Introduction to the Standard Model – part II

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Outline of the lecture

- Lagrangians in particle physics
- The gauge principle
- The Lagrangians of QED and QCD
- Towards Electroweak unification
  1. properties of charged and neutral current interactions
  2. local gauge invariance in the Electroweak $SU(2)_L \otimes U(1)_Y$ theory
  3. structure of the charged current, $W^{\pm}$ bosons
  4. structure of the weak current, the Weinberg rotation, γ and Z.

Lagrangian formalism in field theory

- The Lagrangian formalism allows us to discuss/introduce **symmetries** of a physical system in a straightforward way.
- The dynamics of a physical system is encoded in the Lorentz-invariant **action**

$$S = \int \underbrace{d^4x}_{[M]^{-4}} \underbrace{\mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x))}_{[M]^4}, \quad d^4x = dt\, d^3\vec{x}$$

- The Lagrangian (more accurate: Lagrangian density) is a Lorentz–invariant functional of the fields ϕ_i and their derivatives $\partial_\mu \phi$.
- The principle of **stationary action** requires the variation of the action δS to be zero under small variations $\delta\phi_i, \delta\partial_\mu\phi_i$ of the fields. The condition $\delta S = 0$ determines the Euler–Lagrange equations of motion:

Euler–Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial^\mu \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi_i)} = 0$$

- Knowing the Klein–Gordon and Dirac equations, we can construct the appropriate Lagrangian. They should be constructed as Lorentz-scalars quadratic in the fields.

Lagrangian for Spin-0 particles

- The charged spin-0 particle (e.g. π^+, π^-) is described by a complex scalar field $\phi(x)$. Its Lagrangian reads:

Free Lagrangian for charged spin-0 field

$$\mathcal{L}_0 = \partial^\mu \phi^* \partial_\mu \phi - m^2 \phi^* \phi.$$

- The first term with derivatives is often called the **kinetic** term. The **mass term** is quadratic in the fields. *Pay attention to the sign in front of m^2!*
- You should regard ϕ, ϕ^* as independent variables. Let's take derivatives wrt. ϕ^*:

$$\frac{\partial \mathcal{L}_0}{\partial(\partial^\mu \phi^*)} = \partial_\mu \phi, \quad \frac{\partial \mathcal{L}_0}{\partial \phi^*} = -m^2 \phi \Rightarrow -m^2\phi - \partial^\mu \partial_\mu \phi = 0 \Leftrightarrow (\partial^\mu \partial_\mu + m^2)\phi = 0$$

- The neutral spin-0 particle is described by a real field ϕ. The Lagrangian reads

$$\mathcal{L}_0 = \frac{1}{2}\, \partial^\mu \phi \partial_\mu \phi - \frac{1}{2}\, m^2 \phi^2 \,.$$

- These Lagrangians describe **free fields**. (Self-) interactions will involve higher powers of the field, e.g. by adding a "potential" $V(\phi^* \phi)$.

Lagrangian for Spin-1/2 (Dirac-)particles

- Just like the Dirac equation, we expect the Lagrangian to be **linear** in the four-derivative ∂^μ. It transforms like a Lorentz-vector $\Rightarrow$ Lagrangian must involve the vector $\bar{\psi}\gamma_\mu\psi$, with $\bar{\psi} = \psi^\dagger\gamma_0$.
- Mass m is a Lorentz-scalar $\Rightarrow$ couples to the scalar $\bar{\psi}\psi$.

Lagrangian for the free Dirac-particle

$$\mathcal{L}_0 = \bar{\psi}\Big(i\gamma^\mu\partial_\mu - m\Big)\psi\,.$$

- An important observation: the Lagrangian can be written in terms of the chiral components ψ_L, ψ_R as

$$\mathcal{L}_0 = \bar{\psi}_L i\gamma^\mu\partial_\mu\psi_L + \bar{\psi}_R i\gamma^\mu\partial_\mu\psi_R - m(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L)\,.$$

- For zero mass, left- and right-handed fields **decouple**.
- The mass-term which couples left and right chiralities is called a "Dirac mass". Fundamental Dirac particles in the SM are the charged leptons e, μ, τ and the quarks u, d, s, c, b, t. Neutrinos in the SM come only as left-handed particles and need an extra discussion.
- A (parity violating) theory which involves only one chirality of Dirac particles cannot have a mass term of the Dirac type.

