IVANE JAVAKHISHVILI TBILISI STATE UNIVERSITY

Spherically Symmetric Solutions Of Cotton Gravity

First-year master's student

Vitali Dididze

Supervisor : Achille Stocchi

Dark Matter Problem

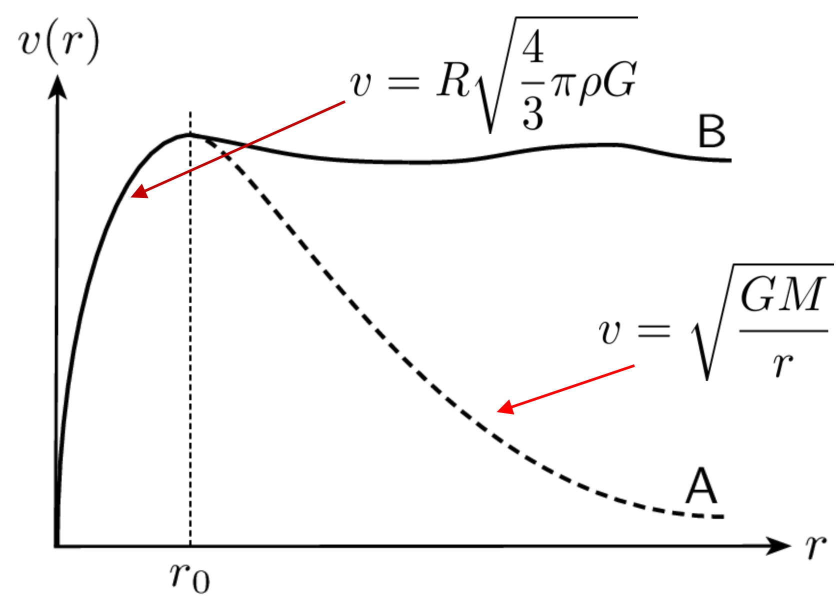
One of the indicators of the existence of dark matter is the galaxy rotation curves.

- If there were no hidden mass, according to classical physics, rotation curves would look like A, but because of the dark matter halos around the galaxies, they appear different $v \approx constant$

Another indication of hidden mass is gravitational lensing angles.

$$\theta = \frac{4GM}{c^2}\frac{1}{b}$$

Calculated by this method, the masses of galaxies are much higher than expected. They far exceed the mass of observed baryonic matter through the galaxies



Alternative Model Of Gravitation

Cotton gravity is a MOND (Modified Newtonian dynamics) type theory.

A new term is added to the gravitational potential that depends linearly on the distance from the point source.

$$\Phi = -\frac{GM}{r} + \frac{\gamma r}{2} \tag{1}$$

Cotton field equations

$$C^{\sigma}{}_{\nu\mu} = 8\pi G\left[\nabla_{\mu}T^{\sigma}_{\nu} - \nabla_{\nu}T^{\sigma}_{\mu} - \frac{1}{3}\left(\delta^{\sigma}_{\nu}\nabla_{\mu}T - \delta^{\sigma}_{\mu}\nabla_{\nu}T\right)\right] \tag{2}$$

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

- Every solution of Einstein's equations satisfies the equations of the new model, both with and without the cosmological term.
- The cosmological term is introduced as an integration constant.

J. Harada, "*Emergence of the Cotton tensor for describing gravity,*" Phys. Rev. D (2021)

Cotton Tensor

The main mathematical object of this theory is the rank 3 Cotton tenor, (Émile Cotton; 1872-1950) which is related to Weyl and Riemann tensors.

$$C_{\nu\rho\sigma} = 2\nabla_\mu W^\mu{}_{\nu\rho\sigma} = \nabla_\rho R_{\nu\sigma} - \nabla_\sigma R_{\nu\rho} - \frac{1}{6}(g_{\nu\sigma}\nabla_\rho R - g_{\nu\rho}\nabla_\sigma R) \quad (3)$$

Properties:

