Introduction to (perturbative) QCD: Lecture 1

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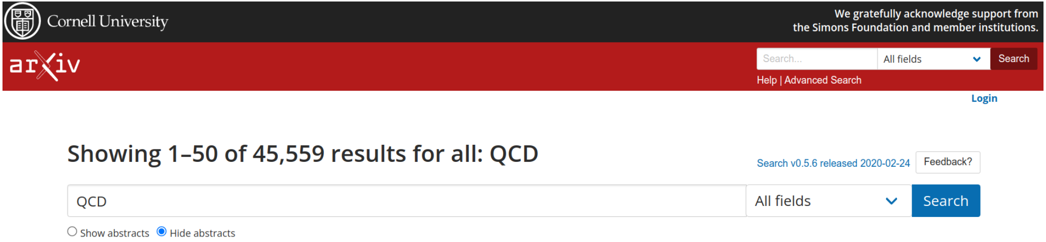
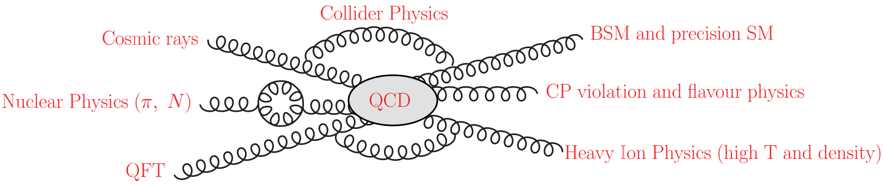
Quantum Chromodynamics (QCD): Why so important?

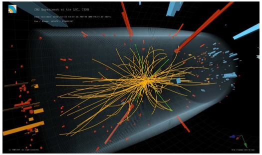
- old topic (early 70's) but still really much alive today
- people are still very active working on QCD



- many connections to other areas in particle physics

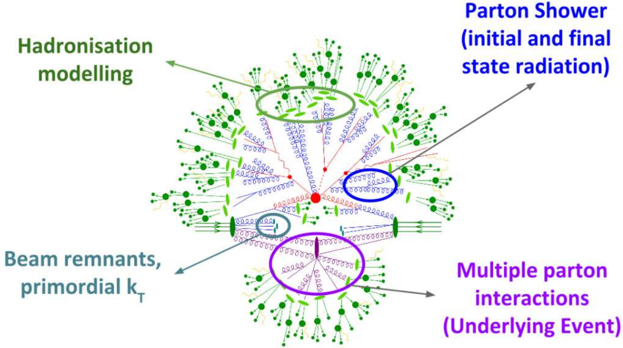


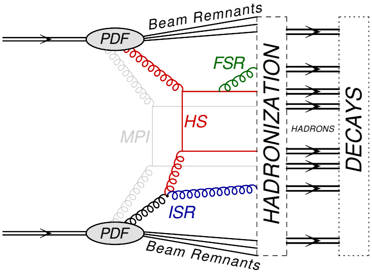
High-energy physics at hadron colliders



LHC: Collision of two protons

- we send protons ($\sim 10^{-15}$ m) at very high energies to probe very short-distance scales ($\sim 10^{-18}$ m)





- understand all the physics behind at different energy scales: from ~ 1 GeV up to ~ 1 TeV

Propose of the short course

- Introduce basic concepts of perturbative QCD (or refresh your knowledge)
- Understand the terminology
- Be familiar with most important developments in the field
- But no historical introduction (lack of time)

Two lectures:

1. **Basics of (perturbative) QCD**: $[SU(3)]_{\text{colour}}$, QCD Lagrangian, Gauge invariance and gauge fixing, Feynman Rules, Colour Algebra, Renormalization and Running Coupling, Asymptotic Freedom, naive Parton Model
2. **Perturbative QCD and the improved Parton Model** : NLO perturbative corrections, IR soft/collinear singularities, Cancellation mechanism and safe observables, Initial-State IR divergences, Universal Factorizaton of Collinear Singularities, Scale-dependent Parton Densities, Scaling Violation, DGLAP evolution equations

References

Textbooks:

- *QCD and Collider Physics*, R.K. Ellis, W. J. Stirling, B. R. Webber, Cambridge University Press (1999)
- *Foundations of Quantum Chromodynamics*, T. Muta, World Scientific (1998)
- *Basics of perturbative QCD*, Yu. Dokshitzer, V.A. Khoze, A.H. Mueller, S.I. Troyan, Edition Frontieres (1991)
- *Applications of Perturbative QCD*, R.D. Field, Addison Wesley, (1989)

Lectures (recorded and online available):

- S. Catani, *Introduction to QCD*, Academic Training Lectures, CERN 1998
- M. Mangano, *QCD*, Academic Training Lectures, CERN 1997
- B. Webber, *Quantum Chromodynamics*, Academic Training Lectures, CERN 1999
- B. Webber, *QCD at High Energy*, Academic Training Lectures, CERN 2008
- G. Zanderighi, *QCD for Postgraduates*, Academic Training Lectures, CERN 2010

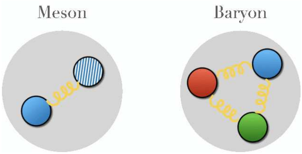
Basics

Quantum Chromodynamics (QCD): theory of strong (or hadronic) interactions

- usually formulated in terms of elementary QCD fields:
  quarks (antiquarks) and gluons
- whose interactions obey principles of a relativistic Quantum Field Theory (QFT)
- with a non-abelian gauge invariance SU(3)

Hadron spectrum fully classified with the following assumptions:

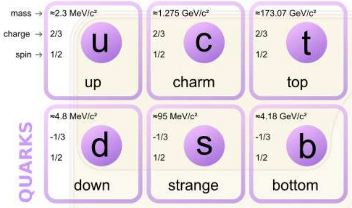
- hadrons: $q\bar{q}$ (mesons) and qqq (baryons):
  made of 1/2 spin quarks
- each quark of a given flavour comes in $N_c = 3$ colours
- colour SU(3) is an exact symmetry
- hadrons are colour neutral (colour singlet under SU(3))



