$ or $B^0 \rightarrow \overline{B}^0$

$$\cosh(\Delta\Gamma t / 2) \rightarrow 1 \text{ when } \Delta\Gamma \rightarrow 0$$

UnMixed : $B^0 \rightarrow B^0$ or $\overline{B}^0 \rightarrow \overline{B}^0$

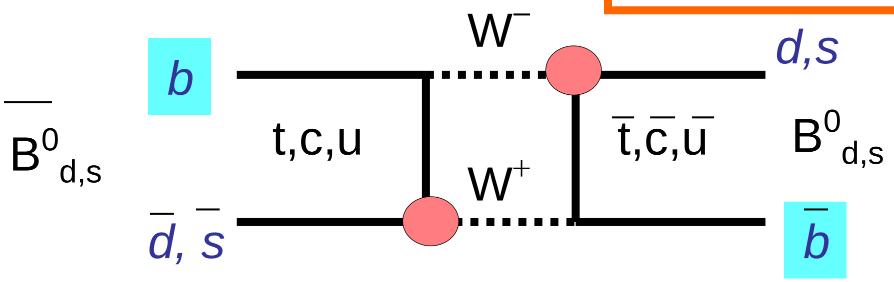
Oscillations are characterized by Δm which is related to Vtd and Vts

The probability that the meson B^0 produced (by strong interaction) at $t = 0$ transforms (weak interaction) into $\overline{B^0}$ (or stays as a B^0) at time t is given by :

$$P_{B_q^0 \to B_q^0(\overline{B_q^0})} = \frac{1}{2} e^{-t/\tau_q} (1 \pm \cos \Delta m_q t)$$

Δm_q is **the oscillation frequency : 1 ps^{-1} = 6.58 10^{-4} eV**

$$\Delta m_B = 2|M_{12}|$$



In SM :ΔF=2 process

GIM mechanism (Rate ~ $m_1^2 - m_2^2$)

Dominated by ***t*** *exchange*

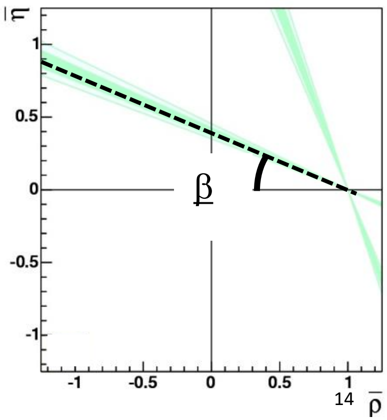
Rate LARGE

V_{td} V_{ts}

Allow to access fundamental parameters of the Standard Model

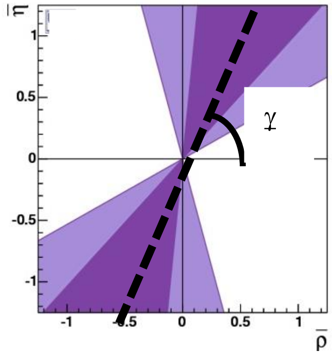
If we can access to the imaginary part of the ampliutude involving V_{td} → access to β angle

$$\beta = \arg\left(\frac{V_{td}V_{tb}^{*}}{V_{cd}V_{cb}^{*}}\right) = \mathrm{atan}\left(\frac{\bar{\eta}}{(1-\bar{\rho})}\right)$$



If we can access to the imaginary part of the amplitude involving V_{ub} → access to γ angle

$$\gamma = \arg\left(\frac{V_{ud}V_{ub}^{*}}{V_{cd}V_{cb}^{*}}\right) = \mathrm{atan}\left(\frac{\bar{\eta}}{\bar{\rho}}\right)$$



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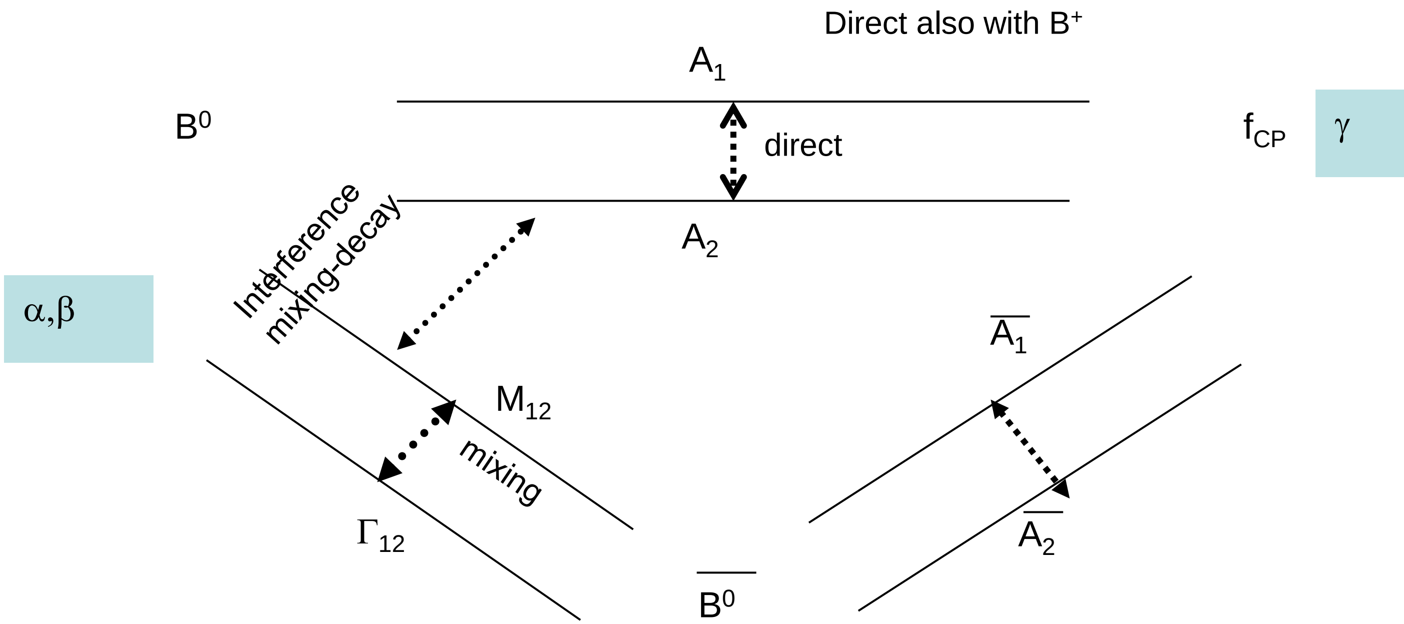
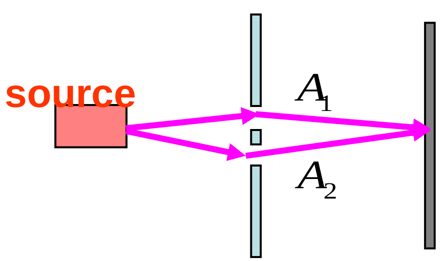
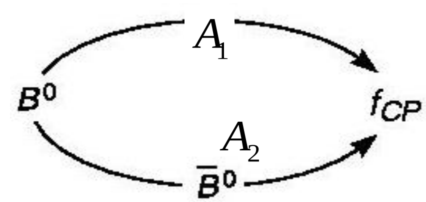
Angles are accessible through CP violating measurements



