Small-instanton induced flavor invariants and the axion potential

Based on 2402.09361

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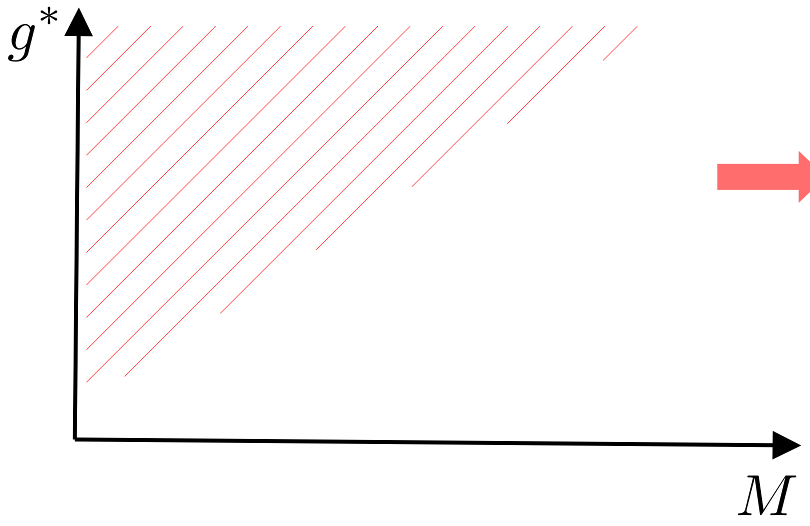
Where is new physics?



SMEFT: test assumptions on the UV

- Positivity
- Weakly-coupled: Trees vs Loops
- Flavor hierarchies
- Mapping models to observables

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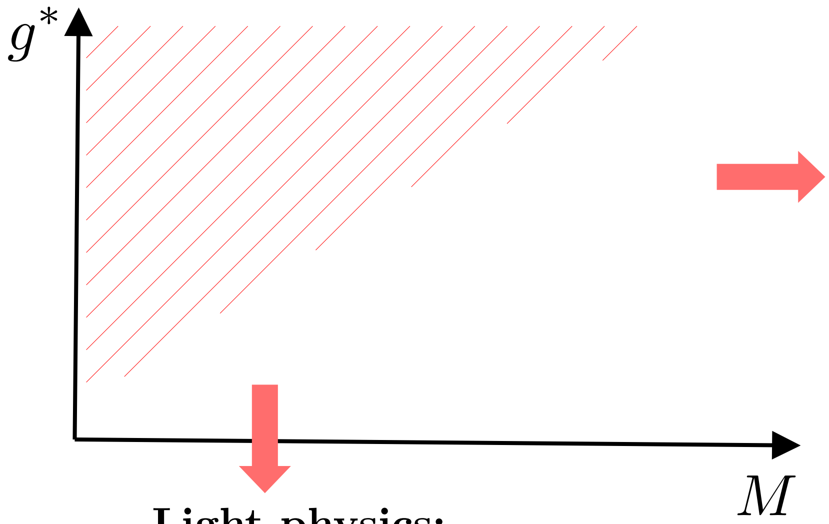

Light physics:

Axion and axion-like particles

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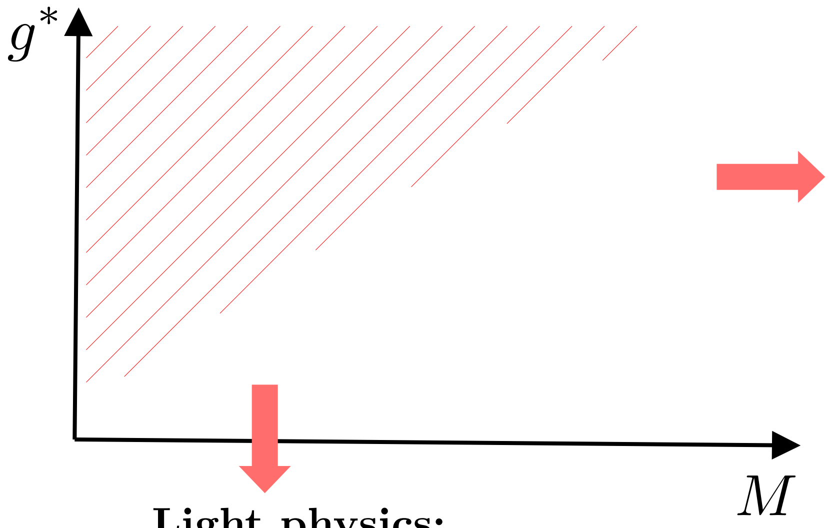

Light physics:

Axion and axion-like particles

SMEFT: test assumptions on the UV

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How does heavy physics affect axion solutions?

The strong CP-problem

$$\mathcal{L} \supset \bar{\theta}\frac{g_s}{16\pi^2}G\tilde{G} \qquad \bar{\theta} = \theta + \arg(\det\mathcal{M}_q)$$

- CP-odd term – contributes to neutron EDM

$$d_n < 3 \times 10^{-26} e \cdot cm \Rightarrow \bar{\theta} < 10^{-10}$$

- Why so small?
  - Symmetry based solution
  - Dynamical solution

The QCD axion

- A dynamical solution: introduce a spontaneously broken $U(1)_A$

$$\mathcal{L} \supset \left(\bar{\theta} + \frac{a}{f_a}\right) \frac{g_s}{16\pi^2} G\tilde{G}$$