Quarks and flavour quantum numbers

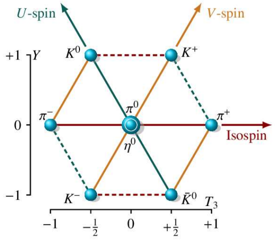
Quark field: $\psi_f(x) \Rightarrow$ usual spin 1/2 fermion field (like electron, but fractional electric charge e_f)

- flavour: $f = 1, ..., N_f$
- $N_f = 6$ different types of quarks
- 3 light quarks: u, d, s
  (mass « 1 GeV ← typical hadronic scale)

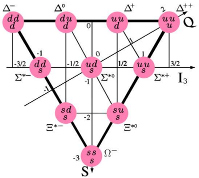


Naive quark model (Gell-Man and Neeman, 1961)

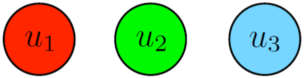
- mass spectrum of ordinary hadronic matter symmetric under $[SU(3)]_{flavour}$
- successful in understanding hadron spectroscopy
- but it requires one additional quantum number



From Baryon spectroscopy: $\mathbf{\Delta^{++}(1230)}$
$\Rightarrow$ spin 3/2 baryon resonance observed in πN-scattering



- $\Delta^{++} = u^{\uparrow} u^{\uparrow} u^{\uparrow}$: observed quantum numbers suggest that the wave function is completely symmetric w.r.t. spin and flavour quantum numbers
- **forbidden by Pauli exclusion principle** (u-quark is fermion)
- need additional quantum number to be antisymmetrized: $\Delta^{++} = \varepsilon_{ijk} u_i^{\uparrow} u_j^{\uparrow} u_k^{\uparrow}$
- ε_{ijk}: completely antisymmetric tensor (Levi-Civita)
- colour indices: $i, j, k = 1, ..., N_c$, with $N_c = 3$ (at least)

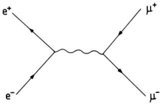


$\Rightarrow$ New quantum number solves spin-statistics problem

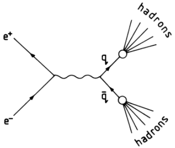
First experimental evidence for colour

From total cross sections in e^+e^--annihilation:

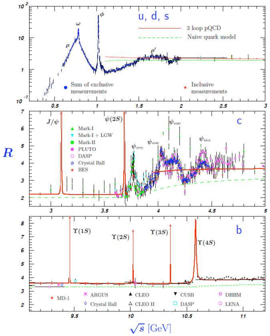
$$R = \frac{\sigma(e^+e^- \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-)} \propto \frac{N_c \sum_f e_f^2}{e_\mu^2} = N_c \sum_f e_f^2$$



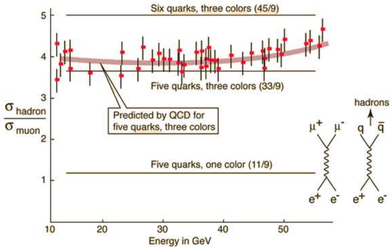
$e^+e^- \to \mu^+\mu^-$



$e^+e^- \to q\bar{q}$



- $n_f = 3 \Rightarrow R = \frac{2}{3} N_c$
- $n_f = 4 \Rightarrow R = \frac{10}{9} N_c$
- $n_f = 5 \Rightarrow R = \frac{11}{9} N_c$
- respective exp. data:
  $R = 2, \frac{10}{3}, \frac{33}{9} \Rightarrow N_c = 3$



$[SU(3)]_{colour}$ is an exact symmetry of the nature

Colour singlet hadrons

Observed hadrons do not carry colour quantum number $\Rightarrow$ Only colourless states can exist in nature (colour confinement)

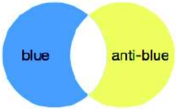
$$\begin{pmatrix}\psi_r\\ \psi_g\\ \psi_b\end{pmatrix}$$

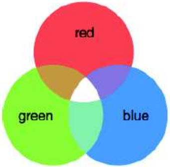
Hadrons are invariant under colour transformations (colour singlets)

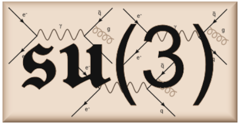
- symmetry transformation: $\psi^i_f \Rightarrow$ triplet under $[SU(3)]_{colour}$ rotations
- quarks $\Rightarrow$ fundamental irrep. (3)
- antiquarks $\Rightarrow$ conjugate irrep. $(\bar{3})$

Transformation properties of different states under $[SU(3)]_{colour}$

- qq: $3 \otimes 3 = \bar{3} \oplus 6$ (no colour singlet)
- $q\bar{q}$: $3 \otimes \bar{3} = 1 \oplus 8$ (singlet: meson)
- qqq: $3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10$ (singlet: baryon)
- other combinations possible: $qqqq\bar{q}$, $q\bar{q}q\bar{q}$, etc.
- but not (exotic): $qq, qqqq, qq\bar{q}$, etc.







Quantum CHROMOdynamics has to be invariant under local $SU(3)$ transformations

3 charges: 'Red, Green, Blue' $R \to G \to B$

$$U|A\rangle = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix} \begin{pmatrix} a_r \\ a_g \\ a_b \end{pmatrix} = \begin{pmatrix} a'_r \\ a'_g \\ a'_b \end{pmatrix} = |A'\rangle$$

Elements of the **special unitary Lie group $SU(N_c)$**:

- $N_c \times N_c$ complex matrices $U(x) \Rightarrow 2 \times N_c^2$ parameters

$$U^\dagger U = UU^\dagger = 1 \quad (unitary) \quad \to N_c^2 \text{ conditions}$$

$$det(U) = 1 \quad (special) \quad \to 1 \text{ condition}$$

- most general parametrization: $U(x) = e^{ig\vartheta^a(x)t^a}$
- t^a are matrix generators of Lie algebra of $SU(N_c)$
- $\vartheta^a(x)$ are independent arbitrary (real) parameters (functions)
- $2N_c^2 - N_c^2 - 1$: there are $a = 1, ..., N_c^2 - 1$
- for $N_c = 3$ there are **8 independent parameters and generators**