Analogy: “Double-Slit” Experiments with Matter and Antimatter





In the double-slit experiment, there are two paths to the same point on the screen.

In the B experiment, we must choose final states that both a B^0 and a B^0 can decay into.

We perform the B experiment twice (starting from B^0 and from B^0). We then compare the results.

Three types of CP violation

$$A = \sum_j A_j \, e^{i\delta_j} \, e^{i\phi_j}$$

$$\bar{A} = \sum_j A_j \, e^{i\delta_j} \, e^{-i\phi_j}$$

A CP conjugate of $\bar{A}$

CP violation if $|A|^2 \neq |\bar{A}|^2$

CP violation CP in mixing :

$$B^0 \longrightarrow \bar{B}^0 \longrightarrow \bar{f}$$

$$\bar{B}^0 \longrightarrow B^0 \longrightarrow f$$

CP violation in decay (« direct CP») :

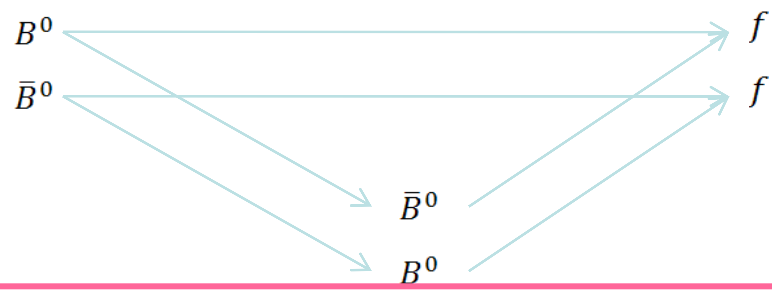
γ measurement

$$B \xrightarrow{A} f$$

$$\bar{B} \xrightarrow{\bar{A}} \bar{f}$$

Only one existing for charged B

CP violation in the interference between mixing and decay :



CP violation in the decay

CP-conjugated amplitudes :

$$\begin{cases} A_f = A(B \to f) \\ \bar{A}_f = A(\bar{B} \to \bar{f}) \end{cases}$$

$$A_f = \sum_j a_j \cdot e^{i\theta_j} e^{i\phi_j}$$

$$\bar{A}_{\bar{f}} = \sum_j a_j \cdot e^{i\theta_j} e^{-i\phi_j}$$

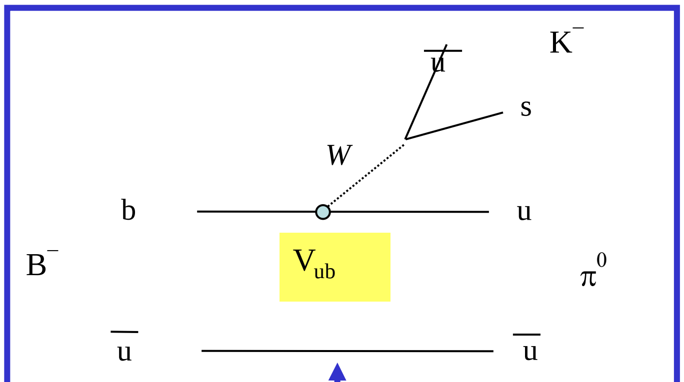
ϕ_j : alters sign under *CP* (weak phase)

