Introduction to (perturbative) QCD: Lecture 2

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Propose of the short course

- Introduce basic concepts of QCD (or refresh your knowledge)
- But no historical introduction (lack of time)
- Understand the terminology
- Be familiar with most important developments in the field

Two lectures:

1. **Basics of (perturbative) QCD**: $[SU(3)]_{\text{colour}}$, QCD Lagrangian, Gauge invariance and gauge fixing, Feynman Rules, Colour Algebra, Renormalization and Running Coupling, Asymptotic Freedom, naive Parton Model
2. **Perturbative QCD and the improved Parton Model** : NLO perturbative corrections, IR soft/collinear singularities, Cancellation mechanism and safe observables, Initial-State IR divergences, Universal Factorizaton of Collinear Singularities, Scale-dependent Parton Densities, Scaling Violation, DGLAP evolution equations

Asymptotic Freedom and Parton Model

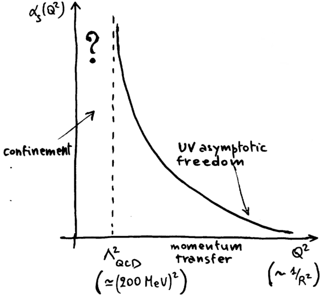
pQCD approach to hadronic physics applies to

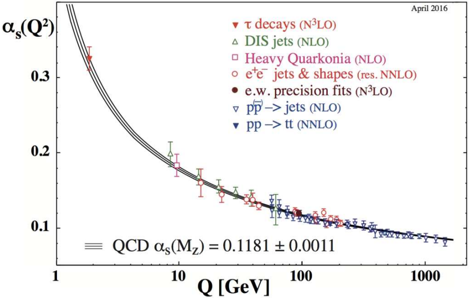
- INCLUSIVE
- HARD-SCATTERING processes

Based on:

- PARTON MODEL (PM)
- ASYMPTOTIC FREEDOM (AF)

- **HARD-SCATTERING** $\Rightarrow$ at least one momentum scale $Q \gg M_{\rm hadron} \sim 1$ GeV
  - at this scale the QCD effective coupling $\alpha_s(Q^2)$ can be sufficiently small to attempt a perturbative expansion

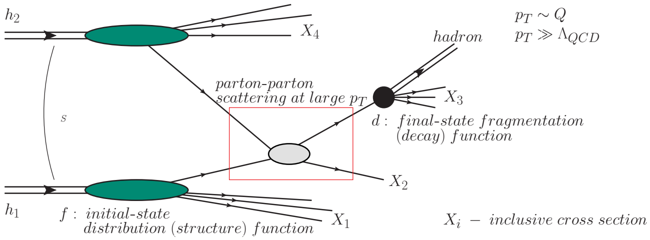




Asymptotic Freedom and Parton Model

- **INCLUSIVE** $\Rightarrow$ Parton Model picture
  - factorization of long-distance and short-distance physics

$$\sigma \sim (f_1 f_2) \otimes \sigma_{hard} \otimes d + \mathcal{O}\left(\left(\frac{1}{p_T}\right)^p\right) \qquad p \geq 1$$



- f_1, d_1: non-perturbative but universal (process independent)
- hard-cross section (pQCD) $\alpha_s(\mu^2)$ sufficiently small at large p_T

$$\alpha_s(p_T^2) \sim \frac{1}{\beta_0 \ln \frac{p_T^2}{\Lambda_{QCD}^2}}$$

- higher-twist or power corrections

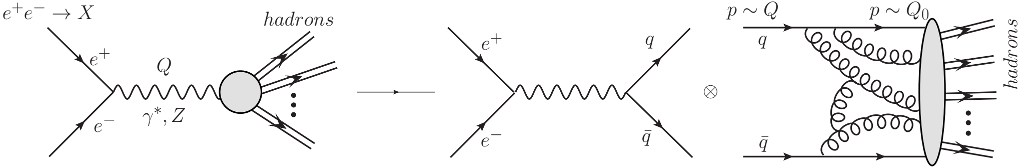
Is this picture correct?

- no rigorous (field theory) proof in the most general case

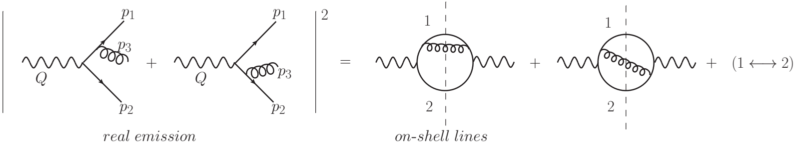
Is this picture self consistent and quantitative?

- improved parton model ("true" perturbative QCD)

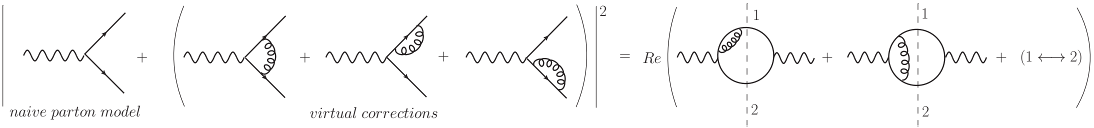
Total cross sections: e^+e^- annihilation



- first perturbative correction (NLO QCD): $\mathcal{O}(\alpha_s)$



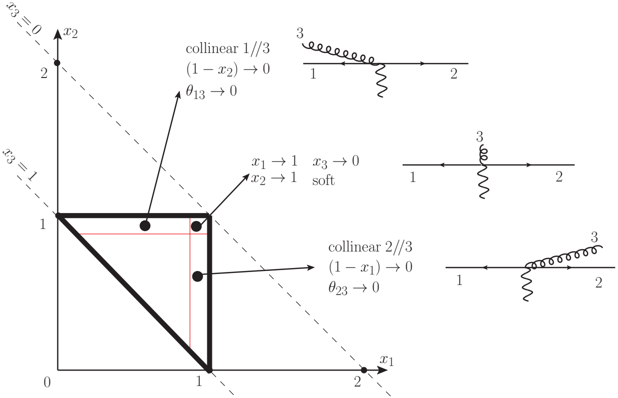
similar diagrams
BUT
dif!ferent kinematics



$$\sigma^{NLO} = \int d\phi_2 \left|\mathcal{M}_0\right|^2 + \int_R d\phi_3 \left|\mathcal{M}_{\mathrm{real}}\right|^2 + \int_V d\phi_2 2\mathrm{Re}(\mathcal{M}_{\mathrm{virtual}}\mathcal{M}_0^*)$$

- **kinematics variables for real emissions** (c.m. frame):

$$x_i = \frac{2p_i \cdot Q}{Q^2} = \frac{E_1}{\frac{1}{2}Q} \qquad x_i > 0$$



- energy fractions x_i lie within a **triangle**

- x_i: energy fractions
- energy conservation:
  $x_1 + x_2 + x_3 = \frac{2(p_1+p_2+p_3)\cdot Q}{Q^2} = 2$
- angles:
  $$2p_1 \cdot p_3 = (p_1 + p_3)^2 = = (Q - p_2)^2 = Q^2 - 2p_2 Q$$
  $$2E_1 E_3(1 - \cos\vartheta_{13}) = = Q^2(1 - x_2) \quad \leftarrow x_i < 1$$
- in particular:
  $\vartheta_{13} \to 0 \Longleftrightarrow x_2 \to 1$