$[SU(3)]_{colour}$ generators

In QCD with $N_c = 3$:

- representations of t^a generators are **hermitian** and **traceless** matrices (3×3)

$$t^a = \begin{bmatrix} m & z_1 & z_2 \\ z_1^* & n & z_3 \\ z_2^* & z_3^* & -m-n \end{bmatrix}$$

- 8 independent real parameters: m, n (2) and two for each complex number z_1, z_2, z_3 (6)

- $[t^a, t^b] = if^{abc}t^c \Rightarrow$ do not commutate $\Rightarrow$ non-abelian group
- structure constants of the $SU(3)$ group: $f^{abc} \neq 0$

- quarks transform in the **fundamental (triplet) representation**: $(t^a_{ij})_F = \frac{1}{2}\lambda^a_{ij}$

$$r = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad b = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad g = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

- λ^a_{ij}: Gell-Mann matrices

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Gluons

The most general (local) colour transformations involves the exchange of

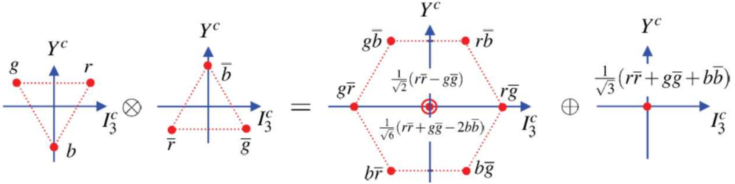
- $A_\mu^a(x)$ with $a = 1, ..., 8$ colour fields $\Rightarrow$ gluons (spin 1 and massless)
- gluons transform in the **adjoint (octet) representation**: $(t^a_{bc})_A = -if^{abc}$

	$\lambda^1 = \begin{pmatrix} 0&1&0\\1&0&0\\0&0&0 \end{pmatrix}$	$\lambda^4 = \begin{pmatrix} 0&0&1\\0&0&0\\1&0&0 \end{pmatrix}$	$\lambda^6 = \begin{pmatrix} 0&0&0\\0&0&1\\0&1&0 \end{pmatrix}$	$\lambda^3 = \begin{pmatrix} 1&0&0\\0&-1&0\\0&0&0 \end{pmatrix}$
	$\lambda^2 = \begin{pmatrix} 0&-i&0\\i&0&0\\0&0&0 \end{pmatrix}$	$\lambda^5 = \begin{pmatrix} 0&0&-i\\0&0&0\\i&0&0 \end{pmatrix}$	$\lambda^7 = \begin{pmatrix} 0&0&0\\0&0&-i\\0&i&0 \end{pmatrix}$	$\lambda^8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1&0&0\\0&1&0\\0&0&-2 \end{pmatrix}$
Gluons:	$r\bar{g}, g\bar{r}$	$r\bar{b}, b\bar{r}$	$g\bar{b}, b\bar{g}$	$\frac{1}{\sqrt{2}}(r\bar{r} - g\bar{g})$ $\frac{1}{\sqrt{6}}(r\bar{r} + g\bar{g} - 2b\bar{b})$

- can be "associated" to Gell-Mann matrices
- carry colour and anticolour

$$R = \begin{bmatrix} 1\\0\\0 \end{bmatrix}, \quad G = \begin{bmatrix} 0\\1\\0 \end{bmatrix}, \quad \overline{R} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad \overline{G} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \Rightarrow \lambda^1 = \begin{bmatrix} 0&1&0\\1&0&0\\0&0&0 \end{bmatrix} \quad R\overline{G} \text{ or } G\overline{R}$$

- gluon colour wave-functions are the same as for mesons



Gauge interactions

GAUGE $\rightarrow$ standard of measure or calibration

- GLOBAL gauge invariance means that the charge is conserved

Global gauge invariance of free-fermion Lagrangian under a gauge transformation

⇓

Changing to local transformation destroys this symmetry

⇓

Adding new vector fields with gauge transformation properties that restore the invariance

⇓

We get the Lagrangian with local gauge invariance and interactions!

QED: no invariance under local transformation

$$\psi(x) \quad \rightarrow \quad \psi'(x) = e^{i\alpha(x)}\psi(x)$$

$$-\bar{\psi}(x)\gamma^{\mu}\frac{\partial\alpha(x)}{\partial x^{\mu}}\psi(x)$$

$$A'_{\mu}(x) = A_{\mu}(x) + \frac{1}{e}\frac{\partial\alpha(x)}{\partial x^{\mu}}$$

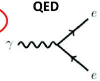
gauge field!

$$\mathcal{L}_{\text{free}}^{\text{gauge}} = -\frac{1}{4}F^{\mu\nu}(x)F_{\mu\nu}(x) \qquad \mathcal{L}_{\text{int}} = e\bar{\psi}(x)\gamma^{\mu}\psi(x)A_{\mu}(x)$$

Local (gauge) U(1)-symmetry

compensating field

QED



LOCAL GAUGE SYMMETRY

$\rightarrow$ calibration convention

- convention can be decided independently in each space-time point

All known forces in Nature are gauge interactions!

The QCD Lagrangian

Main properties of QCD: **NON-ABELIAN** and RENORMALIZABLE gauge theory

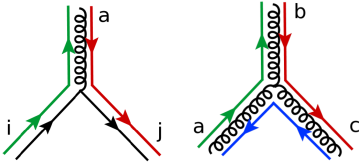
$$\mathcal{L}(x) = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \sum_{f=1}^{N_f} \overline{\psi}^i_f \left[i\gamma^\mu (D_\mu)_{ij} - m_f \delta_{ij} \right] \psi^j_f$$

- covariant derivative: $[D_\mu]_{ij} = \delta_{ij}\partial_\mu - igt^a_{ij}A^a_\mu$
- t^a_{ij} are $SU(N_c)$ generators in the fundamental representation
- $i, j = 1, \dots N_c$ and $a = 1, \dots N_c^2 - 1$
- g is the QCD coupling and ψ^i_f is quark field
- gluon field strength: $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf_{abc}A^b_\mu A^c_\nu$

$\Rightarrow$ Complete analogy with QED

but gluon A^a_μ carries colour charge a (colour-anticolour)