- Zero derivative : $\nabla^\nu C_{\nu\rho\sigma} = 0$
- Zero Trace: $g^{\nu\rho} C_{\nu\rho\sigma} = 0$
- Antisymmetric : $C_{\nu\rho\sigma} + C_{\nu\sigma\rho} = 0$
- cyclic permutations : $C_{\nu\rho\sigma} + C_{\rho\sigma\nu} + C_{\sigma\nu\rho} = 0$ (4)

Conservation of Energy-Momentum

$$\longrightarrow \nabla^\nu C_{\nu\rho\sigma} = 16\pi G \nabla_\mu T^\mu_{\nu\rho\sigma} = 0$$

$$\nabla_\mu T^\mu{}_\rho = 0$$

Vacuum equation: $C_{\nu\rho\sigma} = 0$ (5)

É. **Cotton**, „*Sur les variétés à trois dimensions,*“ Annales de la Faculté des Sciences de Toulouse (1899)

Schwarzschild-like metric

A spherically symmetric Schwarzschild-like solution of (5) is given by:

$$ds^2 = -e^{\nu(r)}dt^2 + e^{-\nu(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (5)$$

Where: $$-g_{00} = 1/g_{11} = e^{\nu(r)} = \underbrace{1 - \frac{2MG}{r} - \frac{\Lambda}{3}r^2}_{\text{Schwarzschild-de Sitter metric}} + \underbrace{\gamma r}_{\text{New linear term}} \quad (6)$$

- This extra linear term has been successfully used to describe the galaxy rotation curves without the dark matter
- γ and Λ are integration constants and they can be estimated using the radius of the observable universe (Hubble horizon).