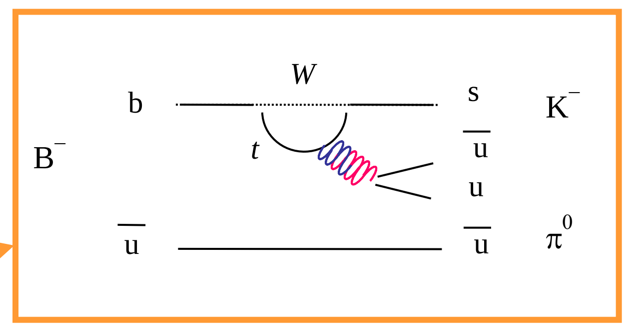
θ_j : *CP* invariant (strong phase)

$$A_{CP} \equiv \frac{\Gamma(\bar{B} \to \bar{f}) - \Gamma(B \to f)}{\Gamma(\bar{B} \to \bar{f}) + \Gamma(B \to f)} = \frac{|\bar{A}_{\bar{f}}|^2 - |A_f|^2}{|\bar{A}_{\bar{f}}|^2 + |A_f|^2}$$

$$A_{CP} = \frac{\sum_{ij} a_i a_j \cdot \sin(\theta_i - \theta_j) \cdot \sin(\phi_i - \phi_j)}{\sum_{ij} a_i a_j \cdot \cos(\theta_i - \theta_j) \cdot \cos(\phi_i - \phi_j)}$$

Direct *CP* violation requires at least two amplitudes with different weak *and* strong phases

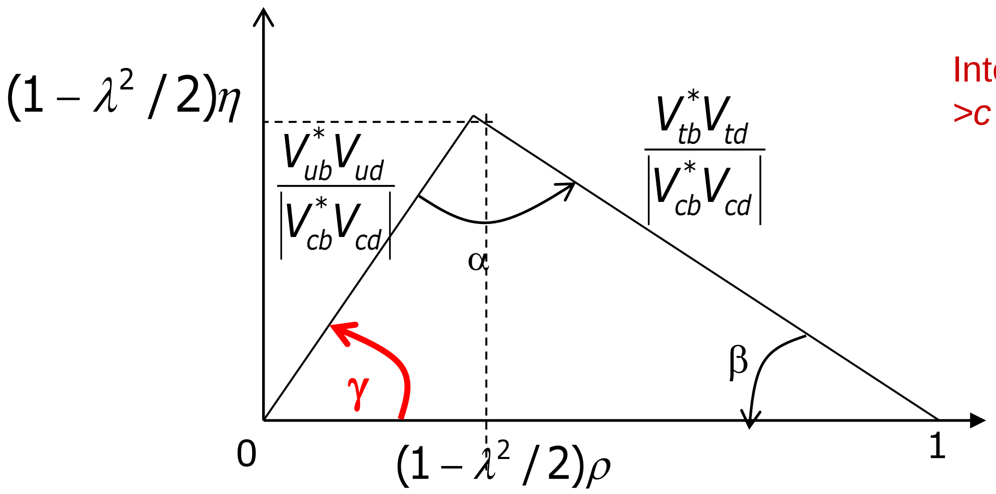




$$A_-(B^- \to f) = a_1 e^{i\theta_1} e^{i\phi_1} + a_2 e^{i\theta_2} e^{i\phi_2}$$

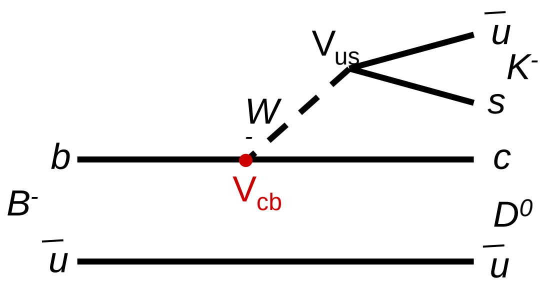
$$A_+(B^+ \to \bar{f}) = a_1 e^{i\theta_1} e^{-i\phi_1} + a_2 e^{i\theta_2} e^{-i\phi_2}$$

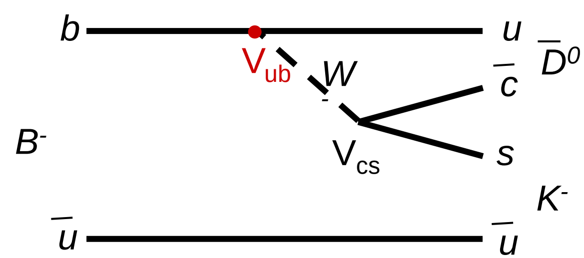
In this case weak phase difference : γ



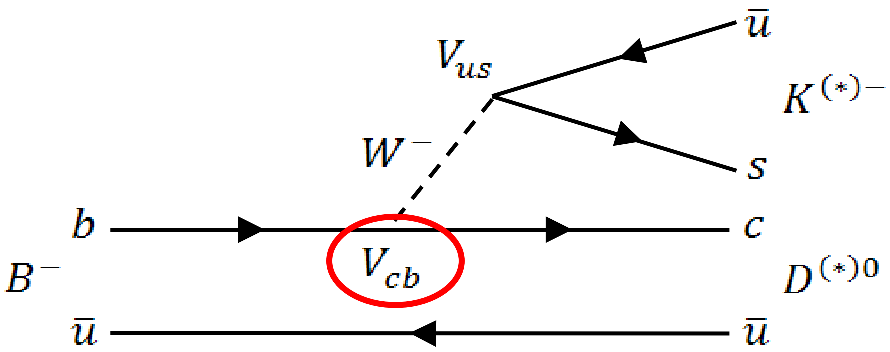
Interferences between *b*->*c* and *b*->*u* transitions