- gluons are charged w.r.t. strong interactions
  - $\rightarrow$ **gluon radiation from quark changes its colour**
  - $\rightarrow$ **gluons interact between themselves**



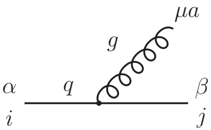
The QCD Lagrangian and interactions

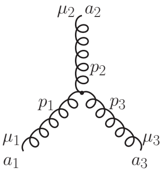
- $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu$ $\Rightarrow$ **source of all peculiar features of QCD**
- split the QCD Lagrangian:

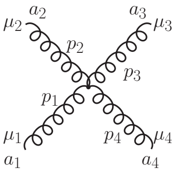
$$\mathcal{L} = \mathcal{L}_{free} + \mathcal{L}_{interaction}$$

$\mathcal{L}_{free}$ is quadratic in A_μ, ψ_f

$$\mathcal{L}_{interaction} = g \sum_{f=1}^{N_f} \overline{\psi}^i_f \gamma^\mu t^a_{ij} A^a_\mu \psi^j_f - g f^{abc} \partial^\mu A^a_\nu A^b_\mu A^{\nu c} - \frac{1}{4} g^2 f^{abc} f^{ade} A^b_\mu A^c_\nu A^{\mu d} A^{\nu e}$$

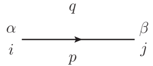




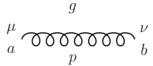


Feynman Rules

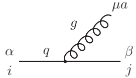
QCD:



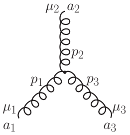
$$\frac{i}{p^2 - m^2 + i\varepsilon}\delta^{ij}(\not{p} + m)_{\alpha\beta}$$



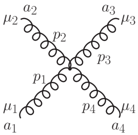
$$\frac{i}{p^2 + i\varepsilon}\delta^{ab} d^{\mu\nu}(p)$$



$$-ig(t^a)_{ij}(\gamma^\mu)_{\alpha\beta}$$

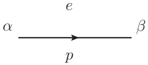


$$-gf^{a_1a_2a_3}[g^{\mu_1\mu_2}(p_1 - p_2)^{\mu_3}$$
$$+g^{\mu_2\mu_3}(p_2 - p_3)^{\mu_1} + g^{\mu_3\mu_1}(p_3 - p_1)^{\mu_2}]$$

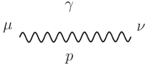


$$-g^2[f^{ba_1a_2}f^{ba_3a_4}(g^{\mu_1\mu_3}g^{\mu_2\mu_4} - g^{\mu_1\mu_4}g^{\mu_2\mu_3})$$
$$+g^2[f^{ba_1a_3}f^{ba_2a_4}(g^{\mu_1\mu_2}g^{\mu_3\mu_4} - g^{\mu_1\mu_4}g^{\mu_3\mu_2})$$
$$+g^2[f^{ba_1a_4}f^{ba_3a_2}(g^{\mu_1\mu_3}g^{\mu_4\mu_2} - g^{\mu_1\mu_2}g^{\mu_4\mu_3})$$

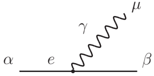
QED:



$$\frac{i}{p^2 - m^2 + i\varepsilon}(\not{p} + m)_{\alpha\beta}$$



$$\frac{i}{p^2 + i\varepsilon} d^{\mu\nu}(p)$$



$$-ie(\gamma^\mu)_{\alpha\beta}$$

Gauge invariance and gauge fixing (quantization)

The QCD Lagrangian:

$$\mathcal{L}_{\text{QCD}} = \mathcal{L}_{\text{classical}} + \mathcal{L}_{\text{gauge-fixing}} + \mathcal{L}_{\text{ghost}}$$

- The gauge-fixing term is needed because of a degeneracy of sets of gluon field configurations that enter the path-integral formulation of QCD and which are equivalent under gauge transformation
- This degeneracy makes it **impossible to write a gluon propagator**.
  Adding gauge fixing term solves the problem.

$$\mathcal{L}_{\text{gauge-fixing}} = -\frac{1}{2\alpha}(\partial_\mu A^{a\mu})(\partial_\nu A^{a\nu})$$

Gauge fixing explicitly breaks gauge invariance. However, in the end physical results are independent of the gauge choice.

Gauge invariance and gauge fixing (quantization)

Specification of gluon (photon) polarization tensor requires gauge choice

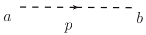
$$d^{\mu\nu}(p) = \sum_\lambda \epsilon^\mu_{(\lambda)}(p)\epsilon^\nu_{(\lambda)}(p) \begin{cases} -g^{\mu\nu} + (1-\alpha)\dfrac{p^\mu p^\nu}{p^2 + i\varepsilon}, & \text{covariant gauges} \\ -g^{\mu\nu} + \dfrac{p^\mu n^\nu + n^\mu p^\nu}{pn} - \dfrac{n^2 p^\mu p^\nu}{(pn)^2}, & \text{axial gauges} \end{cases}$$

- **covariant gauges**: depend on parameter α
  - $\alpha = 0$: Landau gauge
  - $\alpha = 1$: Feynman gauge
- **axial gauges**: depend on an arbitrary vector n^μ
  - big advantage: ghost contributions disappear
  - Light-cone gauge: a special case of axial-gauge with $n^2 = 0$

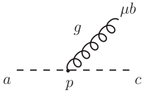
The QCD Lagrangian:

$$\mathcal{L}_{\text{QCD}} = \mathcal{L}_{\text{classical}} + \mathcal{L}_{\text{gauge-fixing}} + \mathcal{L}_{\text{ghost}}$$

- In **covariant gauges** $\Rightarrow$ GHOST contributions (propagates as a scalar but contributes as a fermion)



$\delta^{ab} \frac{i}{p^2 + i\epsilon}$



$g f^{abc} p^{\mu}$

$$\mathcal{L}_{\text{ghost}} = \partial_{\mu} \eta^{a\dagger} (\partial^{\mu} \delta^{ab} + g f_{abc} A^{c\mu}) \eta_b$$