Total cross sections: e^+e^- annihilation

- **real cross section**:

$$\sigma^R = \int_0^1 dx_1 dx_2 dx_3 \delta(2 - x_1 - x_2 - x_3) \left|\mathcal{M}_{\text{real}}(x_1, x_2, x_3)\right|^2$$

$$\left|\mathcal{M}_{\text{real}}(x_1, x_2, x_3)\right|^2 = \sigma_0 C_F \frac{\alpha_s}{2\pi} \frac{x_1^2 + x_2^2}{(1 - x_1)(1 - x_2)} \qquad \rightarrow \text{singular when : } x_{1,2} \rightarrow 1$$

- singularity not integrable $\int_0^1 dx_1 \frac{1}{1-x_1} \rightarrow \infty$ (a disaster for QCD?)

$$\text{InfraRed(IR)} = \begin{cases} \text{soft : } \omega = E_g \rightarrow 0 \sim t \rightarrow \infty & \text{long-distance} \\ \text{collinear : } \vartheta \rightarrow 0 \sim \lambda \rightarrow \infty & \text{physics} \end{cases}$$

- in the real world (QCD): physical cut-off $\epsilon \sim M_h/Q \sim \Lambda_{QCD}/Q$

$$\alpha_s(Q^2) \int_0^{1-\epsilon} \sim \alpha_s(Q^2) \ln \frac{1}{\epsilon} \qquad \leftarrow \text{finite but : } \ln \frac{1}{\epsilon} \sim \ln \frac{1}{\alpha_s(Q^2)}$$

$\sigma \sim \sigma_o(1+\alpha_s \cdot \frac{1}{\alpha_s} + \ldots) \sim \sigma_o(1+1+\ldots) \Rightarrow$ perturbative expansion does not make sense

IR singularities $\Rightarrow$ non-perturbative phenomena are not power suppressed

$\Rightarrow$ factorization between short/long distances breaks down? pQCD inconsistency?

Total cross sections: e^+e^- annihilation

- a closer look at the structure of the IR singularities
  (rewrite numerator using $x_1 + x_2 + x_3 = 2$):

$$x_1^2 + x_2^2 = 1 + (1 - x_1 - x_2)^2 - 2(1 - x_1)(1 - x_2) = 1 + (1 - x_3)^2 - 2(1 - x_1)(1 - x_2)$$

$$\frac{x_1^2 + x_2^2}{(1 - x_1)(1 - x_2)} = \frac{1 + (1 - x_3)^2}{(1 - x_1)(1 - x_2)} - 2 \qquad \longleftarrow \text{non singular}$$

then split in two contributions:

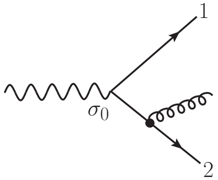
$$\frac{1}{(1 - x_1)(1 - x_2)} = \frac{1}{2 - x_1 - x_2}\left(\frac{1}{1 - x_1} - \frac{1}{1 - x_2}\right) = \frac{1}{x_3}\left(\frac{1}{1 - x_1} - \frac{1}{1 - x_2}\right)$$

$\frac{1}{x_3}$: soft singularity $\qquad$ $\frac{1}{1-x_1}$: collinear for $\vartheta_{23} \to 0$ $\qquad$ $\frac{1}{1-x_2}$: collinear for $\vartheta_{13} \to 0$

$|\mathcal{M}_{\text{real}}|^2 \;\underset{\text{looks like}}{\sim}\;$ [non-singular interference term] + [sum of two independent collinear (and soft) emissions (IR limit $\sim$ classical limit)]

Total cross sections: e^+e^- annihilation

Each of the singular terms



$\sim \sigma_0 d\omega_{23}$

- **probability of collinear splitting**:

$$P_{qg}(x_3) = C_F \frac{\alpha_s}{2\pi} \frac{1+(1-x_3)^2}{x_3}$$

- $\frac{d\vartheta_{23}^2}{\vartheta_{23}^2}$: collinear spectrum
- $\frac{d\omega_3}{\omega_3}$: bremsstrahlung spectrum

$$d\omega_{23} = \frac{dx_1}{1-x_1} dx_3 P_{qg}(x_3) = \frac{d\cos\vartheta_{23}}{1-\cos\vartheta_{23}} dx_3 P_{qg}(x_3) \simeq \frac{d\vartheta_{23}^2}{\vartheta_{23}^2}\frac{d\omega_3}{\omega_3} \qquad [\vartheta_{23} \to 0, \omega_3 = E_3 \to 0]$$

$$\begin{aligned} \sigma^R &= \sigma_0 \int_0^1 dx_1 dx_2 dx_3 \delta(2-x_1-x_2-x_3) \cdot \left(\frac{-C_F\alpha_s}{\pi}\right) \\ &+ \sigma_0 \left\{ \int_{-1}^1 \frac{d\cos\vartheta_{23}}{1-\cos\vartheta_{23}} \int_0^1 dx_3 P_{qg}(x_3) \frac{1}{\frac{1-x_3(1-\cos\vartheta_{23})}{2}} + (1 \Leftrightarrow 2) \right\} \end{aligned}$$

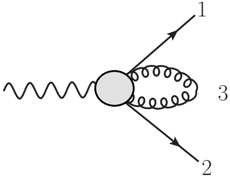
phase-space factor:

$$\frac{1}{\frac{1-x_3(1-\cos\vartheta_{23})}{2}} \leftrightarrow \int_0^1 dx_2 \delta(2-x_1-x_2-x_3)\Big|_{1-x_1=\frac{x_2x_3(1-\cos\vartheta_{23})}{2}}$$

Total cross sections: e^+e^- annihilation

- back to the full cross section:

  add virtual terms $\Rightarrow$ completely analogous to real terms **but different kinematics**



$$\sim \int_0^1 dx_1 dx_2 \delta(2 - x_1 - x_2) \int_0^\infty dx_3 \, |\mathcal{M}_{\mathrm{virtual}}|^2$$

- $\delta(2 - x_1 - x_2)$: momentum conservation

$$\text{loop integral}: \int_0^\infty dx_3 \ldots \simeq \begin{cases} \int_1^\infty dx_3 \ldots & \text{UV region} \\ \int_0^1 dx_3 \ldots & \text{IR behaviour} \end{cases}$$

- UV: renormalized (running) coupling $\otimes$ finite
- **same IR behaviour as real matrix element apart from overall sign and kinematics**