Classical Electrodynamics

- **Photons** are the quanta of the electromagnetic field. Classical electrodynamics comes already as a Lorentz-invariant field theory. You are probably already familiar with its Lagrangian.
- Let's recall the **Maxwell equations**.

$$\vec{\nabla} \cdot \vec{B} = 0, \quad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0,$$
$$\vec{\nabla} \cdot \vec{E} = \rho, \quad \vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{J}.$$

- The first two (homogeneous) equations can be solved by introducing the potentials

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \vec{\nabla} \times \vec{A}.$$

- It is useful to rewrite Maxwell's equations in a *manifestly covariant form*. Firstly $A^\mu = (\phi, \vec{A})$ transforms as a four-vector.
- Lorentz-invariance of electromagnetism requires from us a theory of charged matter for which charge density and current density can be combined into a four–vector current.

$$J^\mu = (\rho, \vec{J})\,.$$

Classical Electrodynamics

- The Lorentz transformation properties fields $\vec{E}, \vec{B}$ allow them to be collected into the antisymmetric **field strength tensor**

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}, \text{ and } \tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}\,.$$

- Then the covariance of the Maxwell equations becomes transparent and simple

$$\partial_\mu F^{\mu\nu} = J^\nu \qquad \partial_\mu \tilde{F}^{\mu\nu} = 0\,.$$

- The conserved current and the *conservation of charge* emerge immediately:

$$\partial_\nu J^\nu = \partial_\nu \partial_\mu F^{\mu\nu} = 0 \Rightarrow \frac{dQ}{dt} = \frac{d}{dt}\int d^3\vec{x}\, J^0 = 0$$

- The wave equation for the 4–potential A^ν reads:

$$\Box A^\nu - \partial^\nu(\partial_\mu A^\mu) = J^\nu \Leftrightarrow (g^{\nu\mu}\Box - \partial^\nu\partial^\mu)A_\mu = J^\nu$$

Lagrangian for A^μ

$$\mathcal{L} = \underbrace{-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}}_{\text{kinetic term}} + \underbrace{A^\mu J_\mu}_{\text{coupling to current}}\,,$$

Gauge transformation

- There is an arbitrariness in the field A^μ. Namely we can perform the **gauge transformation**, which leaves the electric/magnetic field strengths unchanged:

$$A^\mu(x) \to A'^\mu(x) = A^\mu(x) + \partial^\mu \Lambda(x) \quad \Rightarrow F'^{\mu\nu} = F^{\mu\nu}$$

- Taking for example $\partial_\mu A^\mu = 0$ (Lorentz-gauge), we obtain the equation

$$\Box A^\nu = J^\nu$$

- If $J^\nu =$ i.e. there is no external current, we get

$$\Box A^\mu = 0 \quad \Rightarrow \quad \mathrm{Klein-Gordon\,equation\,with}\, m = 0!$$

- All four components satisfy the *massless* Klein-Gordon equation.
- The photon is a massless particle! Electromagnetic fields have long (infinite) range! This is predicted/enforced by gauge invariance.
- The physical photon has only **two** polarization degrees of freedom. Not all four components of A^μ are physical.

Gauge principle and electromagnetic interaction

- **Gauge "symmetry" allows for the introduction of the electromagnetic interaction**
- Consider the Dirac fermion field without interaction.

$$\mathcal{L}_0 = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$

- This Lagrangian is invariant under a "rephasing" of the fermion field

$$\psi \to \psi' = \exp(i\theta)\psi,$$

  with some constant phase θ. We call this a *global $U(1)$ symmetry*.
- **Gauge symmetry** is the requirement of invariance under **local** $U(1)$ transformations. In a gauge theory we can change the phase of the field at **every point** in spacetime without consequences for the physics.

$$\psi \to \psi' = \exp(i\theta(x))\psi \Rightarrow i\partial_\mu \psi \to \exp(i\theta(x))(i\partial_\mu \psi - \partial_\mu \theta \psi) \Rightarrow \mathcal{L}_0 \to \mathcal{L}_0 - \partial_\mu \theta \, \bar{\psi}\gamma^\mu \psi \, .$$