- Axion is the Goldstone boson, whose shift symmetry allows to absorb θ effects:

$$\theta_{\text{eff}} \equiv \frac{\langle a \rangle}{f_a}$$

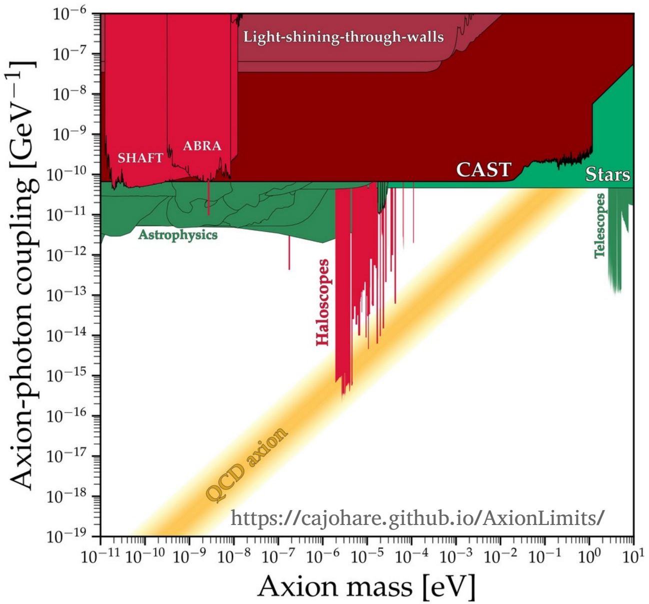
<u>Strong CP-problem is now a question about the vev of the axion</u>

The axion potential

- QCD effects break shift-symmetry and generate an axion potential, with $\langle a \rangle = 0$ Vafa, Witten 94
- Through $\chi_{\rm PT}$ axion mass can be related to pion mass

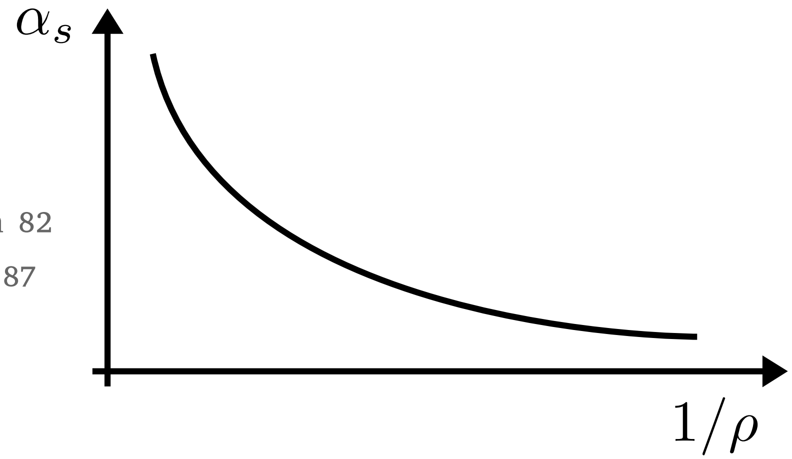
$$m_a f_a = \frac{m_u m_d}{(m_u + m_d)^2} m_\pi^2 f_\pi^2$$

Cortona, Hardy, Vega, Villadoro 1511.02867



How stable is this prediction?

Enhancing the axion mass



Holdom and Peskin 82

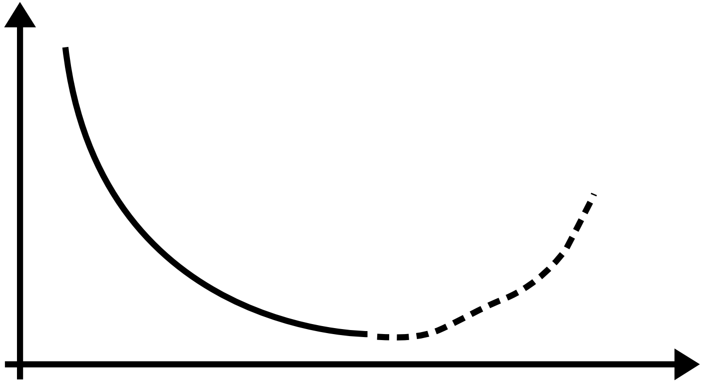
Flynn and Randall 87

Instanton contributions scale as:

$$V(a) \sim e^{-\frac{8\pi}{\alpha_s(1/\rho)}}$$

$$m_a = m_{\rm QCD}$$

Enhancing the axion mass



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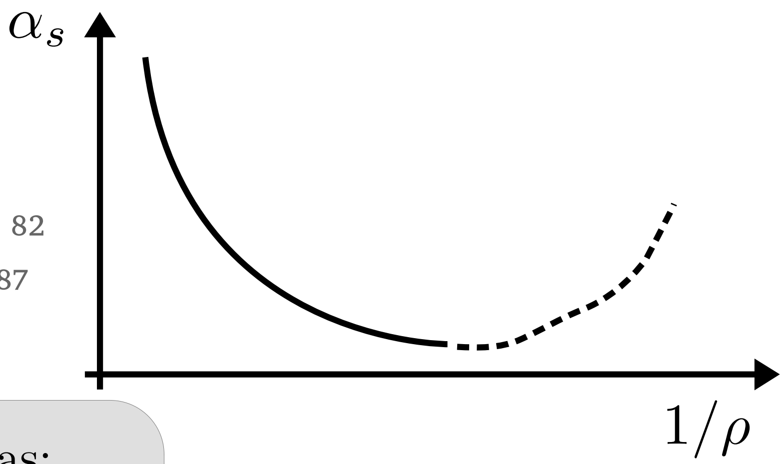
May be relevant. Calculable in one-instaton limit

Enhancing the axion mass



Holdom and Peskin 82

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Instanton contributions scale as:

$$V(a) \sim e^{-\frac{8\pi}{\alpha_s(1/\rho)}}$$

Other ideas:

Hook 1802.10093
Luzio, Gavela, Quilez, Ringwald 2102.00012
Valenti, Vecchi, Xu 2206.04077
Gavela, Quilez, Ramos 2305.15465

$$m_a = m_{\mathrm{QCD}} + m_{\mathrm{UV}}$$

May be relevant. Calculable in one-instaton limit

Product gauge groups

Agrawal and Howe 1710.04213

$$SU(3)_1 \times SU(3)_2 \to SU(3)_c$$

$$\frac{1}{\alpha_1(M)} + \frac{1}{\alpha_2(M)} = \frac{1}{\alpha_s(M)}$$

- Each sector is more strongly coupled than QCD and therefore has more relevant instanton effects
- Stronger effects when one considers k-product of $SU(3)$