- Ghosts are complex scalar fields obeying Fermi statistics
- to cancel unphysical longitudinal degrees of freedom which should not propagate
- ALTERNATIVE: choose an axial gauge and introduce an arbitrary direction (then only two physical polarizations propagate)

Gauge invariance and gauge fixing (quantization)

- **Gauge equivalent configurations are not dynamical degrees of freedom**
- In the quantization procedure:

  $\Rightarrow$ permit propagation of only physical (transverse polarizations)

  degrees of freedom for gluon (photon), i.e. fix the gauge
- Two general options:
  - **physical gauges** $\rightarrow$ select only 2 transverse polarizations (in a given frame)
    - e.g. $n^\mu A^a_\mu = 0$ (axial gauge) where n^μ is an arbitrary fixed four-vector
  - **covariant gauges** $\rightarrow$ propagation of full A^a_μ (physical + unphysical)
    - but introduce another unphysical field (ghost) to cancel unphysical (longitudinal) degrees of freedom of A^a_μ
    - ghost interact in QCD (in QED it decouples and can be forgotten)

Which gauge one should use?

- physical gauges: intermediate steps not explicitly Lorentz invariant
- covariant gauges: introduction of ghost

In practice:

- typically, physical gauges are simpler for lowest-order calculations and for approximate higher-order calculations

Gauge invariance and gauge fixing (quantization)

QCD Lagrangian invariant under $SU(N_c)$ local transformations

- one can redefine the quark and gluon fields independently at every point in space and time without changing the physical content of the theory

$$\vartheta^a \ll 1 \rightarrow \text{ infinitesimal transformation}$$

$$[U(x)]^{ij} = [e^{ig\vartheta^a(x)t^a}]_{ij} \simeq 1 + ig\vartheta^a(x)t^a_{ij} + \mathcal{O}(\vartheta^2)$$

$$\psi^i_f(x) \rightarrow [U(x)]^{ij}\psi^j_f(x) = \psi^i_f(x) + ig\vartheta^a(x)t^a_{ij}\psi^j_f(x)$$

$$A^a_\mu t^a_{ij} \rightarrow [U(x)]^{ik}A^a_\mu t^a_{ke}[U^{-1}(x)]^{ej} + \frac{i}{g}[U(x)]^{ik}\partial_\mu[U^{-1}(x)]^{kj}$$

$$A^a_\mu \rightarrow A^a_\mu - \partial_\mu\vartheta^a - gf^{abc}\vartheta^b A^c_\mu$$

- colour rotation (absent in QED) and longitudinal shift (QED-like)
- in QCD longitudinal component re-interact because of colour rotation

Colour Algebra and Colour Factors

- Calculation of Feynman graphs similar to QED apart from an overall colour factor
- explicit form of colour matrices not important for most practical purposes
- most relevant colour relations:

$$Tr(t^a t^b) = T_R \delta^{ab}$$

$$(t^a t^a)_{ij} = C_F \delta_{ij}$$

$$f^{abc} f^{abd} = C_A \delta^{cd}$$

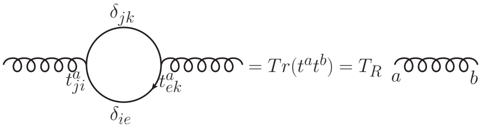
$$T_R = 1/2 \ \ \text{(normalization)}$$

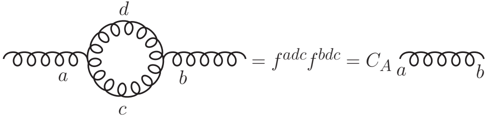
$$C_F = \frac{N_c^2 - 1}{2N_c}$$

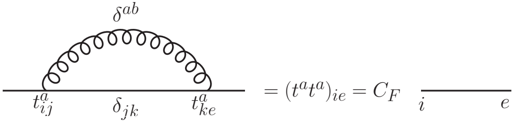
$$C_A = N_c$$

$$\begin{cases} Tr(T^a) = 0 \\ [T^a, T^b] = i f^{abc} T^c \end{cases}$$

either $(T^a)_{ij} = t^a_{ij}$
or $(T^a)_{bc} = i f_{bac}$







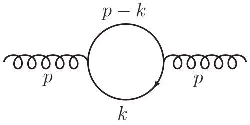
- **Casimirs of the fundamental and adjoint representations**
- $C_F = \frac{4}{3} \rightarrow$ quark colour charge squared
- $C_A = 3 \rightarrow$ gluon colour charge squared

Renormalization and running coupling

Main properties of QCD: NON-ABELIAN and **RENORMALIZABLE** gauge theory

- calculation of radiative corrections → loop contributions (virtual)

$$g^2 \int d^4k \frac{1}{k^2} \frac{1}{(p-k)^2} \to g^2 \int_p^\infty \frac{d^4k}{(k^2)^2} \sim g^2 \ln \frac{\infty}{p}$$



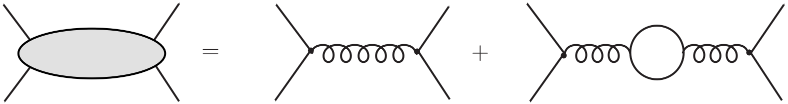
- ultraviolet (UV) region: $k \gg p \to$ **UV divergence**: $\ln \frac{\infty}{p}$
- UV divergence is a property of QCD (and many other QFTs): it arises because we extend our theory up to infinite energies, but each theory is valid only up to a certain scale Λ

- UV singularities can be removed by **RENORMALIZATION PROCEDURE**

bare coupling g_B (masses m_B, ...)	REPLACE ⟹	renormalized coupling g (masses m, ...)
arbitrary parameters in the Lagrangian		physical parameters in physical quantities

Renormalization and running coupling

- $\alpha = \frac{g^2}{4\pi}$ by analogy with fine structure constant in QED
- renormalization procedure works by regularization (1) and substraction (2) (by redefinition (3) of α_B)



(1) $$\sim \alpha_B \left\{ 1 + \alpha_B \beta_0 \int_{p^2}^{\Lambda_{\rm cutoff}} \frac{d^4 k}{(k^2)^2} + \mathcal{O}(\alpha_B^2) \right\}$$