  This is a powerful scheme that **forces us** to introduce a field A^μ (**photon**) which couples to the current $e\bar{\psi}\gamma^\mu\psi$. Add $\mathcal{L}_{\rm int} = eA^\mu \bar{\psi}\gamma_\mu \psi$.
- The photon field (gauge field) transforms as

$$A^\mu \to A^\mu + \frac{1}{e}\partial^\mu \theta$$

- $U(1)$ gauge invariance puts no constraint on the value of the gauge coupling e. In fact we can add an arbitrary number of Dirac fields ψ_i with charges $Q_i e$, transforming as $\psi_i' = \exp(iQ_i\theta(x))\psi$ and interaction terms $Q_i e A^\mu \bar{\psi}_i \gamma_\mu \psi_i$.

More on the gauge principle

- A useful concept is the **covariant derivative**

$$D_\mu \psi(x) = (\partial_\mu - ieA_\mu(x))\psi(x)\,.$$

- It transforms under gauge transformations just like ψ:

$$D_\mu \psi(x) \to \exp(i\theta(x)) D_\mu \psi(x)$$

It can therefore serve as a building block for constructing gauge invariant quantities. For example, we can write the QED Lagrangian as

$$\mathcal{L} = \bar{\psi} i D^\mu \gamma_\mu \psi - m\bar{\psi}\psi = \underbrace{\mathcal{L}_0}_{\text{free Dirac Lagr.}} + e \underbrace{A^\mu \bar{\psi}\gamma_\mu \psi}_{\text{interaction}}$$

- We still need to add a kinetic term for the EM field:

QED Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} i D^\mu \gamma_\mu \psi - m\bar{\psi}\psi$$

- This Lagrangian gives rise to the Maxwell equations with the e.m. fermion current $J^\mu = e\bar{\psi}\gamma^\mu\psi$.
- a mass term for the photon $\propto m_\gamma^2 A^\mu A_\mu$ is **forbidden by gauge invariance!**

- The gauge invariance principle is a powerful scheme for constructing theories with **interactions mediated by vector fields.**
- The interaction term in QED $eA^\mu \bar{\psi}\gamma_\mu \psi$ has the mass dimension $\dim(e) + \dim(A_\mu) + 2\dim(\psi) = \dim(e) + 1 + 2 \cdot 3/2 = \dim(e) + 4$, which means that the em coupling constant e is **dimensionless**!. This is the hallmark of a **renormalizable theory**. In such a theory all divergences that appear in the calculation of Feynman diagrams can be absorbed into a finite number of parameters (masses, couplings) of the theory.
- The SM is a gauge theory with the gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$. Renormalizability, proven by 't Hooft and Veltman is the basis of the SM as a viable theory that makes reliable predictions.
- The strong interactions (Quantum Chromodynamics) are described by the gauge group $SU(3)_c$ introduced by Gell-Mann, Fritzsch and Leutwyler.
- Before we come to the full SM, let us demonstrate the construction of the QCD lagrangian.

- Here we wish to generalize the QED to a gauge theory with the gauge group $SU(N)$. In the SM the cases $N = 2$ (electroweak sector), and $N = N_c = 3$ (QCD) will be relevant.
- We now assume, that our spinor also transforms in some internal space ("color") under a local $SU(N)$:

$$\psi^\alpha(x) \to [\psi'(x)]^\alpha = [U(x)]^\alpha{}_\beta \psi^\beta(x)\,, \quad \alpha, \beta = 1, \dots, N.$$

where $U(x)$ is a unitary $N \times N$ matrix ($U^\dagger U = UU^\dagger = \mathbb{I}$). The derivative of ψ will transform in a complicated way:

$$\partial_\mu \psi \to \partial_\mu(U\psi) = U[\partial_\mu \psi + U^\dagger \partial_\mu U \psi]$$