Csáki, Ruhdorfer, Shirman 1912.02197

CP-violation (SM)

- Breaking of shift-symmetry and CP-violation result in linear term for the potential

$$V(a) = \chi_{\mathcal{O}}(0)\frac{a}{f_a} + \frac{1}{2}\chi(0)\left(\frac{a}{f_a}\right)^2$$

- This linear term shifts the minimum

$$\theta_{\text{ind}} \equiv -\frac{\chi_{\mathcal{O}}(0)}{\chi(0)}$$

<u>CP-violating effects induce contributions to θ</u>

CP-violation (SM)

- CP violation in the SM is parameterized by

$$J_4 = \mathrm{Im}\left(\mathrm{Tr}\left[Y_u Y_u^\dagger, Y_d Y_d^\dagger\right]\right)$$

Jarlskog 85
Bernabeu, Branco, Gronau 85

- The misalignment of the axion potential from SM CPV is too small to be observed:

$$\theta_{\rm eff}^{\rm SM} \sim \frac{G_F^2}{m_c^2} J_{\rm CKM} f_\pi^4 \Lambda_\chi^2 \sim 10^{-19}$$

Georgi, Randall 86
Luzio, Gisbert, Levati, Paradisi, Sørensen 2312.17310

- Had J_4 been larger, PQ solution would not have worked
- Observation of axion and $\theta > 10^{-19}$ implies new sources of CPV.

New sources of CPV – SMEFT

- Using the SMEFT expansion to parameterize new physics:

$$\mathcal{L}_{\text{SMEFT}} = \mathcal{L}_{\text{SM}} + \frac{\mathcal{L}_6}{\Lambda^2} + \mathcal{O}(1/\Lambda^4)$$

$$\mathcal{L}_d = c_i \mathcal{O}_i$$
$$[\mathcal{O}_i] = d$$

- Many potential sources of CPV – can they affect the QCD axion solution?
- Basis of Jarlskog-like CPV invariants for the SMEFT:

$$L_{abcd}(C) = \text{Im}\left[\text{Tr}\left(X_{\text{u}}^a X_{\text{d}}^b X_{\text{u}}^c X_{\text{d}}^d C\right)\right]$$

$$X_{u,d} \equiv Y_{u,d} Y_{u,d}^\dagger$$

Bonnefoy, Gendy, Grojean, Ruderman
2112.03889, 2302.07288

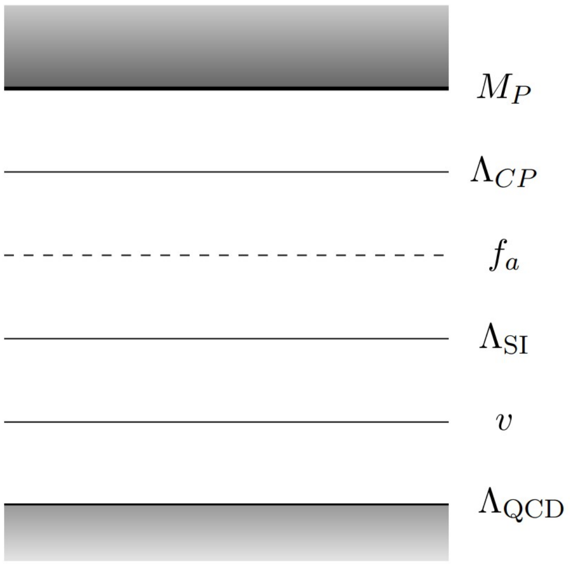
Why does this matter?

- CP-violation effects with only QCD : $\sim \dfrac{\Lambda_{\text{QCD}}}{\Lambda_{\text{CPV}}}$
- In the presence of high-energy instantons, contributions to enhanced EDMs

$$\sim \frac{\Lambda_{\text{SI}}}{\Lambda_{\text{CPV}}} \lesssim 10^{-10}$$

<u>Do these models still work?</u>

<u>Can CPV invariants be useful?</u>



Bedi, Gherghetta, Pospelov
2205.07948

Instanton computations – bosonic contributions

- Vacuum to vacuum transition in instanton background

't Hooft 76
Shifman Vainshtein, Zakharov 79

$$\langle 0|0\rangle\big|_{\text{1-inst.}} = e^{-i\theta_{\text{QCD}}} \int d^4x_0 \int \frac{d\rho}{\rho^5}\, d_N(\rho) \int \prod_{f=1}^{N_f} \left(\rho\, d\xi_f^{(0)} d\bar{\xi}_f^{(0)}\right) e^{-\bar{\psi} J \psi + \text{h.c.}}$$

Belavin, Polyakov, Schwartz, Tyupkin 75

BPST instanton:

- Gauge orientation
- Size, ρ
- Center, x_0

$$\int \mathcal{D}\mathcal{A}\, e^{\mathcal{A}_\mu M_{\mu\nu} \mathcal{A}_\nu} = \frac{1}{\sqrt{\det M}}$$

collective coordinates → Zero modes, divergent path integral

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- Split into zero and non-zero modes

$$\int \mathcal{D}\mathcal{A} \sim \int dx_0^4 \int d\rho\, J(\eta, x_0, \rho) \int \mathcal{D}\tilde{\mathcal{A}}$$

Jacobian from coordinate transformation

Integral over non-zero modes

't Hooft 76

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- Instanton solution will correspond to fermion zero modes

$$-i\not{D}\Big|_{\text{1-inst.}} \psi^{(0)}(x) = 0$$

$$\psi^{(0)}(x)\Big|_{\text{1-inst.}} = \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} = \frac{1}{\pi} \frac{\rho}{\left[(x-x_0)^2+\rho^2\right]^{3/2}} \begin{pmatrix} 0 \\ \varphi \end{pmatrix}$$

Only $\chi_L^\dagger, \chi_R$ will possess zero modes in the instanton solution. In anti-instanton $\chi_R^\dagger, \chi_L$ will possess solution