Total cross sections: e^+e^- annihilation

- back to the full cross section:
  - **SIGN**: it comes from **UNITARITY**

$$\text{probability that everything happens} \quad P = 1 = \underset{(\text{LO})}{1} + \alpha_s(\underset{(+)}{\text{real}} - \underset{(-)}{\text{virtual}})$$

  - **KINEMATICS**:

$$\sigma^R + \sigma^V = \text{finite} + \int_{-1}^{1} \frac{d\cos\vartheta_{23}}{1-\cos\vartheta_{23}} \int_0^1 dx_3 P_{qg}(x_3) \left[\frac{1}{\frac{1-x_3(1-\cos\vartheta_{23})}{2}} - 1 \right] + (1 \Leftrightarrow 2)$$

$$\left[\frac{1}{\frac{1-x_3(1-\cos\vartheta_{23})}{2}} - 1 \right] = \frac{\frac{x_3(1-\cos\vartheta_{23})}{2}}{\frac{1-x_3(1-\cos\vartheta_{23})}{2}} \to x_3\vartheta_{23}^2$$

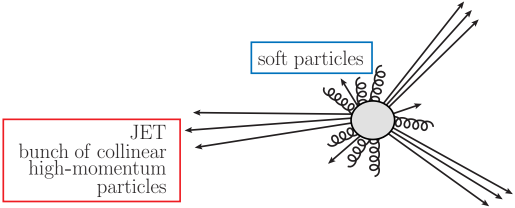
- $x_3 \to 0$ and $\vartheta_{23} \to 0$:
- kinematics differences are irrelevant in IR region

- total cross section is finite because real/virtual cancellation of IR singularities
- pQCD can consistentely applied to total cross section

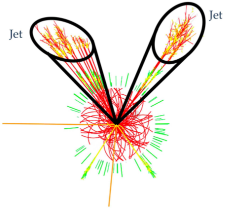
Total cross sections: e^+e^- annihilation

Two comments:

1. Matrix elements enhanced in soft and collinear regions (phase space is flat)



- typical tructure of hadronic final state: jets + soft particles



2. The pQCD approach applicable to total cross section. What about less inclusive quantities and other processes?
   - IR behaviour is UNIVERSAL provided the measured quantity fulfils some SAFETY CRITERIA

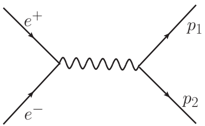
General structure of pQCD cross sections

Cross section: $\sigma = \sigma^{LO} + \sigma^{NLO} + \ldots$

- **Leading-order (LO)**: $\rightarrow$ suppose m final-state partons

$$\sigma^{LO} = \int_m d\sigma^B \quad \longleftarrow \text{Born level cross section}$$

$$d\sigma^B \sim d\phi^m \left|\mathcal{M}_m^{\text{tree}}(\{p_i\})\right|^2 F_J^m(\{p_i\})$$



  - total phase-space: $d\phi^m = \prod_{i=1}^{m} \frac{d^4 p_i}{2\pi^3} \delta_+(p_i^2)\delta^{(4)}(p_{in} - \sum_i p_i)$
  - $\left|\mathcal{M}_m^{\text{tree}}(\{p_i\})\right|^2$: tree-level QCD matrix element (depends on the process)
  - $F_J^m(\{p_i\})$: phase-space (measurement) function that defines the physical quantity we want to compute (including experimental cuts)
  - $F_J^m(\{p_i\}) = 1 \Rightarrow$ total cross section

General structure of pQCD cross sections

Cross section: $\sigma = \sigma^{LO} + \sigma^{NLO} + \dots$

- **Next-to-leading order (NLO)**: $\rightarrow$ add real and virtual contributions

$$\sigma^{NLO} = \int_{m+1} d\sigma^{R} + \int_{m} d\sigma^{V}$$

same structure as Born level but: $\mathcal{M}_m^{\text{tree}} \rightarrow \mathcal{M}_{m+1}^{\text{tree}}$ and $\mathcal{M}_m^{\text{tree}} \rightarrow \mathcal{M}_m^{\text{1-loop}}$

$$\int_{m+1} d\sigma^{R} : \ \text{divergent}$$

  - divergences arise from integration of $\mathcal{M}_{m+1}^{\text{tree}}$ over IR region

$$\int_{m} d\sigma^{V} : \ \text{divergent}$$

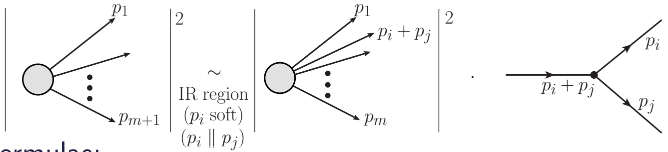
  - divergences arise in loop integral $\mathcal{M}_m^{\text{1-loop}}$ from its IR region

NLO: real emissions

We can easily compute the matrix elements in the IR region:

IR behaviour of QCD matrix elements is UNIVERSAL

real emission :

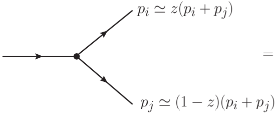


Factorization formulae:

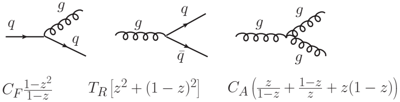
$$\left|\mathcal{M}^{\text{tree}}_{m+1}(p_1, \ldots, p_i, \ldots, p_j, \ldots, p_{m+1})\right|^2 \simeq \left|\mathcal{M}^{\text{tree}}_{m}(p_1, \ldots, p_i + p_j, \ldots, p_m)\right|^2 \cdot V_{ij}$$

- V_{ij}: process independent singular factor

collinear limit :



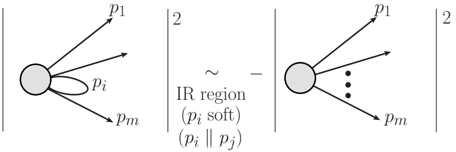
$= V_{ij} = \frac{1}{\theta_{ij}} P_{ij}(z)$ DGLAP splitting function



- analogous formulae for soft limit (eikonal factorization)

We can easily compute the matrix elements in the IR region:
IR behaviour of QCD matrix elements is UNIVERSAL

virtual emission :



$\cdot \int_{\text{loop}} V_{ij}$

Factorization formulae:

$$\left|\mathcal{M}_m^{\text{1-loop}}(p_1, \ldots, p_m)\right|^2 \simeq - \left|\mathcal{M}_m^{\text{tree}}(p_1, \ldots, p_m)\right|^2 \cdot \int_{\text{loop}} V_{ij}$$

inserting into: $\sigma^{NLO} = \text{finite} + \sigma^{NLO}_{\text{IR region}}$

$$\sigma^{NLO}_{\text{IR region}} \simeq \sum_{i,j} \int_m d\phi^{(m)} \left|\mathcal{M}_m^{\text{tree}}\right|^2 \int_{\text{i,j}} V_{ij} \left[F^{m+1}(\ldots, p_i, p_j, \ldots) - F^m(\ldots, p_i + p_j, \ldots)\right]$$