Just like in QED, to cancel the unwanted piece $U^\dagger \partial_\mu U \psi$, we introduce a **gauge field** $\hat{A}_\mu$, which now is a $N \times N$ matrix as well! We package it into the **covariant derivative**

$$\hat{D}_\mu \psi \equiv [\partial_\mu \mathbb{I} - ig\hat{A}_\mu]\psi\,.$$

We require the covariant derivative to transform as

$$\hat{D}_\mu \psi \to U\hat{D}_\mu \psi$$

and this can be achieved, if the gauge field transforms as

$$\hat{A}_\mu \to U\hat{A}_\mu U^\dagger - \frac{i}{g}(\partial_\mu U)U^\dagger = U\hat{A}_\mu U^\dagger + \frac{i}{g} U\partial_\mu U^\dagger$$

$SU(N)$ gauge theory

- We can already construct the fermionic part of the Lagrangian in a gauge invariant way. Namely $i\bar{\psi}\gamma^\mu \hat{D}_\mu \psi$ is gauge invariant!
- We still need a "kinetic term" for the gauge field. Now, in QED:

$$D_\mu D_\nu - D_\nu D_\mu = -ieF_{\mu\nu} = -ie(\partial_\mu A_\nu - \partial_\mu A_\nu)\,.$$

  We use this to **define** the field strength tensor as

$$-ig\hat{F}_{\mu\nu} = \hat{D}_\mu \hat{D}_\nu - \hat{D}_\nu \hat{D}_\mu = -ig(\partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - ig[\hat{A}_\mu, \hat{A}_\nu]).$$

- It is not gauge invariant as in QED, but has a simple transformation property: $\hat{F}_{\mu\nu} \to U\hat{F}_{\mu\nu}U^\dagger$, so that a gauge invariant Lagrangian is:

$$\mathcal{L}_{\rm YM} = -\frac{1}{2} Tr[\hat{F}_{\mu\nu}\hat{F}^{\mu\nu}].$$

- Here *YM* stands for Yang and Mills who were the first to construct a gauge theory of ρ–mesons, gauging in effect the isospin group $SU(2)$. The $SU(N)$ Yang–Mills theory is a theory purely of gluons.
- It turns out, that the "kinetic" term of the nonabelian gauge theory already contains interactions!

$SU(N)$ gauge theory

- Let us try to understand better the degrees of freedom contained in the $N \times N$ matrix $\hat{A}_\mu$.
- You may recall, that we can write any element of $SU(2)$ as an exponential

$$U = \exp\left(i\frac{\sigma^k}{2}\theta^k\right) \approx \mathbb{I} + i\frac{\sigma^k}{2}\theta^k, \quad k = 1,2,3, \quad \left[\frac{\sigma^k}{2}, \frac{\sigma^l}{2}\right] = i\epsilon_{klm}\frac{\sigma^m}{2}\,.$$

- In the same way, for a general $SU(N)$ group, we write

$$U = \exp\left(it^a\theta^a\right) \approx \mathbb{I} + it^a\theta^a, \quad k = 1,\dots,N^2-1, \quad [t^a, t^b] = if_{abc}t^c\,.$$

- We now write our gauge field as $\hat{A}_\mu = \sum_a A^a_\mu\, t^a$. Here $a = 1,\dots,N^2-1$ is the color index of the gauge field. In QCD quarks have $N = N_c = 3$ colors, and we have $N_c^2 - 1 = 8$ gluon fields.

QCD Lagrangian

$$\begin{aligned}\mathcal{L}_{\rm QCD} = \quad & \sum_f \bar{q}^\alpha_f(i\gamma^\mu\partial_\mu - m_f)q^\alpha_f + gA^a_\mu\sum_f \bar{q}^\beta_f\gamma^\mu(t^a)_{\beta_\alpha}q^\alpha_f \\ & -\frac{1}{4}(\partial^\mu A^{a\nu} - \partial^\mu A^{a\mu})(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) \\ & -\frac{1}{2}gf_{abc}(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu})A^b_\mu A^c_\nu + \frac{1}{4}g^2 f_{abc}f_{ade}A^{a\mu}A^{b\nu}A^c_\mu A^d_\nu\,.\end{aligned}$$

Feynman diagrams: vertices of QCD

- In QCD with $N_c = 3$, one uses as generators $t^a = \lambda^a/2$, the Gell-Mann matrices, $a = 1, \ldots, 8$.