For example, In case of the solar system: $\gamma = 1.9 \times 10^{-26} m^{-1}$

J. Harada, "*Emergence of the Cotton tensor for describing gravity,*" Phys. Rev. D (2021)

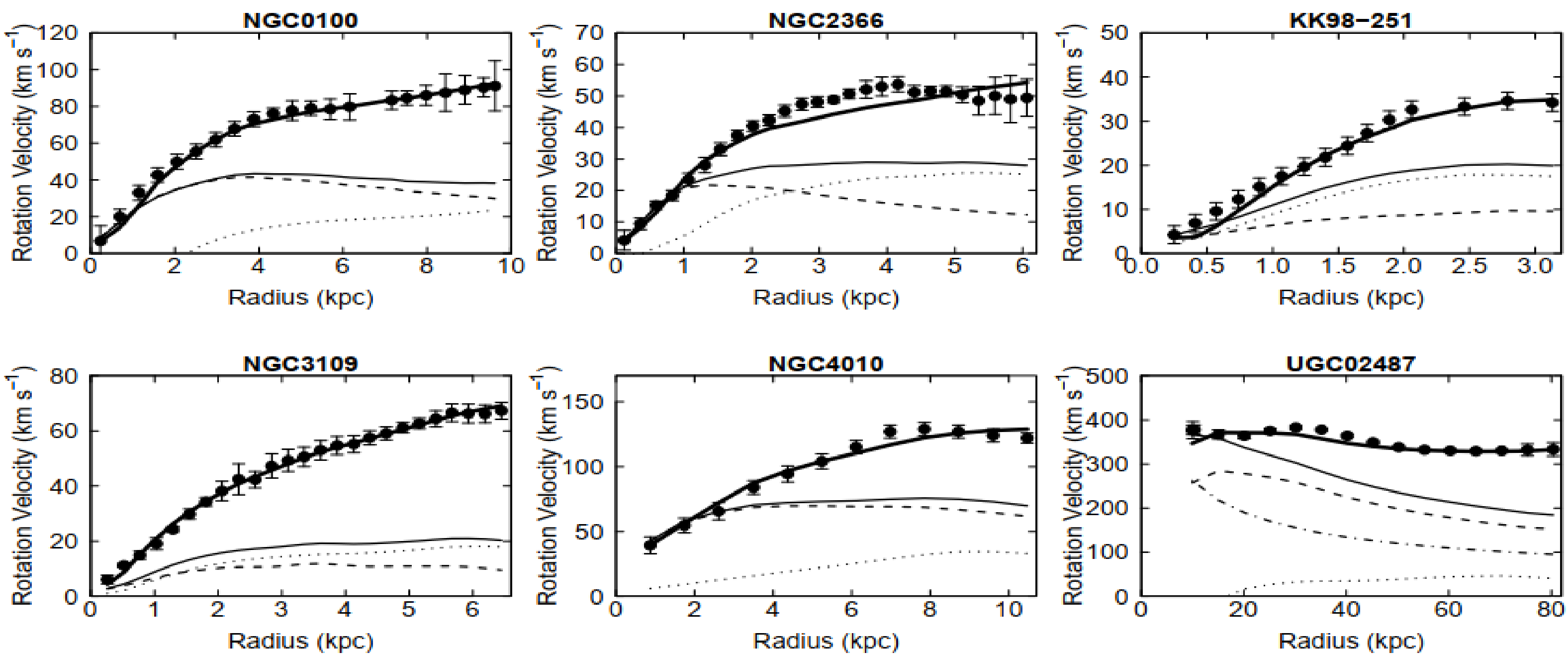
Galaxy Rotation Curves

The equivalent of Poisson's equation in this theory can be expressed as:

$$\left[\nabla^2 - \frac{1}{3|\boldsymbol{r}|^2}\left((\boldsymbol{r}\cdot\nabla)^2 + 4(\boldsymbol{r}\cdot\nabla) + 2\right)\right]\Phi = \frac{16\pi G}{3}\rho \tag{7}$$

And in spherically symmetric case this can be solved to obtain

$$\Phi = -\frac{GM}{r} + \frac{\gamma}{2}r \tag{8}$$



J. Harada, *"Cotton gravity and 84 galaxy rotation curves,"* Phys. Rev. D (2022)

General spherically symmetric vacuum solution

We can consider more general case where $g_{tt}(r) \neq -1/g_{rr}(r)$ is not satisfied

$$ds^2 = -e^{\alpha(r)}dt^2 + e^{-\beta(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{9}$$

If we take previous expression for α this equation can be solved analytically for second unknown function β

We get a large expression which in large distances, where $\frac{2mG}{r} \to 0$ and $\gamma r \leq 1$, can be reduced to

$$\beta = \frac{2GM}{r} + \gamma r + C_2 r^2 \tag{10}$$

Our goal is to fully, analyze the solutions in different limits and their cosmological application, in particular fitting to galaxy rotation curves. We also plan to solve field equations of Cotton gravity in different metrics.

Gogberashvili & Girgvliani, *"General spherically symmetric solution of Cotton gravity,"* CQG (2024)

Velocity squared force

The appearance of a term Cr^2 eventually leads to the velocity dependent part in geodesic equation.

For geodesic equation under the assumption of static space-time and weak gravity

$$\frac{d^2x^i}{dt^2} \approx -\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\Gamma^i{}_{\mu\nu} \approx -\Gamma^i{}_{00} - \Gamma^i{}_{jk}v^j v^k$$

We get what in the spherically symmetric case of Cotton gravity we get additional v^2 dependent term in radial acceleration

$$a_r \approx \frac{1}{2}\left(g'_{00} - g'_{rr}v_r^2\right) \approx -\frac{Gm}{r^2} + \frac{32\pi G\varepsilon}{9} r\, v_r^2$$

Where this new repulsive long-range correction to Classical gravitational force, which in principle could provide explanation for observed acceleration of the universe.

K. Loeve, K. S. Nielsen and S. H. Hansen, "Consistency analysis of a Dark Matter velocity dependent force as an alternative to the Cosmological Constant," Astrophys. J. (2021).

Summary

- Cotton Gravity inherently includes general relativity
- Galaxy rotation curves have been successfully described by Cotton Gravity using a linear term in the Schwarzschild solution (Dark Matter)
- Schwarzschild-like solution gives us many interesting results such as r^2 dependent term in metric. Also using different approximations, we can calculate that there should be naked singularity on black hole photosphere
- Because of the r^2 term, we get long-range repulsive force that is dependent on velocity squared (Dark Energy)
- By using more general metrics, we can obtain results that could have significant implications on cosmological scales.