- differences in total phase space irrelevant in IR region
- same structure as at Born level
- $F^m(\{p_i\})$: phase space restriction for the physical quantity

Infrared safety

SAFETY CRITERIA (Sterman-Weinberg):

IR cancellation $\Leftrightarrow$ $F^{(m+1)}(\ldots, p_i, \ldots, p_j, \ldots) \simeq F^{(m)}(\ldots, p_i + p_j, \ldots)$

(i.e. non-perturbative physics is power suppressed)

IR limit		
soft limit	$p_i \to 0$	IR safety
collinear limit	$p_i \parallel p_j$	collinear safety

Kinoshita-Lee-Nauenberg (KLN) theorem (quantum mechanics) in pQCD:

- perturbative observables must be IR and collinear safe
- for a suitable defined inclusive observable, there is a cancellation between the soft and collinear singularities occurring in the real and virtual contributions
- physical observables always requires the cancellation

In other words:

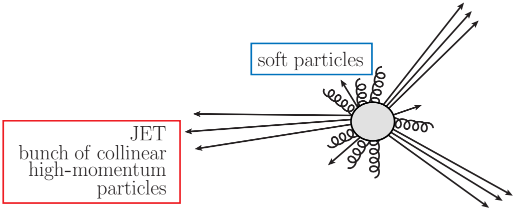
- the measured/computed quantity cannot resolve long-distance phenomena
- its value should remain the same by:
  - adding a (many) soft particles
  - replacing a particle by two (many) collinear particles

Observables that respect the above constraint are called infrared safe observables. Infrared safety is a requirement that the observable is calculable in pQCD.

IR/collinear safe obervables

Examples:

- Event shape distributions (Thrust T, Sphericity S, and many more ...)
- Jet cross section

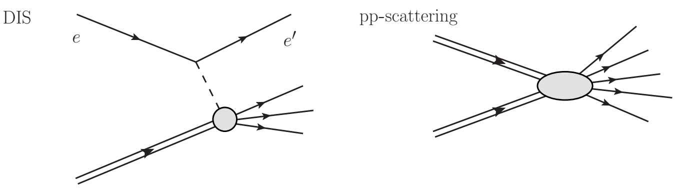


- Qualitative definition: collimated spray of high-energy particles
- Quantitative studies ⇒ precise definition of jet necessary (in particular: low energy particles have to be assigned to jet to have IR safety)

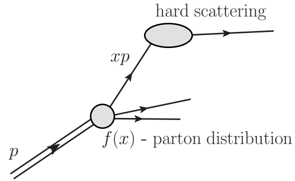
Need a JET ALGORITHM fulfilling the following requirements:

- IR/collinear safe
- simple to implement in the experimental analyses
- simple to implement in theoretical calculations
- small hadronization (non-perturbative) corrections

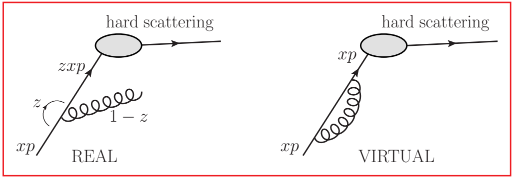
Processes with initial-state hadrons



- parton model:

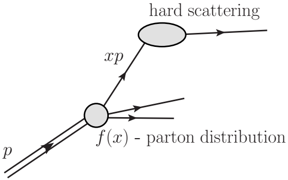


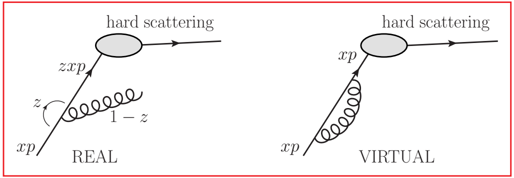
- perturbative calculation of hard scattering $\longrightarrow$ initial-state IR divergences



- kinematics differences (incoming parton have different momenta: xp and zxp):
  - irrelevant in soft limit $z \to 1 \Rightarrow$ **cancellation of soft sinuglarities**
  - relevant in collinear limit $z \neq 1 \Rightarrow$ **no cancellation of collinear singularities**

Processes with initial-state hadrons

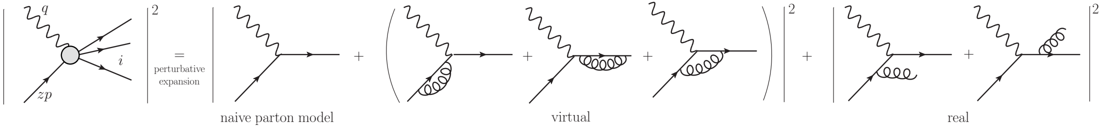




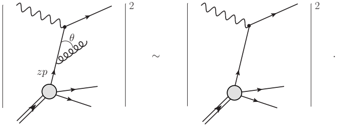
Physical interpretation:

- there are non-perturbative terms that are not power suppressed (initial-state collinear singularities $\sim$ regularized by a physical cut-off related to the hadron size)
- collinear singularities depend on each initial-state hadron and are universal (process independent) $\Longrightarrow$ non-perturbative (singularity) terms can be absorbed (factorized) in non-perturbative parton densities: $f(x) \to f(x, Q^2)$ (scaling violation)
- $f(x, Q^2)$ not computable in absolute value but its Q^2-dependence (rellated to collinear singularities) UNIVERSAL and COMPUTABLE in pQCD (DGLAP evolution equations)

Hard processes with initial-state hadrons in pQCD



- pQCD calculation leads to IR (soft and collinear) divergences
- **completely inclusive final-state** (IR/coll. safe) $\Rightarrow$ cancellation of **soft** and **FINAL-STATE collinear** singularities
- initial-state: one single parton (not fully inclusive) $\Rightarrow$ uncancelled **INITIAL-STATE collinear** singularities $\Rightarrow$ general feature of any IR/coll. safe hard scattering processes with colliding hadrons



$$\alpha_s(Q^2)\int_0^1 \frac{d\theta^2}{\theta^2}$$

- $k_\perp \sim \theta P \sim \theta Q$
- $\displaystyle\int_0^1 \frac{d\theta^2}{\theta^2} = \int_0^Q \frac{dk_\perp^2}{k_\perp^2} \longrightarrow \int_{Q_0}^Q \frac{dk_\perp^2}{k_\perp^2}$
- regularized by physical cutoff Q_0

($Q_0 \sim 1$ GeV, $1/Q^0 \rightarrow$ average distance between partons inside hadron)

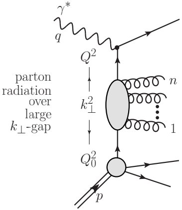
Hard processes with initial-state hadrons in pQCD

physical interpretation: $\Rightarrow$ sensitivity to **IR cut-off**

$\Rightarrow$ **long-distance phenomena included in QCD corrections**

First problem: How to remove sensitivity to IR cutoff?
(i.e. How to identify partonic subprocesses dominated by short-distance interactions?)