(2) $$\sim \alpha_B \left\{ 1 + \alpha_B \beta_0 (\ln \frac{\Lambda_{\rm cutoff}}{\mu^2} + \ln \frac{\mu^2}{p^2}) + \mathcal{O}(\alpha_B^2) \right\}$$

(3) $$\sim \alpha(\mu^2) \left\{ 1 + \beta_0 \alpha(\mu^2) \ln \frac{\mu^2}{p^2} + \mathcal{O}(\alpha_\mu^2) \right\}$$

$$\alpha(\mu^2) \equiv \alpha_B \left[1 + \beta_0 \alpha_B \ln \frac{\Lambda_{\rm cutoff}}{\mu^2} + \mathcal{O}(\alpha_B^2) \right]$$

- $\Lambda_{\rm cutoff}$: UV regularization
- β_0: coefficient of UV behaviour
- $\boldsymbol{\mu^2}$: **renormalization scale** (arbitrary but diagram independent)
- $\alpha(\mu^2)$: UNIVERSAL (cutoff/process independent) but (renormalization) SCALE DEPENDENT

Renormalization and running coupling

- Gauge invariance $\rightarrow$ UV divergent terms have the same symmetry as bare Lagrangian and can be absorbed by redefinition of bare quantities
- must use a gauge invariant regularization
- QCD: **Dimensional Regularization** (DR)
  $\rightarrow$ the **most convenient safe regularization**

$$\int \frac{d^4 k_{[4]}}{(2\pi)^4} \rightarrow \mu_{DS}^{4-d} \int \frac{d^d k_{[d]}}{(2\pi)^d}\,, \quad d \equiv 4-2\epsilon < 4$$

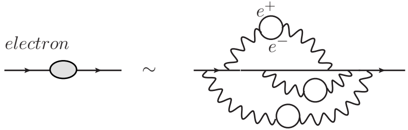
$$\int_0^1 \frac{dx}{x} \rightarrow \int_0^1 \frac{dx}{x^{1-\epsilon}} = \frac{1}{\epsilon}$$

- d-dimensional integrals are more convergent if one reduces the number of dimensions
- scale μ_{DS} needed to preserve the correct dimensions
- divergent integrals lead to poles of the form $\frac{1}{\epsilon}$
- RENORMALIZATION can be carried out at any order of the power expansion in α

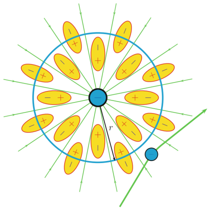
Renormalization and running coupling

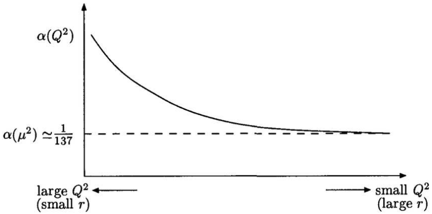
Theory can consistently be defined (renormalization) at the quantum level (remove UV divergences) by introducing renormalization scale-dependent coupling $\alpha(\mu^2)$

- **It makes physical sense** $\rightarrow$ in any physical measurement we observe interactions (coupling) at a certain scale
- QED provides us with an intuitive picture of running coupling



- quantum fluctuations
- the vacuum around the electron is polarized by virtual e^+e^- pairs that produce the dielectric effect
- screening of electron charge at large distances





- $\alpha(r) = \frac{e^2_{eff}(r)}{4\pi}$
- smaller r $\rightarrow$ less screening
- less screening $\rightarrow$ larger e_{eff}
- energy (momentum) $\sim \frac{1}{r}$

Renormalization group equation and asymptotic freedom

- **Size of the running coupling not predicted by the theory**
- **BUT** its scale dependence unambiguously predicted
- From lowest-order definition:

$$\alpha(\mu^2) = \alpha_B \left[1 + \beta_0 \alpha_B \ln \frac{\Lambda_{\text{cutoff}}}{\mu^2} + \mathcal{O}(\alpha_B^2)\right]$$

$$\frac{d \ln \alpha(\mu^2)}{d \ln \mu^2} = -\beta_0 \alpha^2(\mu^2) + \mathcal{O}(\alpha^3(\mu^2))$$

- β_0: coefficient characteristic of UV behaviour of the theory
- any unphysical parameter (bare quantities, UV regulator) disappear

- all-order generalization $\rightarrow$ RENORMALIZATION GROUP EQUATION

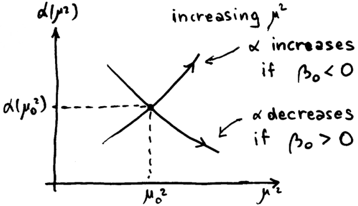
$$\frac{d \ln \alpha(\mu^2)}{d \ln \mu^2} = \beta(\alpha(\mu^2)) \qquad \beta(\alpha) = -\sum_{n=1}^{\infty} \beta_{n-1} \alpha^{n+1}$$

- β: theory dependent function

- R.G. equation predicts scale dependence of running coupling
- only extra input needed $\rightarrow$ **initial condition**: $\alpha(\mu_0^2)$ **at a given input scale**

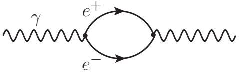
Small-coupling solution of R.G. equation:

- suppose $\alpha(\mu_0^2)$ at a given μ_0 is small ($\alpha(\mu_0^2) \ll 1$)
  - use perturbation theory: $\frac{d\alpha(\mu^2)}{d \ln \alpha(\mu^2)} \approx -\beta_0 \alpha^2(\mu^2)$
  - according to the sign of β_0 the coupling $\alpha(\mu^2)$ may increase or decrease e.g. at large distances (small scales)



actual calculations:

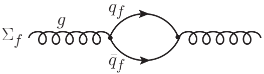
QED



$\beta_0 = -\frac{1}{3\pi}$

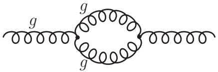
negative (as expected from screening argument)

QCD two different contributions



$\beta_0 = -\frac{1}{3\pi} T_R N_f$

negative (QED-like)



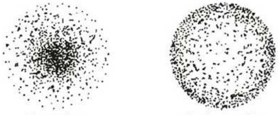
$\beta_0 = +\frac{11}{12\pi} C_A$

POSITIVE!