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$\lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_8 = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{-2}{\sqrt{3}} \end{pmatrix}$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Figure: Gell-Mann Matrices, $\lambda^a/2$ generate $SU(3)$

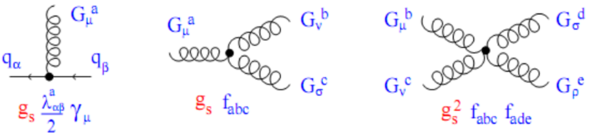
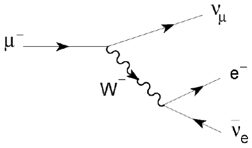


Figure: Interaction vertices in QCD

Towards electroweak unification

- To understand the need for the gauge structure of electroweak currents, one should start from the description of the observed phenomena. See for example the textbook *Leptons and Quarks by L.B. Okun, North Holland Publishing (1982)*.
- Here we enter the electroweak theory "through the roof", by constructing weak quark and lepton currents positing the relevant gauge group. We follow the presentation in the lecture notes of Alexander Korchin presented at TESHEP many times.
- A lot of empirical information is known about the dynamics of flavor changing weak processes:
  1. neutron β decay: $n \to p e^- \bar{\nu}_e$.
  2. Muon decay: $\mu^- \to e^- \bar{\nu}_e \nu_\mu$.
  3. charged pion decay: $\pi^- \to \mu^- \bar{\nu}_\mu$.
  4. neutrino scattering data ("inverse β-decay"): $\bar{\nu}_e p \to e^+ n, \nu_e n \to e^- p, \bar{\nu}_\mu p \to \mu^+ n, \nu_\mu n \to \mu^- p$.
  5. absence of processes like: $\nu_e p \nrightarrow e^+ n, \bar{\nu}_e n \nrightarrow e^- p, \bar{\nu}_\mu p \nrightarrow e^+ n, \nu_\mu n \to e^- p$.
- Observations:
  1. only left–handed fermion and right–handed antifermion chiralities participate.
  2. universal strength of the interaction
  3. neutrinos are left–handed (ν_L) and anti–neutrinos are right–handed (ν_R).
  4. existence of different neutrino types $\nu_e \neq \nu_\mu \neq \nu_\tau$.
  5. there are *separately conserved* lepton numbers L_e, L_μ, L_τ, which distinguish leptons from antileptons, neutrinos from antineutrinos.

- The low energy information and the absence of flavor changing neutral currents (FCNC) transitions like $\mu^- \to e^- e^- e^+$ was sufficient to determine the structure of the Electroweak (EW) theory.
- The intermediate vector bosons $W^{\pm}$ and Z were theoretically introduced and their masses correctly estimated, before their experimental discovery at the CERN SPS in 1983.
- The Z boson was thoroughly studied at the CERN e^+e^- collider LEP. Nowadays $W^{\pm}$ and Z bosons are copiously produced at the LHC and we know much about their properties.



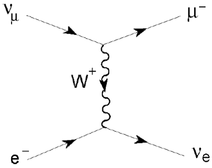


Figure: Processes with $W^{\pm}$ exchange: muon decay $\mu^- \to e^- \bar{\nu}_e \nu_\mu$ and a charge exchange reaction $\nu_\mu e^- \to \mu^- \nu_e$.

$W^{\pm}$ boson interactions

- Only left–handed fermions and right–handed antifermions couple to the $W^{\pm}$. **This means a 100% breaking of two discrete symmetries**: Parity P, which swaps left–handed to right–handed, and Charge conjugation C, which exchanges particles to antiparticles.
- The combined transformation CP, which relates a particle in a right-handed coordinate system to an antiparticle in a left–handed system appears to be conserved at this point. It later also turns out to be violated, but much weaker than P and C alone.
- The $W^{\pm}$ bosons couple to fermionic doublets where the charges of the partners in the doublet differ by one unit (e.g. ν_e and e^-).
- All fermion doublets couple to $W^{\pm}$ bosons with the same universal strength.
- Decay channels of the W^-–boson:

$$W^- \to e^- \bar{\nu}_e, \mu^- \bar{\nu}_\mu, \tau^- \bar{\nu}_\tau, d' \bar{u}, s' \bar{c}.$$

  The "on-mass shell" production of the t–quark is kinematically forbidden, since $m_t = 173\,\mathrm{GeV} > M_W = 80.4\,\mathrm{GeV}$.
- What do we mean by d', s'?