- **higher-order corrections**



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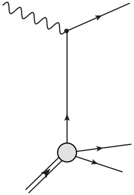


$$\left[\alpha_s(Q^2)\int_{Q_0^2}^{Q^2}\frac{dk_\perp^2}{k_\perp^2}\right]^n$$

- logarithmic collinear spectrum

$$\alpha_s(Q^2)\int_{Q_0^2}^{Q^2}\frac{dk_\perp^2}{k_\perp^2} = \alpha_s(Q^2)\ln\frac{Q^2}{Q_0^2} \simeq \mathcal{O}(1) \qquad \ln\frac{Q^2}{Q_0^2} \sim \frac{1}{\alpha_s(Q^2)} \quad \text{since } Q_0 \sim M_{\text{hadron}} \sim \Lambda_{QCD}$$

- one parton emission $\rightarrow$ corrections of $\mathcal{O}(1)$ $\Rightarrow$ not power suppressed
- resume these corrections to all-order in perturbation theory (include many emissions)

Hard processes with initial-state hadrons in pQCD

Second problem: Reliable estimate requires resummation of log corrections to all orders in α_s

- both problems (sensitivity to IR cutoff, all-order resummation) solved by
  UNIVERSAL FACTORIZATION OF COLLINEAR SINGULARITIES

- exploit analogy with renormalization
- since parton densities are not calculable in pQCD, assume

$$\text{naive parton model} \quad f(z) \longrightarrow f^0(z) \quad \text{bare parton density}$$

- $f^0(z) \Rightarrow$ parton density at some non-perturbative scale Q_0
- then, absorb collinear singularities in redefinition ($\sim$ renormalization) of parton density

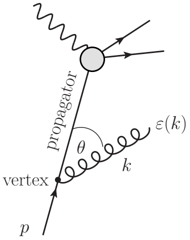
$$f^0 \, \Gamma\left(\alpha_s(Q^2), \frac{Q^2}{Q_0^2}\right) \equiv f(Q^2) \quad \text{true (physical) parton density}$$

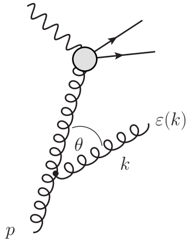
- overall singular factor: $\Gamma\left(\alpha_s(Q^2), \frac{Q^2}{Q_0^2}\right)$ contains all $\left[\alpha_s(Q^2) \cdot \ln \frac{Q^2}{Q_0^2}\right]^n$
- this procedure works if Γ is an universal factor (depends only on hadron) (independent of hard scattering and factorizable from it)

Universal factorization of collinear singularities

- heuristic argument (detailed proof much more involved) based on power counting for collinear singularities in physical gauge

- correction to partonic cross section due to radiation of collinear parton of momentum k
- to identify collinear singularities look for $1/Q^2$ behaviour in squared matrix element





phase-space factor
$\frac{d^3k}{2k^0} \sim dk^0 k^0 d\varphi d(\cos\theta)$
φ azimuthal angle
θ emission angle
$\theta \to 0: \quad d(\cos\theta) \sim d\theta^2$

MATRIX ELEMENT:

propagator:

$$\frac{1}{(p-k)^2} = -\frac{1}{2p\cdot k} = -\frac{1}{2p^0k^0(1-\cos\theta)} \underset{\theta\to 0}{\propto} \frac{1}{\theta^2}$$

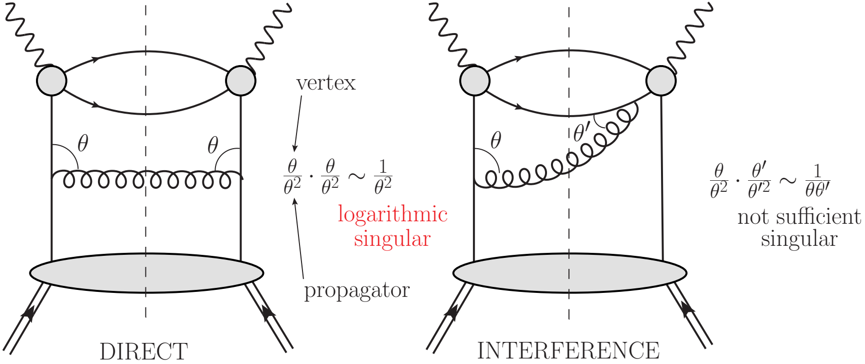
vertex (physical gauge):

$$p_i \cdot \varepsilon(p_j) \underset{\theta\to 0}{\propto} \theta$$

- because physical (transverse) polarization ($p \cdot \varepsilon(p) = 0$)
- $p_i \underset{\theta\to 0}{\propto} p + \mathcal{O}(\theta)$

Universal factorization of collinear singularities

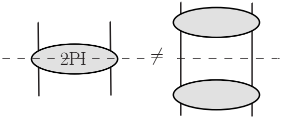
- squaring the matrix element (DIS case)

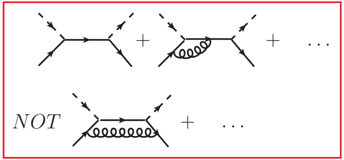


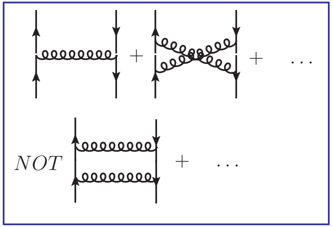
- only direct contributions can lead to collinear singularities in physical gauge
  (in covariant gauge interferences also contribute but final gauge invariant result is the same)

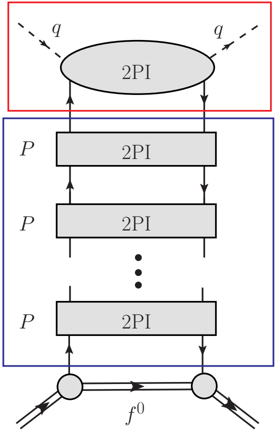
Universal factorization of collinear singularities

- decompose DIS partonic cross section in **two-particle irreducible (2PI) subgraphs**
- 2PI $\Rightarrow$ cannot be disjoint by cutting only two lines







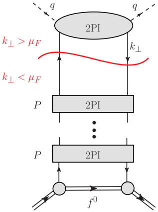


process dependent but no collinear singularities

leads to collinear singularities but process independent (universal $2 \to 2$ parton scattering processes)

Universal factorization of collinear singularities

- introduce arbitrary factorization scale μ_F and split last (upper) $k_\perp$-integration