Shifman 2012

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't Hooft 76
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$$\langle 0|0\rangle\Big|_{\text{1-inst.}} = e^{-i\theta_{\text{QCD}}}\int d^4x_0 \int \frac{d\rho}{\rho^5}\, d_N(\rho) \int \prod_{f=1}^{N_f}\left(\rho\, d\xi_f^{(0)}\, d\bar{\xi}_f^{(0)}\right) e^{-\bar{\psi}J\psi+\text{h.c.}}$$

- Eigenmode expansion: $\psi_f(x) = \sum_k \xi_f^{(k)}\psi^{(k)}$

$$\mathcal{D}\psi\mathcal{D}\overline{\psi} = d\xi^{(0)}d\bar{\xi}^{(0)}\,\mathcal{D}\tilde{\psi}\mathcal{D}\overline{\tilde{\psi}}$$

Integral over non-zero modes

't Hooft 76

$$\int d\xi_f^{(0)}\xi_f^{(0)} = 1 \implies \int d\xi_f^{(0)}\psi_f = \psi_f^{(0)}$$

Instanton computations

- Fermion zero modes give rise to determinant-like structures:

$$\int d^3\xi_1 d^3\xi_2 \; e^{\xi_1 A \xi_2} = \det A\,,$$

$$\int d^3\xi_1 d^3\xi_2 \; e^{\xi_1 A \xi_2} \xi_1 B \xi_2 = \frac{1}{2}\epsilon^{i_1 i_2 i_3}\epsilon^{j_1 j_2 j_3} A_{i_1 j_1} A_{i_2 j_2} B_{i_3 j_3}$$

$$\det(A) = \frac{1}{n!}\epsilon_{i_1,\ldots,i_n}\epsilon_{i_1,\ldots,i_n} A_{i_1 j_1} \ldots A_{i_n j_n}$$

Trace-like invariants will not match these structures making connection to observables less direct

Determinant-like invariants

	$U(3)_{\mathrm{Q}}$	$U(3)_{\mathrm{u}}$	$U(3)_{\mathrm{d}}$	$U(3)_{\mathrm{L}}$	$U(3)_{\mathrm{e}}$
$e^{i\theta_{\mathrm{QCD}}}$	$\mathbf{1}_{+6}$	$\mathbf{1}_{-3}$	$\mathbf{1}_{-3}$	$\mathbf{1}_{0}$	$\mathbf{1}_{0}$
Y_{u}	$\mathbf{3}_{+1}$	$\bar{\mathbf{3}}_{-1}$	$\mathbf{1}_{0}$	$\mathbf{1}_{0}$	$\mathbf{1}_{0}$
Y_{d}	$\mathbf{3}_{+1}$	$\mathbf{1}_{0}$	$\bar{\mathbf{3}}_{-1}$	$\mathbf{1}_{0}$	$\mathbf{1}_{0}$
Y_{e}	$\mathbf{1}_{0}$	$\mathbf{1}_{0}$	$\mathbf{1}_{0}$	$\mathbf{3}_{+1}$	$\bar{\mathbf{3}}_{-1}$

SM invariant with

$$J_\theta = \mathrm{Im}[e^{-i\theta_{\mathrm{QCD}}} \det(Y_{\mathrm{u}} Y_{\mathrm{d}})]$$

$\mathcal{O}^{(1)}_{\mathrm{ququd}} = \bar{Q}u\bar{Q}d$ has 81 CPV phases

$$\mathcal{I}(C^{(1,8)}_{\mathrm{ququd}}) = \mathrm{Im}\left[e^{-i\theta_{\mathrm{QCD}}}\epsilon^{ABC}\epsilon^{abc}\epsilon^{DEF}\epsilon^{def}Y_{\mathrm{u},Aa}Y_{\mathrm{u},Bb}C^{(1,8)}_{\mathrm{ququd},CcDd}Y_{\mathrm{d},Ee}Y_{\mathrm{d},Ff}\right]$$

- Redundant in regards to Trace-like basis!

Determinant-like invariants

$$X_{u,d} \equiv Y_{u,d} Y_{u,d}^{\dagger}$$

- $\mathcal{O}_{\text{quqd}}^{(1)} = \bar{Q}u\bar{Q}d$

$$\mathcal{A}^{a_1,b_1,c_1,d_1}_{a_2,b_2,c_2,d_2}(C^{(1,8)}_{\text{quqd}}) = \text{Im}\Big[e^{-i\theta_{\text{QCD}}}\epsilon^{ABC}\epsilon^{abc}\epsilon^{DEF}\epsilon^{def}Y_{\text{u},Aa}Y_{\text{u},Bb}\left(X_{\text{u}}^{a_1}X_{\text{d}}^{b_1}X_{\text{u}}^{c_1}X_{\text{d}}^{d_1}\right)_C^{\ \ C'} \times C^{(1,8)}_{\text{quqd},C'cD'd}\left(X_{\text{u}}^{a_2}X_{\text{d}}^{b_2}X_{\text{u}}^{c_2}X_{\text{d}}^{d_2}\right)_D^{\ \ D'}Y_{\text{d},Ee}Y_{\text{d},Ff}\Big],$$

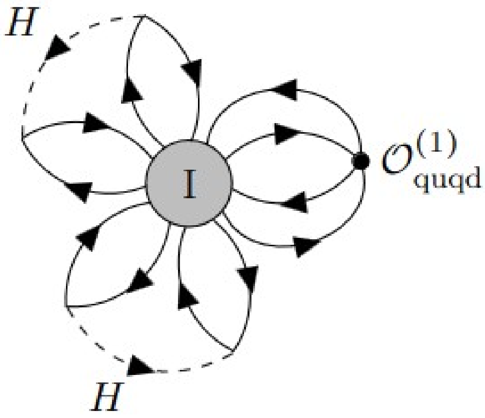
- $\mathcal{O}_{\text{Hq}}^{(1)} = \left(H^{\dagger}i\overleftrightarrow{D}_{\mu}H\right)\left(\bar{Q}\gamma^{\mu}Q\right)$