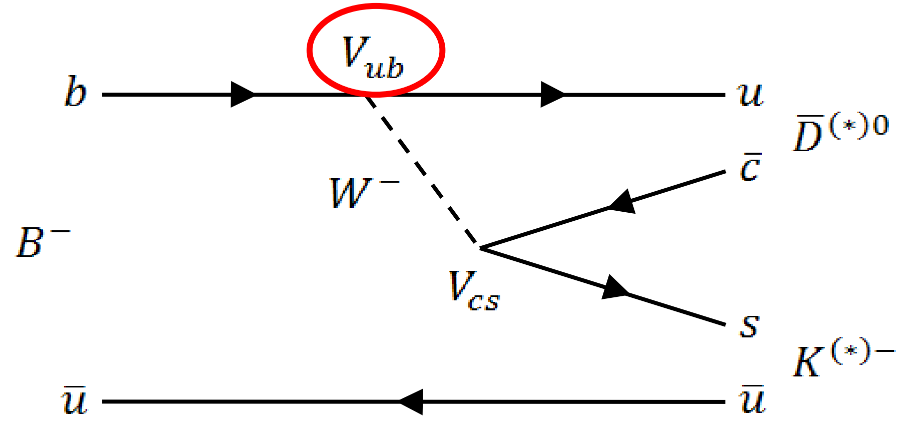
$$\gamma = \arg\left(-\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}\right)$$





Need to have modes for which D^0 and $\overline{D^0}$ are undistinguishable …





C P

$$A\left(B^- \to D^{(*)0} K^{(*)-}\right) = a$$

$$\bar{A}\left(B^+ \to \bar{D}^{(*)0} K^{(*)+}\right) = a$$

$$A\left(B^- \to \bar{D}^{(*)0} K^{(*)-}\right) = a\, r_B\, e^{i\delta}\, e^{-i\gamma}$$

$$\bar{A}\left(B^+ \to D^{(*)0} K^{(*)+}\right) = a\, r_B e^{i\delta} e^{+i\gamma}$$

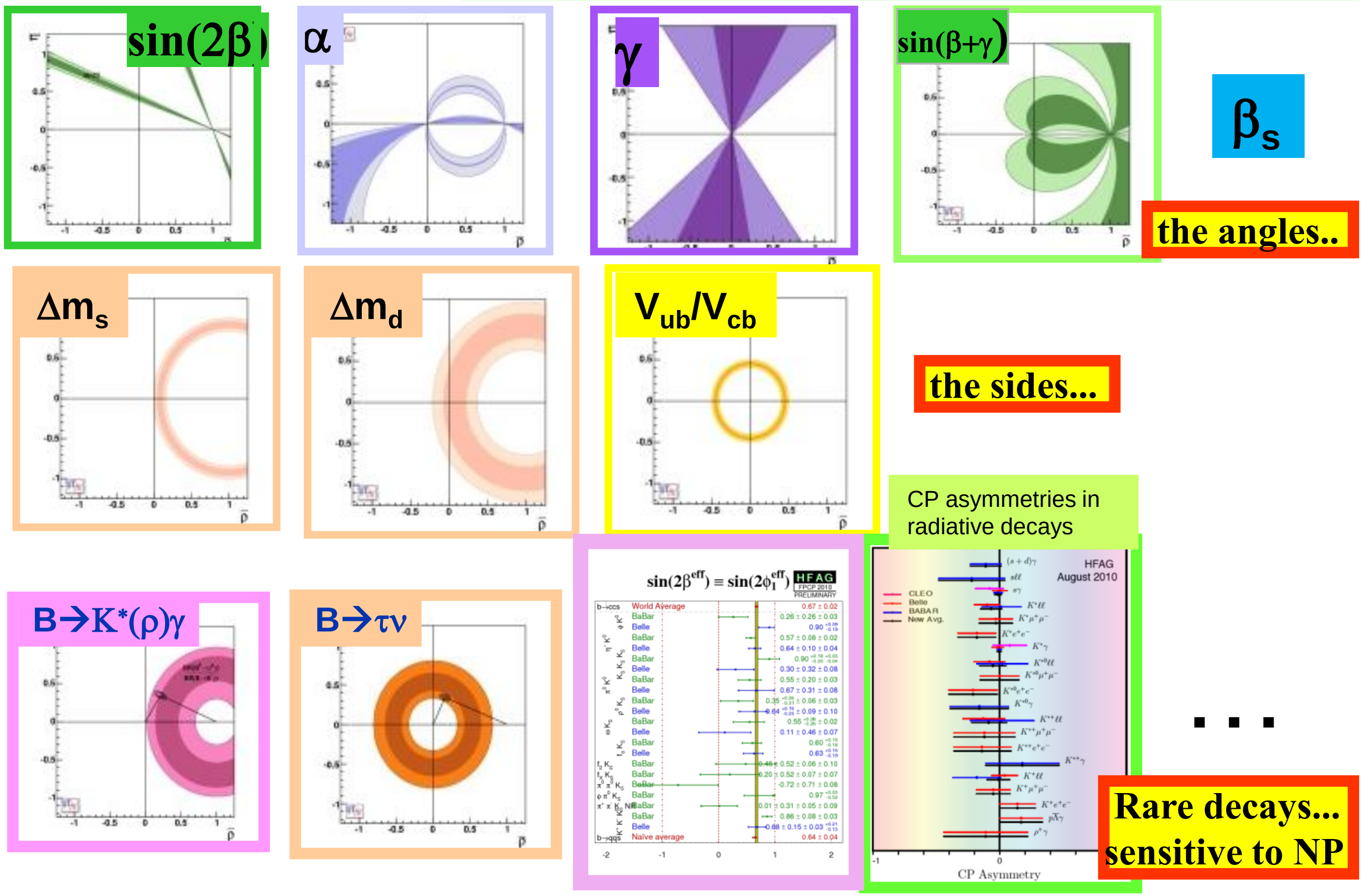
$$r_B = \left|\frac{A(B^- \to \bar{D}^0 K^-)}{A(B^- \to D^0 K^-)}\right|$$