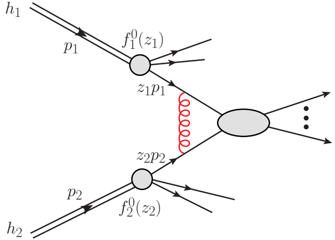
$$\int_{Q_0^2}^{Q^2} \frac{dk_\perp^2}{k_\perp^2} \ldots = \begin{cases} \displaystyle\int_{\mu_F^2}^{Q^2} \frac{dk_\perp^2}{k_\perp^2} \ldots \longrightarrow \hat{\sigma}_{\mathrm{hard}}(\mu_F^2) \\ \displaystyle\int_{Q_0^2}^{\mu_F^2} \frac{dk_\perp^2}{k_\perp^2} \ldots \longrightarrow \Gamma\left(\frac{\mu_F^2}{Q_0^2}\right) f^0 \equiv f(\mu_F^2) \end{cases}$$

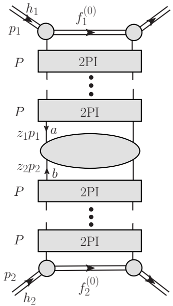
- UNIVERSAL FACTORIZATION FORMULA (factorization theorem of coll. sing.)

$$\sigma(p, Q^2) = \sum_{a=q,\bar{q},g} \int_0^1 dz\, \hat{\sigma}_{\mathrm{hard},a}(zp, \alpha_s(Q^2), \mu_F^2)\, f_a(z, \mu_F^2)$$

- $\hat{\sigma}_{\mathrm{hard},a}(zp, \alpha_s(Q^2), \mu_F^2)$: properly defined (subtracted) partonic cross section (no col. sing., computable order by order in pQCD)
- $f_a(z, \mu_F^2)$: process independent and scale-dependent parton densities (unlike those of naive parton model)
- Q: hard scale

Universal factorization of collinear singularities





- same argument applies to hard-scattering processes in hadron-hadron collisions
- only one new feature w.r.t. lepton-hadron collisions: diagrams with interferences between the two colliding partons
- in physical gauge, again collinear suppressed by power counting

FACTORIZATION FORMULA:

$$\sigma(p_1, p_2, Q^2) \;=\; \sum_{a,b} \int_0^1 dz_1 \int_0^1 dz_2 \; f_{1,a}(z_1, \mu_F^2)\, f_{2,b}(z_2, \mu_F^2)$$

$$\times \;\hat{\sigma}_{\mathrm{hard},ab}(z_1 p_1, z_2 p_2, \alpha_s(Q^2), \mu_F^2)$$

Main features of factorization theorem

1. Introduction of arbitrary factorization scale μ_F
2. Scale dependent parton densities
3. Scale dependence of $f(z, Q^2)$ calculable in pQCD

Physical cross section $\sigma(Q^2)$ cannot depend on μ_F: $\sigma(Q^2) \sim \hat{\sigma}_{\rm hard}(\alpha_s(Q^2), \mu_F^2) \cdot f(\mu_F^2)$
$\Rightarrow$ μ_F dependence in $\hat{\sigma}_{\rm hard}(\alpha_s(Q^2), \mu_F^2)$ and $f(\mu_F^2)$ compensate

- in principle: choose arbitrary μ_F
- **in practice**:

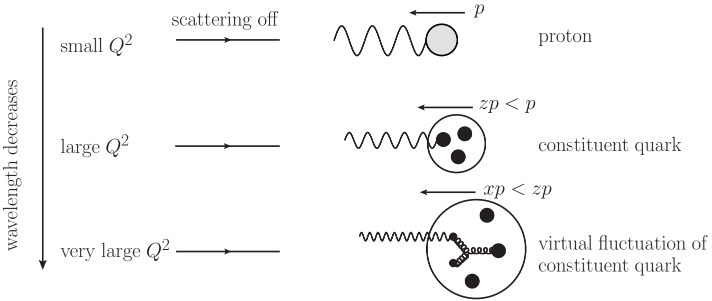
$$\hat{\sigma}_{\rm hard}(\alpha_s, \mu_F) = \alpha_s \left[\hat{\sigma}^{(0)} + \alpha_s \hat{\sigma}^{(1)}(Q/\mu_F) + \ldots + \alpha_s^n \hat{\sigma}^{(n)}(Q/\mu_F) + \ldots\right]$$

- $\hat{\sigma}^{(n)}(Q/\mu_F)$ contain terms $\alpha_s^n (\ln \frac{Q}{\mu_F})^n$ from integration of log collinear spectrum when $k_\perp > \mu_F$
- if μ_F very different from $Q \Rightarrow \left|\ln \frac{Q}{\mu_F}\right| \gg 1$
  $\Rightarrow$ **reliability of fixed-order expansion spoiled**
- **set**: $\mu_F \sim Q$

Main features of factorization theorem

Scale dependence of $f(z, Q^2)$ comes from resummation of large collinear logs

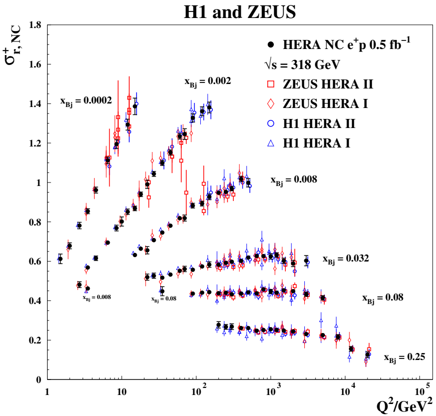
- physical consequence: SCALING VIOLATION (e.g. in DIS, violation of Bjorken scaling at large Q^2, structure functions $F_i(x, Q^2)$ depends logarithmically on Q^2)
- **physical picture**: resolution power of hard-scattering probe increases with Q^2

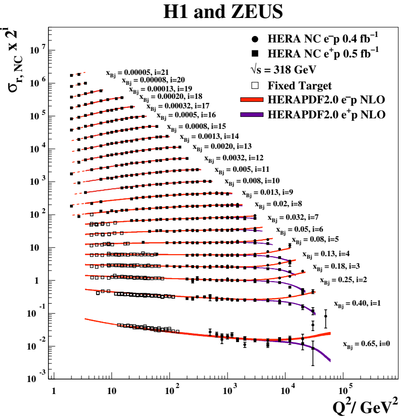


- increasing $Q^2 \Rightarrow$ more probable to scatter off lower-momentum parton
- shift of partons from higher to lower x
- scaling violation: positive at small-x and negative at large-x

Main features of factorization theorem

SCALING VIOLATION:



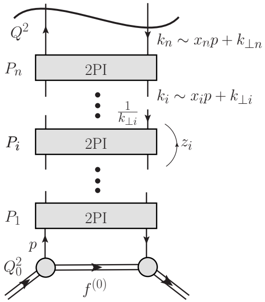


- large-x values: dominated by valence quarks ($Q^2 \uparrow \quad \rightarrow \quad F_2 \downarrow$)
- small-x values: dominated by gluons and sea quarks ($Q^2 \uparrow \quad \rightarrow \quad F_2 \uparrow$)

Main features of factorization theorem

- scale-dependent parton densities $f(x, Q^2)$ not calculable in pQCD
- can be extracted from experimental data (in principle from a single experiment)
- MORE IMPORTANT: **Scale dependence predicted (calculable) by pQCD**
- need experimental information from a one experiment at one input scale

Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution equations



- 2PI kernel: $P_i = P(\alpha_s(k_{\perp i}^2), z_i)$
- z_i: fraction of longitudinal momentum transferred along the cascade $\Rightarrow x_i = \prod_{j=1}^{i} z_i$
- propagator: $\frac{1}{k_i^2} \sim \frac{1}{k_{\perp i}^2}$
- logarithmically dominant kinematical region:

$$Q^2 > k_{\perp n}^2 > k_{\perp n-1}^2 > \ldots > k_{\perp 1}^2 > Q_0^2 \quad k_\perp\text{-ordering}$$

$$f(x, Q^2) \simeq f^{(0)}(x) + \int_{Q_0^2}^{Q^2} \frac{dk_{\perp n}^2}{k_{\perp n}^2} \int_x^1 P_n(\alpha_s(k_{\perp n}^2), z_n)\, f\left(\frac{x}{z_n}, k_{\perp n}^2\right)$$

DGLAP evolution equations

$$f(x, Q^2) \simeq f^{(0)}(x) + \int_{Q_0^2}^{Q^2} \frac{dk_{\perp n}^2}{k_{\perp n}^2} \int_x^1 P_n(\alpha_s(k_{\perp n}^2), z_n)\, f\left(\frac{x}{z_n}, k_{\perp n}^2\right)$$

Taking derivative of $f(x, Q^2)$ w.r.t. Q^2:

$$\frac{df(x, Q^2)}{d \ln Q^2} = \int_x^1 \frac{dz}{z} P(\alpha_s(Q^2), z)\, f\left(\frac{x}{z}, Q^2\right)$$

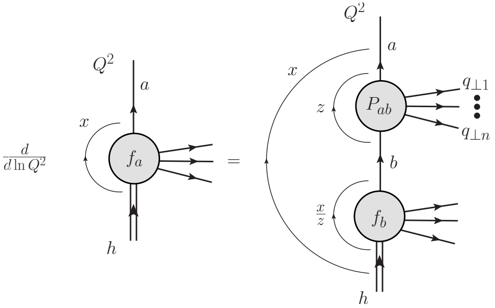
- 1st order differential equation $\Rightarrow$ can be solved by giving initial condition $f\left(\frac{x}{z}, Q_0^2\right)$ from experiment (parton densities at a single input scale Q_0)
- Q_0 no longer arbitrary IR cutoff
- **probability** $\boldsymbol{P(\alpha_s(Q^2), z)}$ $\Rightarrow$ perturbatively calculable (no collinear singularities) as power series expansion in α_s

DGLAP evolution equations

Probabilistic interpretation $\Rightarrow$ system of coupled equations w.r.t. flavours of partons

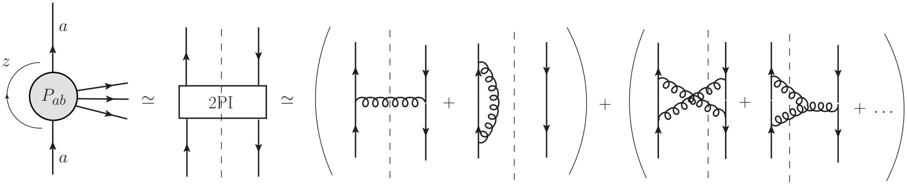
$$\begin{cases} \dfrac{df_q(x,Q^2)}{d\ln Q^2} = P_{qq}\otimes f_q + P_{q\bar q}\otimes f_{\bar q} + P_{qg}\otimes f_g & \text{similiar with } q \Leftrightarrow \bar q \\ \dfrac{df_g(x,Q^2)}{d\ln Q^2} = P_{gq}\otimes f_q + P_{g\bar q}\otimes f_{\bar q} + P_{gg}\otimes f_g \end{cases}$$

where convolution $f\otimes g \equiv \int_0^1 \frac{dz}{z} f\left(\frac{x}{z}\right)\, g(z)$ ($\Rightarrow$ longitudinal momentum conservation)



- P_{ab}: probability of parton evolution $b(p) \to a(zp)$ by radiating a bunch of partons with $q'_\perp s$ of the same order ($q_{\perp 1} \sim q_{\perp 2} \sim \ldots \sim Q$) in the rapidity interval $\Delta y = \frac{1}{z}$

DGLAP evolution equations



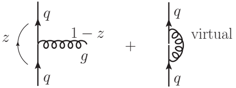
$$P_{ab}(\alpha_s, z) \quad = \quad \frac{\alpha_s}{2\pi} P_{ab}^{(LO)}(z) \quad + \quad \left(\frac{\alpha_s}{2\pi}\right)^2 P_{ab}^{(NLO)}(z) + \ldots$$

- solving DGLAP equations with $P_{ab}^{(LO)}$, $P_{ab}^{(NLO)}$, $P_{ab}^{(NNLO)}$, ...
  $\Rightarrow$ equivalent to resumme large:
  leading logs $\alpha_s^n \ln^n \frac{Q}{Q_0}$, next-to-leading logs $\alpha_s^n \ln^{n-1} \frac{Q}{Q_0}$, ...
- emission of an additional parton without $k_\perp$-ordering costs a power of α_s (with no enhancing $\ln Q$ factor)

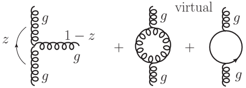
DGLAP evolution equations

DGLAP probabilities at LO:

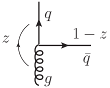
$$P_{qq}(z) = C_F\left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\right]$$



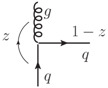
$$\begin{aligned} P_{gg}(z) \quad &= \quad 2C_A\left[\frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z)\right] \\ &+ \quad \delta(1-z)\frac{1}{6}(11C_A - 2N_f) \end{aligned}$$



$$P_{qg}(z) = T_R\left[z^2 + (1-z)^2\right]$$

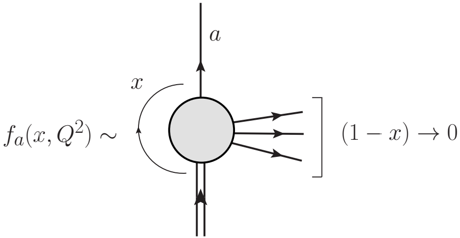


$$P_{gq}(z) = C_F\left[\frac{1+(1-z)^2}{z}\right]$$



- $P_{ab}(z)$ positive for $z < 1$ (as it should be for probabilities)
- $z \to 1$: soft singularity $\Rightarrow P_{qq} \sim 2C_F(\frac{1}{1-z})_+$ and $P_{gg} \sim 2C_A(\frac{1}{1-z})_+$
- $z \to 0$: enhancement of $P_{gg} \sim \frac{2C_A}{z}$ and $P_{gq} \sim \frac{2C_F}{z}$