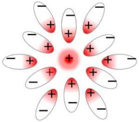
Gluons are charged $\Rightarrow$ the virtual gg-pair **spreads gluon charge over large distances**

Renormalization group equation and asymptotic freedom

$\Rightarrow$ **ANTI-SCREENING** of gluon colour charge



Screening Anti screening



$$\beta_0 = \begin{cases} -\dfrac{1}{3\pi} & \text{QED} \\ \dfrac{1}{12\pi}(11N_c - 2N_f) & \text{QCD} \end{cases}$$

- negative: QED coupling increases at short distances (large momenta)
- positive (for $N_f \leq 16$): QCD coupling decreases at short distances (large momenta)

- **ASYMPTOTIC FREEDOM**: $\alpha_S(\mu^2) \to 0$ when $\mu^2 \to \infty$ $\quad [\alpha(\mu^2) \equiv \alpha_s(\mu^2)]$
- peculiar feature of QCD (non-abelian gauge theory)
- extremely important for physics of strong interactions
  - at large transferred momentum (short distances), hadrons behave as a collection of free (weakly interacting) partons (elementary constituents: quarks and gluons)
  - in this regime, one can use simple method, i.e. perturbation theory, to make theoretical QCD predictions (no necessity of solving exactly the theory)

Renormalization group equation and asymptotic freedom

Solution of R.G. eq. in terms of $\boldsymbol{\alpha(\mu_0^2)} \longrightarrow$

$$\alpha(\mu^2) = \frac{\alpha(\mu_0^2)}{1 + \beta_0 \alpha(\mu_0^2) \ln \frac{\mu^2}{\mu_0^2}} \left(1 + \mathcal{O}(\alpha(\mu_0^2))\right)$$

- QCD: $\beta_0 > 0 \Rightarrow$ lowest-order expression for $\alpha(\mu^2)$ diverges at low scale

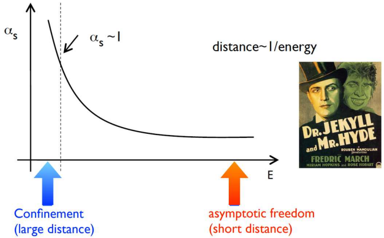
$$1 + \beta_0 \alpha(\mu_0^2) \ln \frac{\mu^2}{\mu_0^2} \approx 0$$

- fundamental QCD scale:

$$\Lambda_{QCD} \approx \mu_0 \exp\left(-\frac{1}{2\mu_0 \alpha(\mu_0^2)}\right)$$

$$\alpha(\mu^2) = \frac{1}{\beta_0 \ln \frac{\mu^2}{\Lambda^2_{QCD}}} \left[1 + \mathcal{O}\left(\frac{1}{\ln \frac{\mu}{\Lambda_{QCD}}}\right)\right]$$

The two faces of QCD



- Λ_{QCD} indicates a scale at which α_s becomes large and perturbative theory is not applicable any longer
- $\Lambda_{QCD} \approx 200$ MeV is measurable but it is not an observable (its value depends on: pert. order, renorm. scheme, N_f)
- increase of α_s at low scales consistent with CONFINEMENT
- rigorous proof of QCD confinement still missing

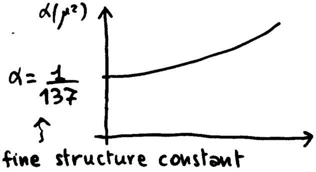
Renormalization group invariance

actual size of $\alpha(\mu_0^2)$ not predicted from the theory $\Rightarrow$ fundamental parameter of the theory $\Rightarrow$ input from experimental data

Since **renormalization-scale is arbitrary** $\Rightarrow$ How is it related to physical scales?

QED: natural answer $\Rightarrow$ free electrons are observables

- in the static limit $\mu \to 0$ ($\mu \leq m_e$ electron mass)
- $\alpha(\mu^2 \approx 0)$ measurable from Coulomb interaction
- $\alpha(\mu^2 \approx 0) = \alpha = 1/137$



QCD: free quarks, gluons not observed $\Rightarrow$ exploit **renormalization group invariance**

- renormalization scale μ is arbitrary, thus physics cannot depend on it
- any physical observable R at the physical scale Q^2 cannot depend on μ^2 (e.g. $\sigma_{\text{tot}}(e^+e^- \to \text{hadrons})$ with $\sqrt{s} = Q$)

$$R(Q^2) = R_{\text{theory}}(\alpha_s(\mu^2), \mu^2, Q^2)$$

- μ^2 dependence compensates

Renormalization group invariance and running coupling

In principle: no need of any $\alpha_s(\mu^2)$

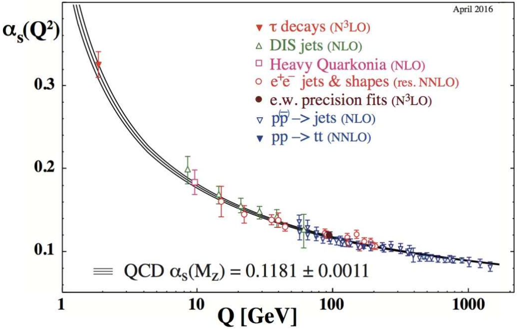
- chose one physical quantity R_0 at scale Q_0 and from its theoretical expression obtain $\alpha_s(\mu^2)$ as a function of $R_0(Q_0^2)$ $\Rightarrow$ insert this function in: $R_{\rm theory}[R_0(Q_0^2), Q_0^2, Q^2]$ (any th. prediction in terms of a single exp. input $(R_0(Q_0^2))$)

In practice: compute $R_{\rm theory}(\alpha_s(\mu^2), \mu^2, Q^2)$ with sufficient theoretical accuracy

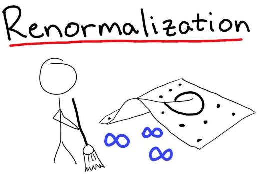
- set $\mu^2 \sim Q^2$ ("natural" physical scale) and extract $\alpha_s(Q^2)$ from comparison with $R_{\rm exp}(Q^2)$