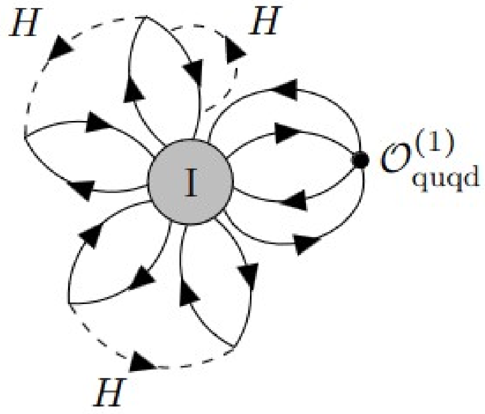
$$\mathcal{I}_{abcd}(C^{(1,3)}_{\text{Hq}}) \equiv \text{Im}\left[e^{-i\theta_{\text{QCD}}}\epsilon^{IJK}\epsilon^{ijk}Y_{\text{u},Ii}Y_{\text{u},Jj}\left(X_{\text{u}}^{a}X_{\text{d}}^{b}X_{\text{u}}^{c}X_{\text{d}}^{d}\,\underline{C^{(1,3)}_{\text{Hq}}Y_{\text{u}}}\right)_{Kk}\det Y_{\text{d}}\right]$$

- $\mathcal{O}_{\text{lequ}}^{(1)} = \left(\bar{L}e\right)\left(\bar{Q}u\right)$

$$\mathcal{I}^{f}_{abcd}(C^{(1,3)}_{\text{lequ}}) \equiv \text{Im}\left[e^{-i\theta_{\text{QCD}}}\epsilon^{IJK}\epsilon^{ijk}Y_{\text{u},Ii}Y_{\text{u},Jj}\left(X_{\text{u}}^{a}X_{\text{d}}^{b}X_{\text{u}}^{c}X_{\text{d}}^{d}\right)_K^{\ \ L}\underline{\left(Y_{\text{e}}^{\dagger}X_{\text{e}}^{f}\right)^{mN}}C^{(1,3)}_{\text{lequ},NmLk}\det Y_{\text{d}}\right]$$

Determinant-like invariants

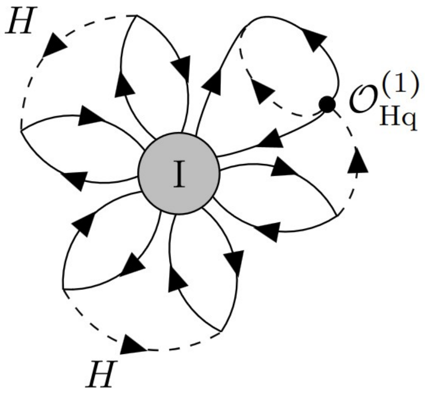


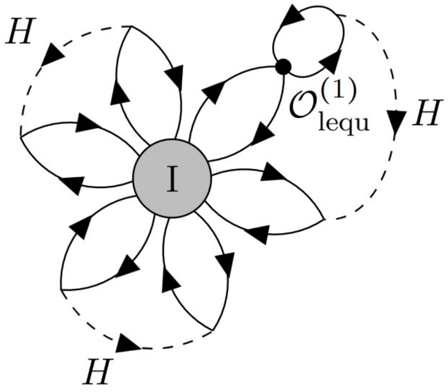


$$\mathcal{A}^{a_1,b_1,c_1,d_1}_{a_2,b_2,c_2,d_2}(C^{(1,8)}_{\rm quqd}) = {\rm Im}\Big[e^{-i\theta_{\rm QCD}}\epsilon^{ABC}\epsilon^{abc}\epsilon^{DEF}\epsilon^{def}Y_{{\rm u},Aa}Y_{{\rm u},Bb}\left(X_{\rm u}^{a_1}X_{\rm d}^{b_1}X_{\rm u}^{c_1}X_{\rm d}^{d_1}\right)_C{}^{C'}$$

$$\times C^{(1,8)}_{{\rm quqd},C'cD'd}\left(X_{\rm u}^{a_2}X_{\rm d}^{b_2}X_{\rm u}^{c_2}X_{\rm d}^{d_2}\right)_D{}^{D'}Y_{{\rm d},Ee}Y_{{\rm d},Ff}\Big],$$

Determinant-like invariants

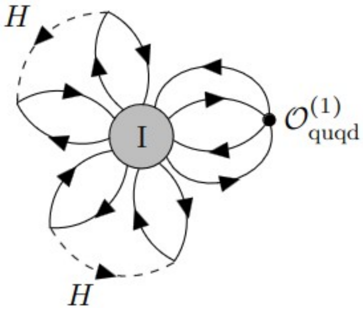




$$\mathcal{I}_{abcd}(C^{(1,3)}_{\mathrm{Hq}}) \equiv \mathrm{Im}\left[e^{-i\theta_{\mathrm{QCD}}}\epsilon^{IJK}\epsilon^{ijk}Y_{\mathrm{u},Ii}Y_{\mathrm{u},Jj}\left(X^a_{\mathrm{u}}X^b_{\mathrm{d}}X^c_{\mathrm{u}}X^d_{\mathrm{d}}\,C^{(1,3)}_{\mathrm{Hq}}\,Y_{\mathrm{u}}\right)_{Kk}\det Y_{\mathrm{d}}\right]$$

$$\mathcal{I}^f_{abcd}(C^{(1,3)}_{\mathrm{lequ}}) \equiv \mathrm{Im}\left[e^{-i\theta_{\mathrm{QCD}}}\epsilon^{IJK}\epsilon^{ijk}Y_{\mathrm{u},Ii}Y_{\mathrm{u},Jj}\left(X^a_{\mathrm{u}}X^b_{\mathrm{d}}X^c_{\mathrm{u}}X^d_{\mathrm{d}}\right)_K{}^L\left(Y^\dagger_{\mathrm{e}}X^f_{\mathrm{e}}\right)^{mN}C^{(1,3)}_{\mathrm{lequ},NmLk}\det Y_{\mathrm{d}}\right]$$