Sensitivity on γ depends strongly (linearly !) on the r_B value…

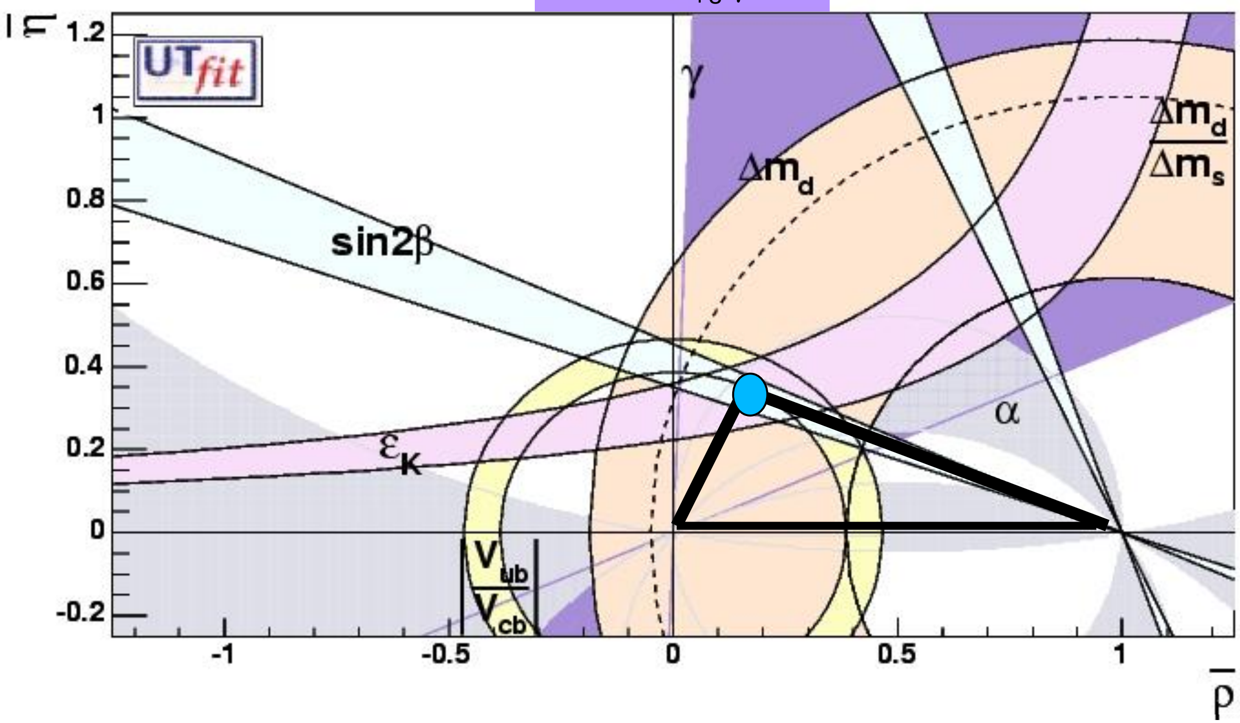
Direct CP violation occurs because there are two different ways of reaching the same final state

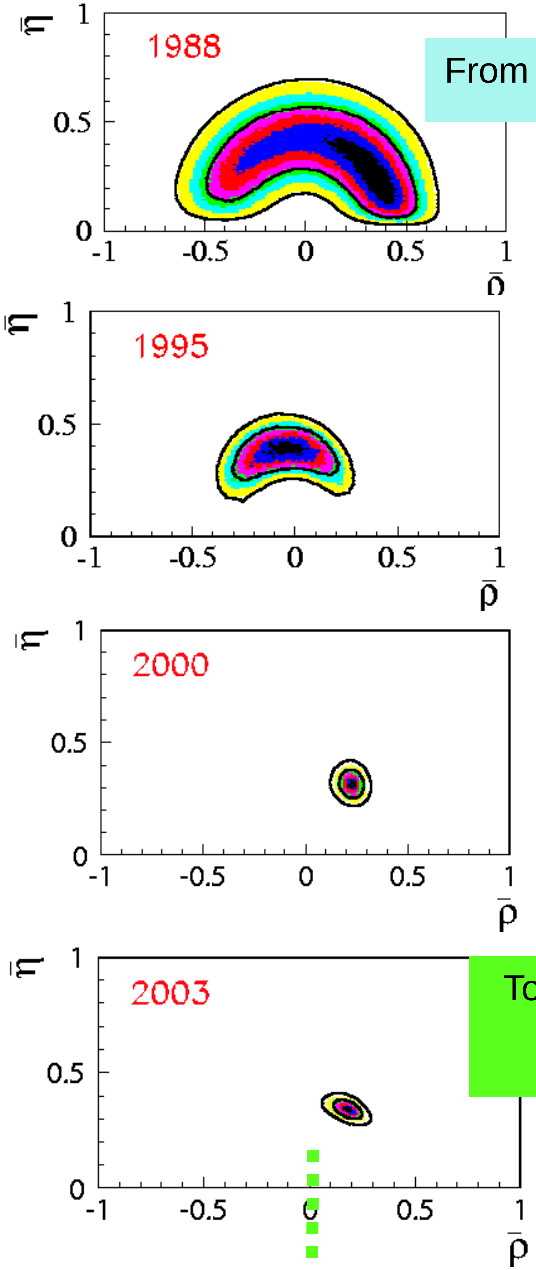
What happened since….