Thank you
for your attention

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One approach to develop alternative model of gravity, without fundamentally revising existing theory, involves using the once-contracted differential Bianchi identity:

$$\nabla_\alpha R^\alpha{}_{\beta\nu\mu} = \nabla_\mu R_{\nu\beta} - \nabla_\nu R_{\mu\beta} \ , \tag{1}$$

where ∇_μ denotes the covariant derivative associated with the Levi-Civita connection. By replacing the Ricci tensors at the right side using the standard Einstein equations,

$$R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) \ , \tag{2}$$

we obtain the third order to the metric tensor differential equations (first order with respect of the Riemann tensor). If additionally we express the Riemann tensor by the Weyl tensor, $W_{\alpha\beta\nu\mu}$, the condition (1) obtains the form of so-called quasi-Maxwellian equations of gravity:

$$\nabla^\alpha W_{\alpha\mu\nu\sigma} = 4\pi G \, M_{\sigma\mu\nu} \ , \tag{3}$$

where the gravitational 'current',

$$M_{\sigma\mu\nu} = \nabla_\mu \left(T_{\sigma\nu} - \frac{1}{3} \, g_{\sigma\nu} T \right) - \nabla_\nu \left(T_{\sigma\mu} - \frac{1}{3} \, g_{\sigma\mu} T \right) \ , \tag{4}$$

is covariantly conserved quantity,

$$\nabla^\sigma M_{\sigma\mu\nu} = 0 \ . \tag{5}$$

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TABLE I. Parameters for galaxies. Column (1) gives the galaxy name. Column (2) gives the numerical Hubble type adopting the following scheme: 0 = S0, 1 = Sa, 2 = Sab, 3 = Sb, 4 = Sbc, 5 = Sc, 6 = Scd, 7 = Sd, 8 = Sdm, 9 = Sm, 10 = Im. Column (3) gives the assumed distance. Column (4) gives the assumed inclination angle (i). The parenthesis shows the error. Columns (3) and (4) are obtained from the SPARC [12]. Column (5) gives the constant in Eq. (24). Column (6) gives the mass-to-light ratio at 3.6 μm band ($\Upsilon_\star$) of the stellar disk. Column (7) gives the baryonic mass ($M_{\rm bar}$). The baryonic mass is a sum of the stellar mass $M_\star$ and the gas mass $M_{\rm gas}$. The stellar mass is a sum of the masses of the stellar disk $M_{\rm disk}$ and the central bulge $M_{\rm bulge}$. Column (8) gives the disk fraction ($f_{\rm disk} = M_{\rm disk}/M_{\rm bar}$). Column (9) gives the bulge fraction ($f_{\rm bulge} = M_{\rm bulge}/M_{\rm bar}$). The mass-to-light ratio for the bulge is assumed to be $1.4\Upsilon_\star$ for all galaxies with the bulge. Column (10) gives the gas fraction ($f_{\rm gas} = M_{\rm gas}/M_{\rm bar}$). Gas mass is estimated by $M_{\rm gas} = 1.33 M_{\rm HI}$, where $M_{\rm HI}$ is the H I mass and 1.33 represents an enhancement factor to account for the cosmic abundance of helium. Columns (5), (6), (7), (8), (9), and (10) are obtained in this work. Column (11) gives the references for the radial H I surface density profiles ($\Sigma_{\rm HI}$) used in this work: Al15 [13], An22 [14], Ba05 [15], Ba06 [16], BC04 [17], BW94 [18], Ca90 [19], CB89 [20], Co91 [21], Co00 [22], CP90 [23], Fr02 [24], Fr11 [25], Ga02 [26], Ge04 [27], Ha14 [28], Ho01 [29], JC90 [30], Ke07 [31], Le14 [32], MC94 [33], No05 [34], Rh96 [35], SG06 [36], Sw02 [37], VH93 [38], VS01 [39].