Weak vs. mass eigenstates

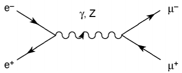
- The weak doublet partners of u, c, t quarks appear to be mixtures of the three quarks with charge $-1/3$:

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \mathrm{V} \begin{pmatrix} d \\ s \\ b \end{pmatrix}, \quad \mathrm{V}^\dagger \mathrm{V} = \mathrm{V}\mathrm{V}^\dagger = \mathbb{I}$$

  The so-called weak eigenstates d', s', b' are different from the mass eigenstates d, s, b. For two families V contains the well-known Cabibbo-angle. For three families V is the CKM (Cabibbo-Kobayashi-Maskawa) matrix and contains the key to CP violation in the SM.

- The experimental evidence for neutrino oscillations shows that the weak eigenstates ν_e, ν_μ, ν_τ are mixtures of mass eigenstates ν_1, ν_2, ν_3. A corresponding mixing matrix (PMNS, Pontecorvo-Maki-Nakagawa-Sakata) exists. However neutrino masses are tiny $|m^2_{\nu_3} - m^2_{\nu_2}| \sim 2.5 \cdot 10^{-3}\,\mathrm{eV}$, $m^2_{\nu_2} - m^2_{\nu_1} \sim 8 \cdot 10^{-5}\,\mathrm{eV}$

Properties of the neutral current



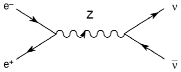


Figure: Feynman diagrams for $e^+e^- \to \mu^+\mu^-$ and $e^+e^- \to \nu\bar{\nu}$

- All interaction vertices of the neutral current are **flavor conserving**.
- γ and Z couple to a fermion and its antifermion, i.e. $\gamma f\bar{f}$ and $Zf\bar{f}$.
- Interactions depend on the fermion electric charge Q_f, and fermions with the same Q_f have exactly the same universal couplings.
- Neutrinos being neutral do not couple to γ but they do couple to Z: $Z\nu\bar{\nu}$.
- Photons couple equally to left- and right-handed fermions, while for Z bosons the coupling strengthes differ (parity violation!).
- Only the left-handed neutrino couples to the Z.
- Experiments at the Z resonance pole reveal that there are three different light neutrino species.

Local invariance in the $SU(2)_L \otimes U(1)_Y$ theory

- How to accomodate all these properties of charged and neutral electroweak currents? (Glashow, Salam, Weinberg). It must be more elaborate than QED or QCD:
  1. different properties of left- and right-handed fermions (left handed fermions in doublets (ν_e, e^-) etc.
  2. massive gauge bosons $W^\pm$, Z in addition to a massless photon.
- The simplest group with doublet representations is $SU(2)$. It has three generators, so we will get three gauge bosons. As we want to have four, we would need an additional $U(1)$, remember the photon had an $U(1)_{\rm em}$ gauge symmetry. The proposed gauge group is the product:

$$G = SU(2)_L \otimes U(1)_Y$$

- Let us consider a single family of quarks and leptons, and introduce the notation:

$$\psi_1(x) \equiv Q_L(x) = \begin{pmatrix} u \\ d \end{pmatrix}_L, \quad \psi_2(x) = u_R, \quad \psi_3(x) = d_R\,.$$

  and for the leptons:

$$\psi_1(x) \equiv L = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L, \quad \psi_2(x) = \nu_{eR}, \quad \psi_3(x) = e^-_R\,.$$

Lagrangian of free quarks: global symmetry

- Let us start with the **free Lagrangian** for just one quark family:

$$\mathcal{L}_0 = \bar{Q}_L i\gamma^\mu \partial_\mu Q_L + \bar{u}_R i\gamma^\mu \partial_\mu u_R + \bar{d}_R i\gamma^\mu \partial_\mu d_R = \bar{u} i\gamma^\mu \partial_\mu u + \bar{d} i\gamma^\mu \partial_\mu d = \sum_i \bar{\psi}_i \gamma^\mu \partial_\mu \psi_i .$$