DGLAP: large-x limit



- $x \to 1$: only soft-parton (gluon radiation)
- evolution dominated by by

$$P_{qq} \sim 2C_F \left(\frac{1}{1-z}\right)_+ \qquad P_{gg} \sim 2C_A \left(\frac{1}{1-z}\right)_+$$

- note: large log

$$\int_x^1 dz \left(\frac{1}{1-z}\right)_+ = -\int_0^x \frac{dz}{1-z} = \ln(1-x)$$

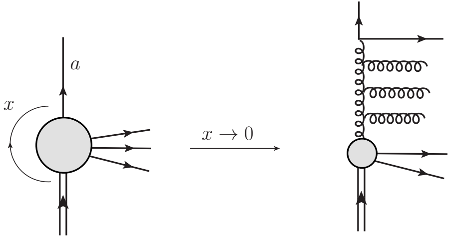
- both for quarks and for gluons

$$f_a(x, Q^2) \simeq f_a(x, Q_0^2) \cdot (1-x)^{p_a} \qquad f_a(x, Q_0^2) \sim (1-x)^\eta$$

$$(1-x)^{p_a} = \exp\left\{p_a \ln(1-x)\right\} = \exp\left\{\int_x^1 dz \frac{2C_a}{(1-z)_+} \int_{Q_0^2}^{Q^2} \frac{dq^2}{q^2} \frac{\alpha_s(q^2)}{2\pi}\right\}$$

- $\displaystyle\int_{Q_0^2}^{Q^2} \frac{dq^2}{q^2} \alpha_s(q^2) = \frac{1}{\beta_0} \ln \frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}$
- $p_a = \frac{C_a}{\pi\beta_0} \ln \frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)} \Rightarrow p_a > 0$ for $Q > Q_0 \Rightarrow$ power suppression increases with Q^2

DGLAP: small-x limit



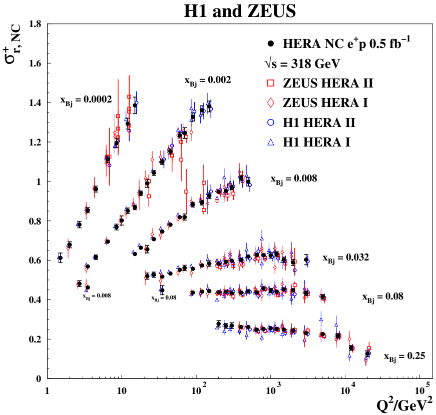
- $x \to 0$: multiple soft gluon exchanges in the evolution dominate (with radiation of many hard gluons)
- evolution driven by gluon density

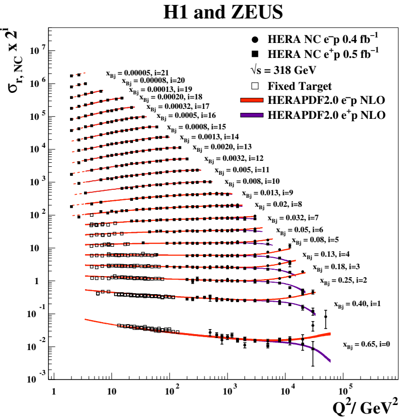
$$P_{gg} \sim \frac{2C_A}{z}$$

$$xf_g(x, Q^2) = xf_g(x, Q_0^2) \times \exp\left\{ \sqrt{\frac{2C_A}{\pi} \frac{1}{\beta_0} \ln \frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)} \ln \frac{1}{x}} \right\}$$

- $\sqrt{\ln \frac{1}{x}}$: strong rise of gluon density
- faster than any power of $\ln \frac{1}{x}$, though slower than any power of $\frac{1}{x}$
- $\sqrt{\frac{1}{\beta_0} \ln \frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}}$: steepness increases with Q^2
- gluon self interactions $\Rightarrow$ peculiar feature of non-abelian gauge theory

DGLAP and experimental data



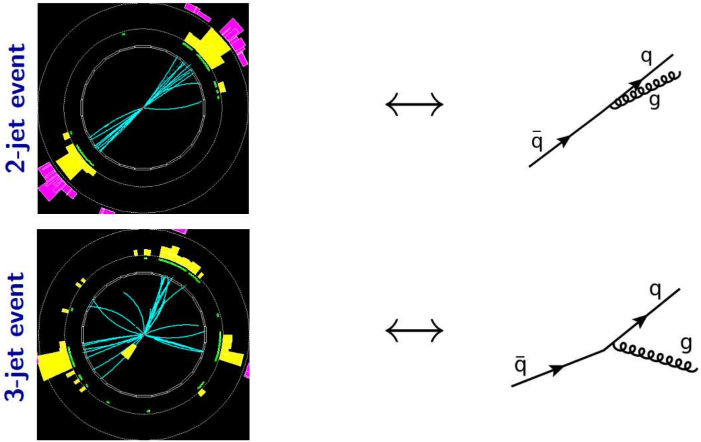


- large-x: $Q^2 \uparrow \;\rightarrow\; F_2 \downarrow$ small-x: $Q^2 \uparrow \;\rightarrow\; F_2 \uparrow$
- DGLAP evolution equations give quantitative description of observed scaling violation
- scale dependence well described by the pQCD

THANK YOU!

BACKUP SLIDES

IR/collinear safe obervables



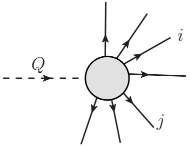
Need a quantitative measure that would allow us to classify events as 2- or 3-jets, both in theoretical calculations and in experiment.

Jet algorithms

CLUSTERING:

- define "distance" between two particles: $y_{ij} = \frac{d_{ij}}{Q^2}$
- d_{ij}: dimensionful resolution variable (distance measure)
- merge particles with minimum y_{ij} until a fixed resolution y_{cut}:

  $y_{ij} > y_{\mathrm{cut}} \quad \forall_{ij}$



Compute distances between particles for all particle pairs: $d_{ij} = \min\left(p_{ti}^{2p}, p_{tj}^{2p}\right)\Delta R_{ij}^2/R^2$ and the particle-beam distances for all particles: $d_{iB} = p_{ti}^{2p}$, where R is a jet radius and $\Delta R_{ij} = \sqrt{(y_i - y_j)^2 + (\phi_i - \phi_j)^2}$ is a distance between particles in the $(y - \phi)$-plane.

Find smallest d_{ij} and d_{iB}:

- $d_{ij} < d_{iB}$: recombine the two particles and add the particle ij to the list of particles
- $d_{ij} > d_{iB}$: call i particle a jet and remove form particle list

k_t algorithm:
$(p = 1)$
$d_{ij} = \min\left(p_{ti}^2, p_{tj}^2\right)\Delta R_{ij}^2/R^2$

Cambridge/Aachen algorithm:
$(p = 0)$
$d_{ij} = \Delta R_{ij}^2/R^2$

anti-k_t algorithm:
$(p = -1)$
$d_{ij} = \min\left(\frac{1}{p_{ti}^2}, \frac{1}{p_{tj}^2}\right)\Delta R_{ij}^2/R^2$