Computation



$$\chi^{(1)\,(\text{UV})}_{\text{quqd}}(0) = \frac{i}{\Lambda^2_{\not{CP}}}\left(\mathcal{A}^{0000}_{0000}\left(C^{(1)}_{\text{quqd}}\right) + \mathcal{B}^{0000}_{0000}\left(C^{(1)}_{\text{quqd}}\right)\right)\int \frac{d\rho}{\rho^5}\, d_N(\rho)\frac{2!}{(6\pi^2)^2}\frac{2}{5\pi^2\rho^2}$$

- Result proportional to invariants
- Integral over instanton size dependent on UV theory

NDA

- Result proportional to invariants
  - Easily test flavor assumptions of the WC
- Refine naive dimensional analysis estimates

Csáki, D'Agnolo, Kuflik, Ruhdorfer, 2311.09285

- Extend to other operators
  - While $\mathcal{O}_{quqd}$ seems to give the larger contribution, might be a suppressed in the UV

Flavor scenarios

- Results in flavor-invariant makes flavor assumption testing straightforward
  - Anarchic scenario – All flavor entries are $\mathcal{O}(1)$
    - Large flavor-chaging interactions
  - Minimal Flavor Violation (MFV) – WC are polynomials in Yukawa couplings
    - Less constrained

UV scenarios

- Product gauge groups

  Scalars which spontaneously break product gauge group to color group, vev $\langle \sigma \rangle$