Many new (or more precise) measurements
to constraint UT parameters and test New Physics



B→DK : ϕ_3/γ

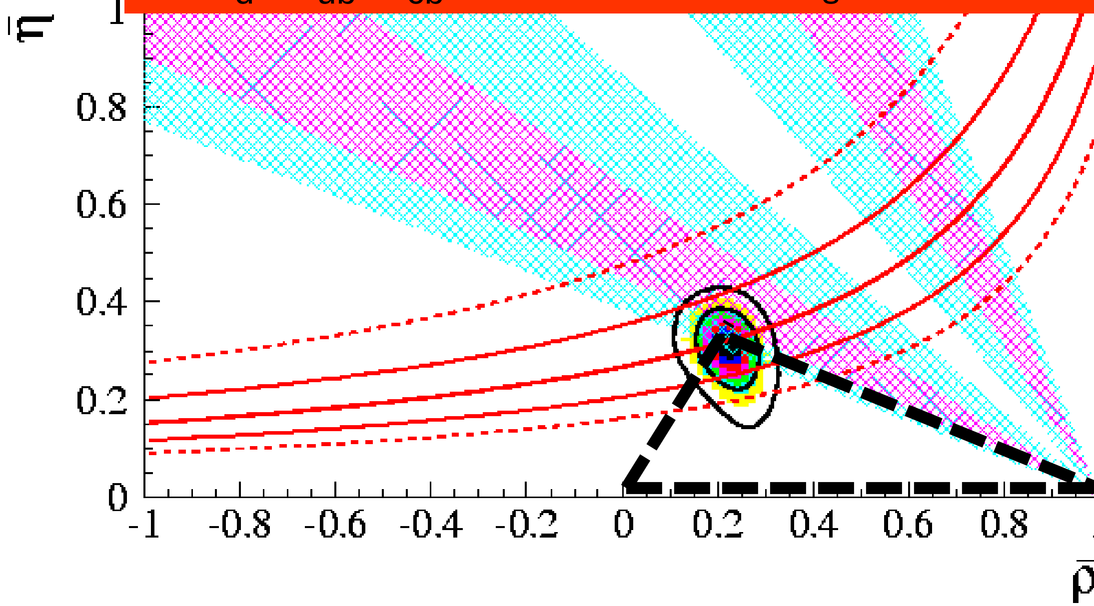




From Childhood

To precision era

Dominated by
Δm_d, V_{ub}, V_{cb}, εK, limit on Δm_s and Lattice

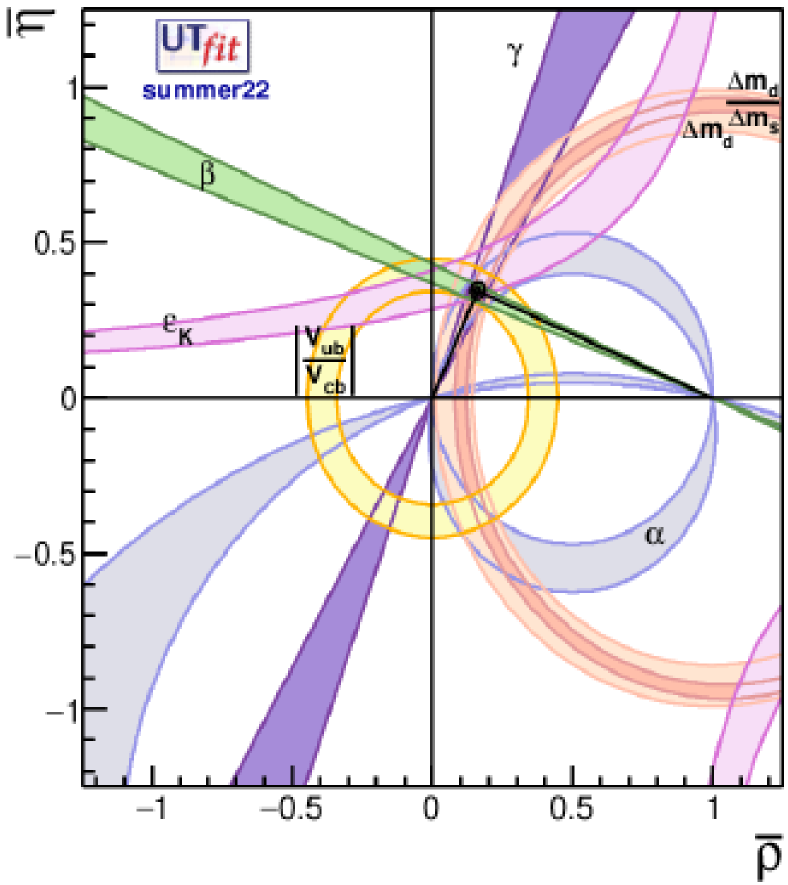


In ~2000 the first fundamental
test of agreement between
direct and indirect measurements of sin2β

Global Fit within the SM

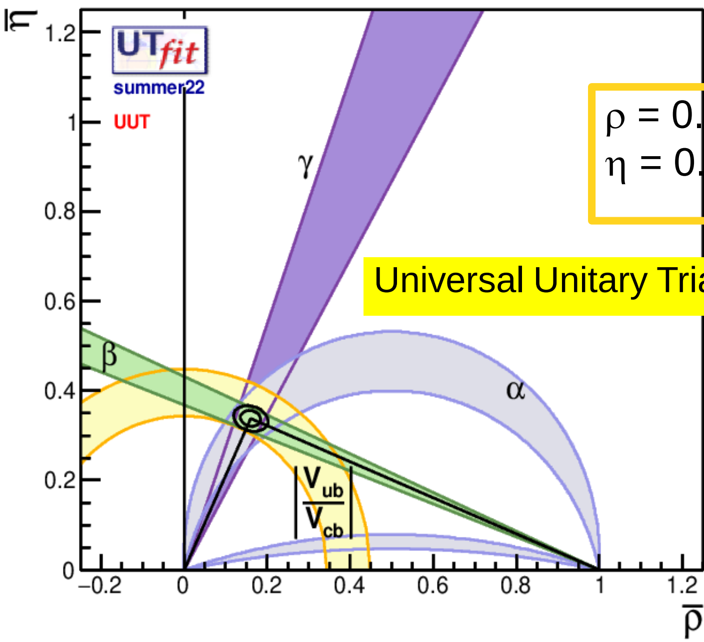
Coherent picture of FCNC and CPV processes in SM

All the constraints Look compatibles !