Name	Type	D	i	$\gamma_{\rm galaxy}/2$	$\Upsilon_\star$	$M_{\rm bar}$	$f_{\rm disk}$	$f_{\rm bulge}$	$f_{\rm gas}$	Ref.
		(Mpc)	(°)	(km^2 s^{-2} pc^{-1})	($M_\odot/L_\odot$)	($10^9 M_\odot$)				
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
DDO064	10	6.80	60(5)	0.75	0.50	0.391	0.208	0.000	0.792	Sw02
DDO154	10	4.04	64(3)	0.39	0.65	0.385	0.084	0.000	0.916	CB89
DDO161	10	7.50	70(10)	0.28	0.34	2.778	0.066	0.000	0.934	Co00
DDO168	10	4.25	63(6)	0.67	0.50	0.432	0.212	0.000	0.788	Ho01
DDO170	10	15.40	66(7)	0.31	0.50	1.033	0.244	0.000	0.756	Ho01
ESO079-G014	4	28.70	79(5)	1.38	0.67	37.932	0.886	0.000	0.114	Ge04
ESO116-G012	7	13.00	74(3)	1.15	0.77	4.837	0.683	0.000	0.317	Ge04
ESO444-G084	10	4.83	32(6)	1.20	0.50	0.209	0.168	0.000	0.832	Co00

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$$ds^2 = [1 - A(r)]\, dt^2 - \frac{1}{[1 - B(r)]} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \ .$$

$$B(r) = \frac{r^4 \left(\Lambda r^3 - \gamma r^2 - 2Gm + r\right) C_2 - \left(3\gamma r^2 + 9Gm - 4r\right) \left(\Lambda r^3 - \gamma r^2 - 2Gm + r\right) C_1}{r \left(2r - \gamma r^2 - 6Gm\right)^2} +$$
$$+ \frac{\left(51\gamma^2\Lambda + 68\Lambda^2 + 60\gamma\Lambda^2 Gm\right) r^7 - \left(51\gamma^3 + 60\Lambda Gm\gamma^2\right) r^6}{17r \left(2r - \gamma r^2 - 6Gm\right)^2} -$$
$$- \frac{\left(120\Lambda G^2m^2\gamma + 42\gamma^2 Gm - 80\Lambda Gm - 68\gamma\right) r^4}{17r \left(2r - \gamma r^2 - 6Gm\right)^2} -$$
$$- \frac{\left(180\Lambda G^2m^2 + 208\gamma Gm\right) r^3 + \left(56Gm - 300\gamma G^2m^2\right) r^2 - 360G^3m^3}{17r \left(2r - \gamma r^2 - 6Gm\right)^2} \ ,$$

For the pure Schwarzschild case with $\gamma = \Lambda = 0$, the metric functions

$$A_{\rm Sch}(r) = \frac{2Gm}{r} \ ,$$
$$B_{\rm Sch}(r) = \frac{r^4(r - 2Gm)C_2 + (4r - 9Gm)(r - 2Gm)C_1}{4r(r - 3Gm)^2} + \frac{2Gm(45G^2m^2 - 7r^2)}{17r(r - 3Gm)^2} \ ,$$

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$$a_r \approx \frac{1}{2}\left(g'_{00} - g'_{rr} v_r^2\right) \approx -\frac{Gm}{r^2} + \frac{32\pi G\varepsilon}{9}\, r\, v_r^2$$

In Cotton gravity, the extra long-range curvature can be interpreted as an effective perfect fluid filling the universe, giving rise to this additional force. The appearance of such correction is natural, as velocity-squared forces are known to exist in other physical systems, such as drag forces on objects moving at high speeds relative to a surrounding fluid.

Velocity-dependent forces, like the Coriolis and Lorentz forces, are well-known, but velocity-squared forces are relatively rare. The concept of a gravitational force proportional to the square of the velocity was initially proposed by Schrödinger, and more recently, a repulsive force proportional to the squared velocity dispersion of a structure was derived by contracting the relativistic generalization of angular momentum.