$$d_N(\rho) \to d_N(\rho)\, e^{-2\pi^2 \rho^2 \sum |\langle \sigma \rangle|^2}$$

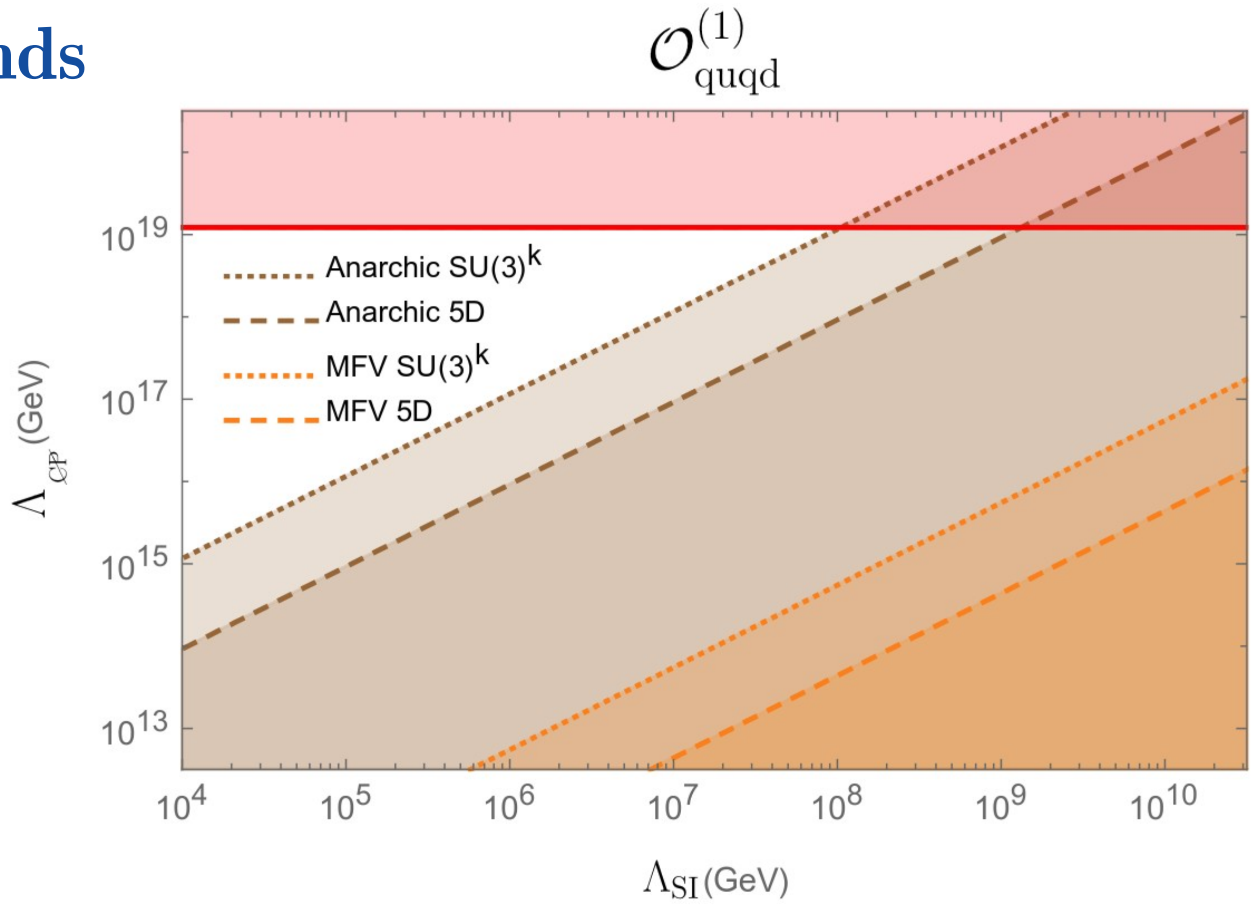
- 5D instantons

  Uplift BPST instanton to an extra dimension of size R

  Gherghetta, Khoze, Pomarol, Shirman, 2001.05610

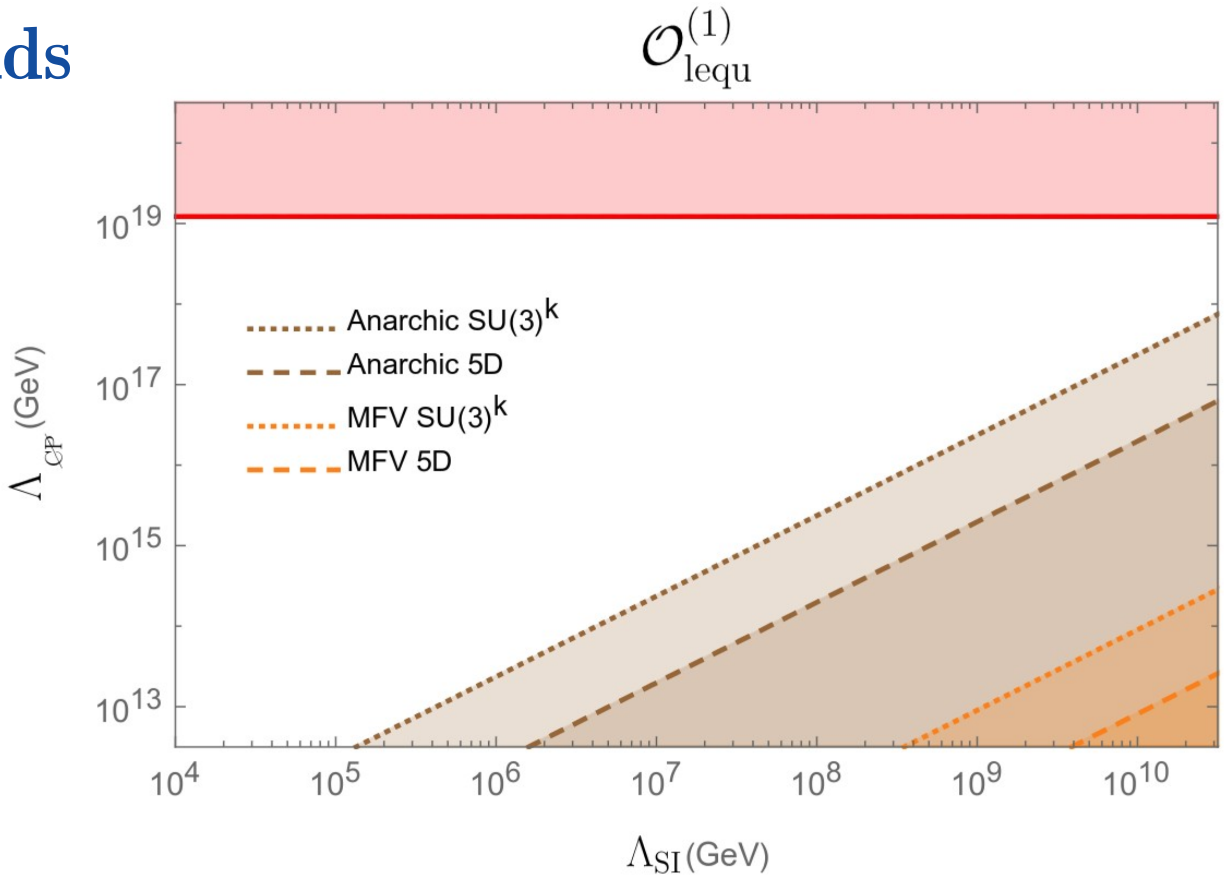
$$d_N(\rho) \to d_N(\rho)\, e^{R/\rho}$$

Bounds



$\mathcal{O}^{(1)}_{\text{quqd}}$
10^{19}
10^{17}
10^{15}
10^{13}
$\Lambda_{\not{C}\not{P}}$ (GeV)
Anarchic SU(3)^k
Anarchic 5D
MFV SU(3)^k
MFV 5D
10^4
10^5
10^6
10^7
10^8
10^9
10^{10}
Λ_{SI} (GeV)

Bounds



$\mathcal{O}_{\rm lequ}^{(1)}$
Anarchic SU(3)^k
Anarchic 5D
MFV SU(3)^k
MFV 5D
10^{19}
10^{17}
10^{15}
10^{13}
$\Lambda_{\not{CP}}$ (GeV)
10^4
10^5
10^6
10^7
10^8
10^9
10^{10}
$\Lambda_{\rm SI}$ (GeV)

Axion EFT – Shift-breaking effects

- Shift-symmetric ALP, derivative basis:

$$\mathcal{L} = \mathcal{L}_{\rm SM} + \frac{1}{2}(\partial_\mu a)(\partial^\mu a) + \frac{\partial_\mu a}{f} \sum_{\psi \in {\rm SM}} \bar{\psi} c_\psi \gamma^\mu \psi$$

- Shift-breaking ALP effects, Yukawa-like basis:

$$\mathcal{L} = \mathcal{L}_{\rm SM} + \frac{1}{2}(\partial_\mu a)(\partial^\mu a) - \frac{a}{f}\left(\bar{Q}\tilde{Y}_u \tilde{H} u + \bar{Q}\tilde{Y}_d H d + \bar{L}\tilde{Y}_e H e + {\rm h.c.}\right)$$

Axion EFT – Shift-breaking effects

- Yukawa-like basis is the more general, and can capture shift-symmetric effects
  - Performing a field-redefinition on the fermions $\psi' \equiv e^{-i\frac{a}{f}c_\psi}\psi$

$$\tilde{Y}_{u,d} = i(Y_{u,d}c_{u,d} - c_Q Y_{u,d}) \ , \quad \tilde{Y}_e = i(Y_e c_e - c_L Y_e)$$

Chala, G. G., Ramos, Santiago 2012.09017

- To verify whether a specific value of $\tilde{Y}$ respects shift-symmetry one needs invariants!

Bonnefoy, Grojean, Kley 2206.04182

Axion EFT – Shift-breaking invariants

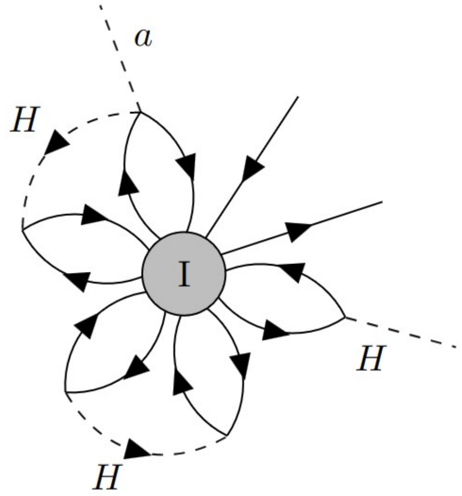
- There are 13 invariants in the fermionic sector
  - Only 1 is CP-even:

Bonnefoy, Grojean, Kley 2206.04182

$$I_{ud}^{(4)} = \mathrm{Im}\,\mathrm{Tr}\left(\left[X_u, X_d\right]^2 \left(\left[X_d, \tilde{Y}_u Y_u^\dagger\right] - \left[X_u, \tilde{Y}_d Y_d^\dagger\right]\right)\right)$$

If small instantons enhance axion mass, will they enhance other shift-breaking invariants?