CKM matrix is the dominant source of flavour mixing and CP violation

Some interesting configurations

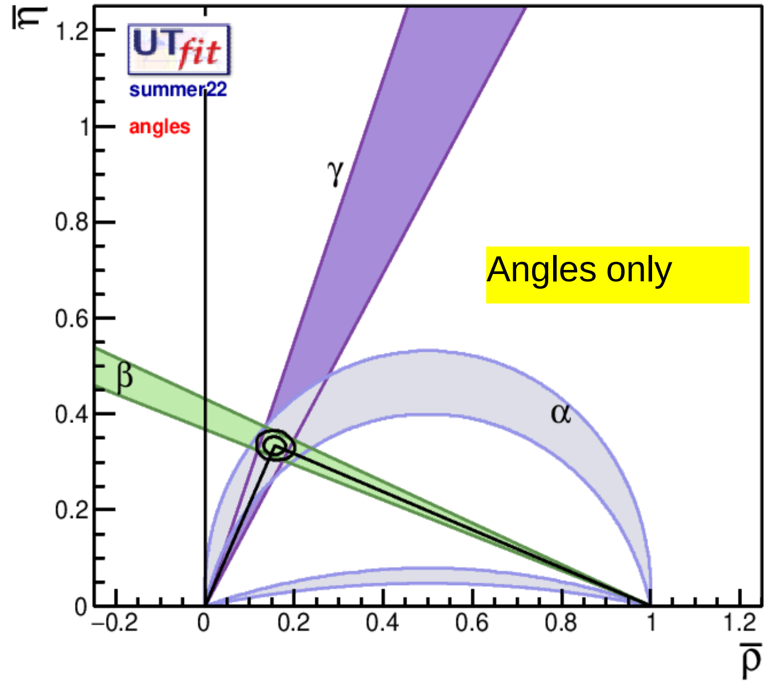


Universal Unitary Triangle

$\rho = 0.160 \pm 0.017$
$\eta = 0.338 \pm 0.011$

~11%

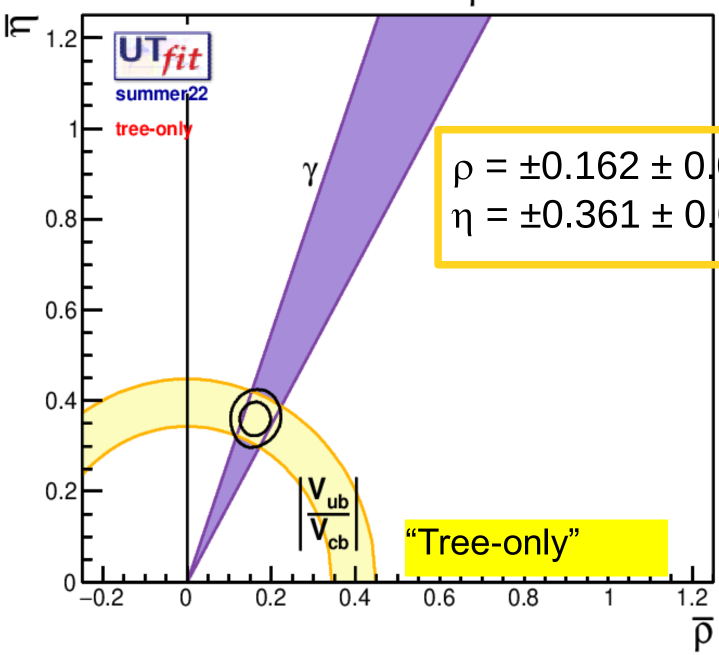
~3%



Angles only

Tree-level processes: Semileptonic and DK B decays

→ reference for model building

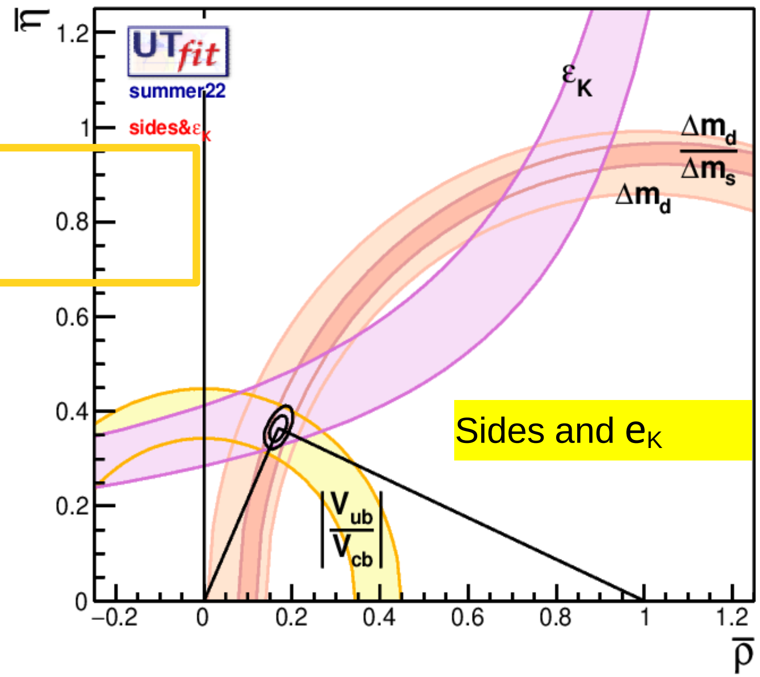


"Tree-only"