- $\mu_0 = M_Z$: conventional reference scale
- world average: $\alpha_s(M_Z) = 0.1181$
- τ decays: $M_\tau = 1.78$ GeV $\to \alpha_s \approx 0.35$

Measurements of the running coupling



Renormalization and running coupling



few summarizing remarks:

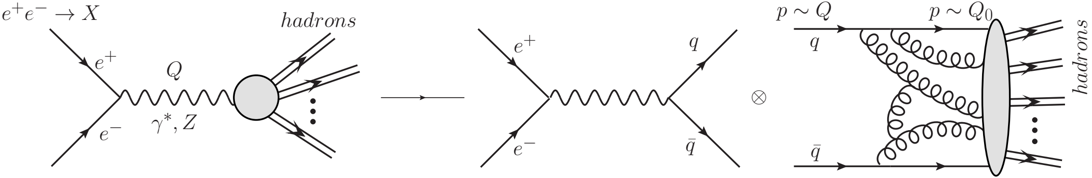
RENORMALIZATION IS NOT A TRICK

- RUNNING COUPLING $\Rightarrow$ introduction of scale-dependent coupling makes physical sense
- RENORMALIZATION GROUP EQUATION $\Rightarrow$ absolute size of coupling not predicted (input) but its scale dependence unambiguously predicted
- RENORMALIZATION GROUP INVARIANCE $\Rightarrow$ size of coupling at a reference scale unambiguously measurable

The Parton Model

If we are not interested in details of hadronic (sufficiently inclusive) process at small scale $\Longrightarrow$ PARTON PICTURE (factorization of short-distance and long-distance processes)

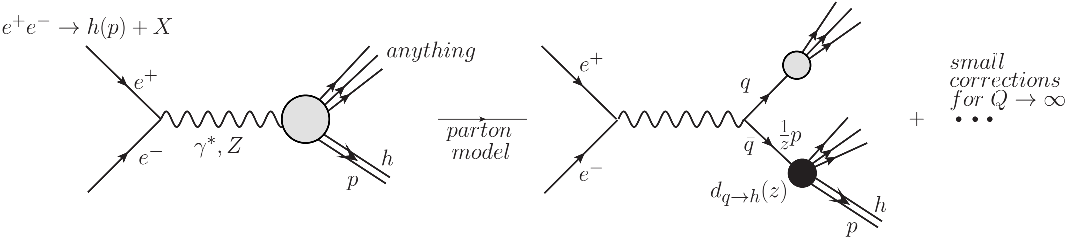
Example: Total hadronic cross section in e^+e^- annihilation (completely inclusive over final state)



- parton model (lowest order QCD): $\sigma(LO) = \frac{4\pi\alpha^2}{Q^2} N_c \sum_f e_f^2$
- interactions at high momentum scale: $\propto \alpha_s^n(Q^2) \sim (\frac{1}{\ln Q})^n$ (inverse powers of $\ln Q$)
- hadronization (conversion into hadrons): $1 + \mathcal{O}(\frac{Q_0}{Q})^p$ (inverse powers of Q)
- $Q^2 \to \infty$ ($Q^2 \gg \Lambda_{QCD}$): forget about corrections and simply compute $\sigma(e^+e^- \to q\bar{q})$

- also less inclusive cross sections with triggered hadrons in final or initial state

Example: Single-particle inclusive cross section in e^+e^- (one hadron observed in the final state)



$$\frac{d\sigma_h}{d^3p} \sim \sum_{partons} \int_0^1 \frac{dz}{z^2} \left(\frac{d\sigma_{parton}}{d^3p_{parton}} \right)_{\frac{1}{z}p = parton} \cdot \; d_{parton\to h}(z)$$

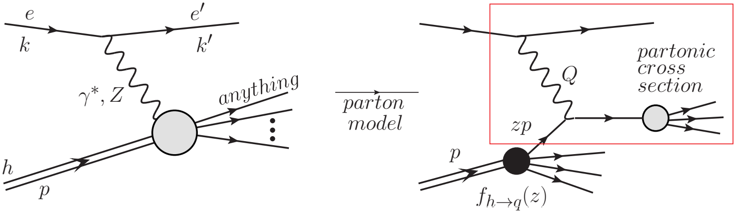
- z: momentum fraction lost in fragmentation process
- $d_{parton\to h}(z)$: parton fragmentation function (probability that a given parton fragments into the observed hadron)

The Parton Model

- also less inclusive cross sections with triggered hadrons in final or initial state

Example: Deep-inelastic lepton-hadon scattering (DIS) (one hadron in the initial state)

$e(k) + h(p) \to e'(k') + X$



$k - k' = q$

$Q^2 = -q^2 \gg \Lambda^2_{QCD}$

$$\sigma_h(p) \sim \sum_{partons} \int_0^1 dz \cdot f_{h\to parton}(z) \cdot \sigma_{parton}(p_{parton} = zp)$$

- z: fraction of hadron momentum carried by the parton
- $f_{h\to parton}(z)$: parton density
- $\Rightarrow$ probability to find a given parton into the initial-state hadron

The Parton Model

Inclusive hadronic processes at large transferred momentum:

- theory predictions in terms of computable partonic cross sections
- few non-perturbative quantities

PARTON MODEL (PM) $\Rightarrow$ compute hadronic cross section in terms of:

Short-distance phenomena

- partonic cross section

$\otimes$ (convolution)

Long-distance phenomena

- parton densities (PDFs)
- fragmentation functions (FFs)

- PDFs and FFs: cannot be computed in QCD perturbation theory
- UNIVERSAL: depend only on the hadron and not on the process
- in principle, computable by non-perturbative methods
- IF NOT: extract them from a single process by comparison with experimental data and use to predict any other process

PM introduced before QCD (Feynman, Bjorken, Gribov, ...) and justified "a posteriori" because of QCD and its asymptotic freedom

THANK YOU!

Next lecture:

How this naive parton model picture can be consistently and quantitatively improved?

$\Rightarrow$ "true" perturbative QCD