- $\mathcal{L}_0$ is invariant under the following **global** transformations in flavor space:

$$\begin{aligned} \psi_1 &\xrightarrow{G} \psi_1' = \exp(iy_1\beta)\, U_L\, \psi_1 \quad \text{doublet} \\ \psi_2 &\xrightarrow{G} \psi_2' = \exp(iy_2\beta)\, \psi_2 \quad \text{singlet} \\ \psi_2 &\xrightarrow{G} \psi_3' = \exp(iy_3\beta)\, \psi_3 \quad \text{singlet}\,. \end{aligned}$$

- Here the y_i are called **hypercharges** and are the charges under the $U(1)_Y$ symmetry. The $SU(2)L$ transformation U_L acts only on the doublet fields:

$$U_L = \exp\left(i\frac{\sigma^i}{2}\alpha^i\right).$$

  There are in total 4 $\beta, \alpha^i, i = 1, 2, 3$ gauge parameters.

- We have not added a mass term like $-m_u \bar{u} u = -m_u(\bar{u}_R u_L + \bar{u}_L u_R)$, because it mixes left and right handed fields and **breaks the symmetry**!

Local gauge invariance under $SU(2)_L \otimes U(1)_Y$

- We now require the Lagrangian to be invariant under **local** $SU(2)_L \otimes U(1)_Y$ transfromations. As we have learned we should replace ordinary derivatives by **covariant derivatives**, and as we have four gauge parameters, we require four gauge bosons:

$$D_\mu \psi_1(x) = \Big(\partial_\mu + ig\hat{W}_\mu(x) + ig' y_1 B_\mu(x)\Big)\psi_1(x)$$
$$D_\mu \psi_2(x) = \Big(\partial_\mu + ig' y_2 B_\mu(x)\Big)\psi_2(x)$$
$$D_\mu \psi_3(x) = \Big(\partial_\mu + ig' y_3 B_\mu(x)\Big)\psi_3(x)$$

where g and g' are the gauge coupling constants, and

$$\hat{W}_\mu = \frac{\sigma^i}{2} W^i_\mu = \frac{1}{2}\begin{pmatrix} W^3_\mu & W^1_\mu - iW^2_\mu \\ W^1_\mu + iW^2_\mu & -W^3_\mu \end{pmatrix}, \quad (i = 1, 2, 3)\,.$$

- Now, the four gauge fields W^1, W^2, W^3, B should be related to the gauge bosons $W^\pm, Z, \gamma$, but *this relation is not straightforward*.
- As usual, we introduce the gauge-field–fermion interaction by the covariant derivative

$$\mathcal{L}_0 \to \mathcal{L} = \sum_{j=1}^{3} i\bar{\psi}_j(x)\gamma^\mu D_\mu \psi_j(x) = \mathcal{L}_0 + \mathcal{L}_{\rm int}\,.$$

Lagrangian of the $SU(2)_L \otimes U(1)_Y$ theory

- You already know how to construct the gauge-field kinetic term and self–interactions using the covariant derivatives. The corresponding field-strength tensors read

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$
$$\hat{W}_{\mu\nu} = \partial_\mu \hat{W}_\nu - \partial_\nu \hat{W}_\mu + ig[\hat{W}_\mu, \hat{W}_\nu]\,.$$

or

$$\hat{W}_\mu = \sum_{i=1}^{3} \frac{\sigma^i}{2} W^i_\mu \quad \Rightarrow \quad W^i_{\mu\nu} = \partial_\mu W^i_\nu - \partial_\nu W^i_\mu - g\epsilon_{ijk} W^j_\mu W^k_\nu\,.$$

- Then the gauge field part of our Lagrangian is

$$\mathcal{L}_{\rm gf} = -\frac{1}{4} B^{\mu\nu} B_{\mu\nu} - \frac{1}{2} Tr(\hat{W}_{\mu\nu}\hat{W}^{\mu\nu}) = -\frac{1}{4} B^{\mu\nu} B_{\mu\nu} - \frac{1}{4} \sum_{i=1}^{3} W^i_{\mu\nu} W_i^{\mu\nu}\,.$$

- Gauge symmetry forbids a mass term for the gauge bosons.
- Fermion mass terms are also not possible, because the left- and right-handed fields have different transformation properties under the gauge group!
- In conclusion: the $SU(2)_L \otimes U(1)$ Lagrangian describes only *massless* fermions and gauge bosons.
- **Bad news!** ... but the problem will be solved later by the Higgs mechanism.