$\rho = \pm 0.162 \pm 0.024$
$\eta = \pm 0.361 \pm 0.025$

~15%

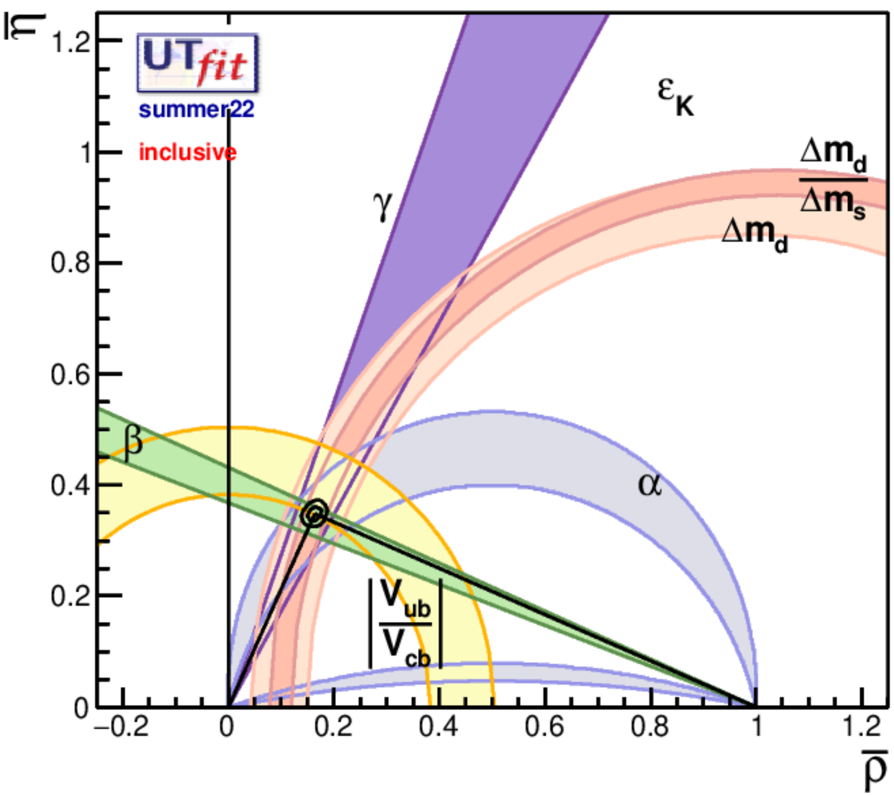
~7%



Sides and e_K

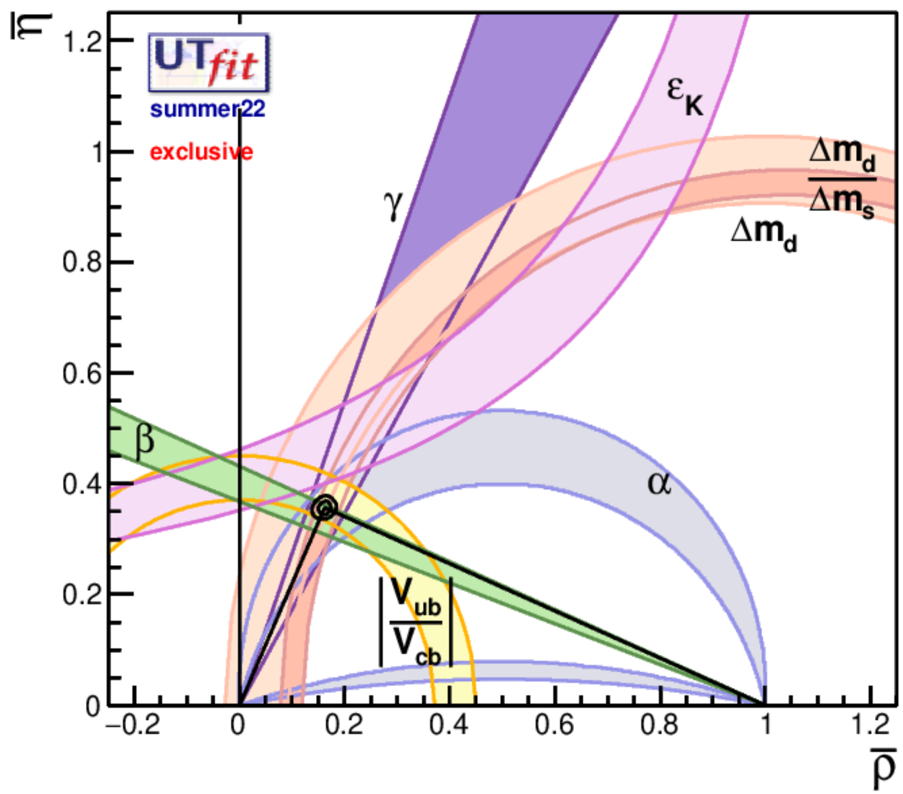
Inclusive vs Exclusive

only inclusive values



$\rho = 0.164 \pm 0.009$
$\eta = 0.348 \pm 0.009$
$\sin2\beta = 0.753 \pm 0.028$

only exclusive values



$\rho = 0.162 \pm 0.009$
$\eta = 0.356 \pm 0.009$
$\sin2\beta = 0.755 \pm 0.020$