Axion EFT – generating shift-breaking invariants



Work in progress

If small instantons enhance axion mass, will they enhance other shift-breaking invariants?

Conclusions

- Small-instantons can be responsible for relevant UV contributions to the axion mass
- In the presence of CP violating physics, this enhancement will also induce contributions to nEDM
  - The estimation of these effects can be made easier with the help of determinant-like invariants
- Shift-breaking effects can also be generated in ALP coupling to fermions

Thanks

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Instanton computations – bosonic contributions

- Vacuum to vacuum transition in instanton background

't Hooft 76
Shifman Vainshtein, Zakharov 79

$$\langle 0|0\rangle\Big|_{\text{1-inst.}} = e^{-i\theta_{\rm QCD}} \int d^4x_0 \int \frac{d\rho}{\rho^5}\, d_N(\rho) \int \prod_{f=1}^{N_f} \left(\rho\, d\xi_f^{(0)} d\bar{\xi}_f^{(0)}\right) e^{-\bar\psi J\psi + \text{h.c.}}$$

<u>Instanton configuration</u>

$$G^a_{\mu\nu}\Big|_{\text{1-inst.}} = -4\eta_{a\mu\nu}\frac{\rho^2}{[(x-x_0)^2+\rho^2]^2}\,.$$

$$\int \mathcal{D}\mathcal{A}\, e^{\mathcal{A}_\mu M \mathcal{A}_\mu} = \frac{8\pi}{\sqrt{\det M}}$$

collective coordinates ⟶ diverges for zero modes

Instanton computations – zero modes measure

$$S = S_{\rm cl} + \frac{1}{2}\phi^A_{\rm qu} M_{AB}\left(\phi_{\rm cl}\right)\phi^B_{\rm qu}$$

$$\phi^A_{\rm qu} = \sum_\alpha \xi_\alpha F^A_\alpha \qquad M_{AB}F^B_\alpha = \epsilon_\alpha F^A_\alpha \qquad \langle F_\alpha | F_\beta \rangle = \delta_{\alpha\beta} u_\alpha$$

$$S = S_{\rm cl} + \frac{1}{2}\sum_\alpha \xi_\alpha \xi_\alpha \epsilon_\alpha u_\alpha$$

$$[{\rm d}\phi] \equiv \prod_{\alpha=0}^{\infty} \sqrt{\frac{u_\alpha}{2\pi}}\,{\rm d}\xi_\alpha \qquad \int [{\rm d}\phi]\, {\rm e}^{-S[\phi]} = \int \sqrt{\frac{u_0}{2\pi}}{\rm d}\xi_0\, {\rm e}^{-S_{\rm cl}} ({\rm det}' M)^{-1/2}$$

Vandoren, van Nieuwenhuizen 0802.1862

Fermion zero-modes

$$i\gamma_\mu \mathcal{D}_\mu u_0 = 0$$

$$\gamma_\mu = \begin{pmatrix} 0 & -i\sigma_\mu^- \\ i\sigma_\mu^+ & 0 \end{pmatrix}, \qquad \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu},$$

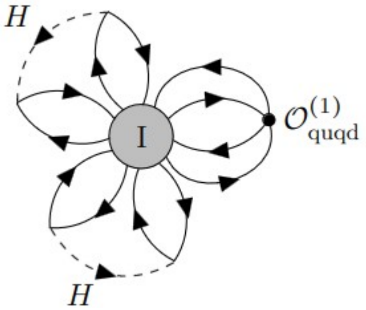
$$u_0 = \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix}, \qquad \sigma_\mu^+ \mathcal{D}_\mu \chi_L = 0, \quad \sigma_\mu^- \mathcal{D}_\mu \chi_R = 0,$$

$$-\mathcal{D}_\mu^2 \chi_L = 0, \qquad \left\{ -\mathcal{D}_\mu^2 + 4\boldsymbol{\sigma\tau} \frac{\rho^2}{[(x-x_0)^2 + \rho^2]^2} \right\} \chi_R = 0.$$

$$u_0(x) = \frac{1}{\pi} \frac{\rho}{(x^2+\rho^2)^{3/2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varphi,$$

Shifman, Advanced Topics in Quantum Field Theory 2012

Computation



$$\chi^{(1)}_{\rm ququd}(0)^{\rm 1-inst.} = -i \lim_{k\to 0} \int d^4x\, e^{ikx} \left\langle 0 \left| T\left\{ \frac{1}{32\pi^2} G\widetilde{G}(x), \frac{C^{(1)}_{\rm ququd}}{\Lambda^2_{\not{CP}}} \mathcal{O}^{(1)}_{\rm ququd}(0) \right\} \right| 0 \right\rangle ,$$

$$= e^{-i\theta_{\rm QCD}} \int d^4x_0 \int \frac{d\rho}{\rho^5} d_N(\rho) \int \mathcal{D}H\mathcal{D}H^\dagger\, e^{-S_0[H,H^\dagger]} \int \prod_{f=1}^{3} \left(\rho^2\, d\xi^{(0)}_{u_f} d\xi^{(0)}_{d_f} d^2\bar{\xi}^{(0)}_{Q_f} \right)$$

$$\times\, e^{\int d^4x (\bar{Q} Y_{\rm u} \widetilde{H} u + \bar{Q} Y_{\rm d} H d + {\rm h.c.})(x)} \frac{1}{32\pi^2} \int d^4x\, G\widetilde{G}(x) \left(\frac{C^{(1)}_{\rm ququd}}{\Lambda^2_{\not{CP}}} \bar{Q} u \bar{Q} d(0) + {\rm h.c.} \right) ,$$