Non-equilibrium phase of QCD
Lecture 1: what is a relativistic fluid?

Michal P. Heller

GHENT UNIVERSITY

Funded by the European Union

European Research Council
Established by the European Commission

thermalization review: **2005.12299** with Berges, Mazeliauskas & Venugopalan
relativistic hydro review:. **1707.02282** with Florkowski & Spaliński

some original results / perspectives from:

1103.3452 with Janik & Witaszczyk

2007.05524 with Serantes, Svensson, Spaliński & Withers

Take homes

Lecture 1 take homes

Background: relativistic hydro → reduction in number of dofs to conserved currents dynamics

→ reasonably behaved simple equations of motion enforcing such a reduction with time

Take home Ia: applicability of hydro = hydro constitutive relations approx. hold

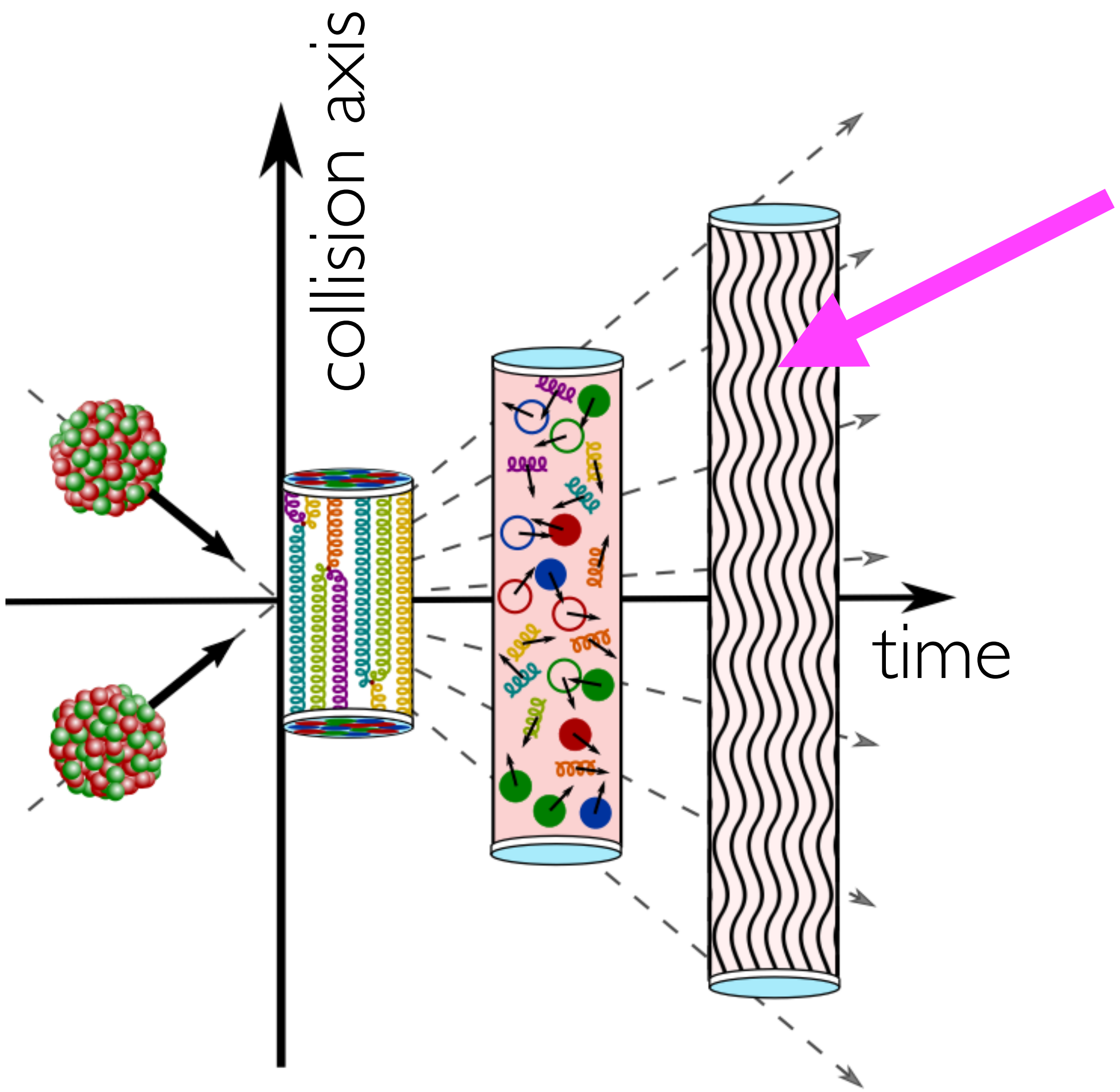
Take home Ib: towards evolution equations for relativistic hydro

Take home Ic: first indication hydro dofs cannot exist alone

Introduction

Why we talk about relativistic hydro here?

ultrarelativistic
heavy-ion collisions
at RHIC and LHC



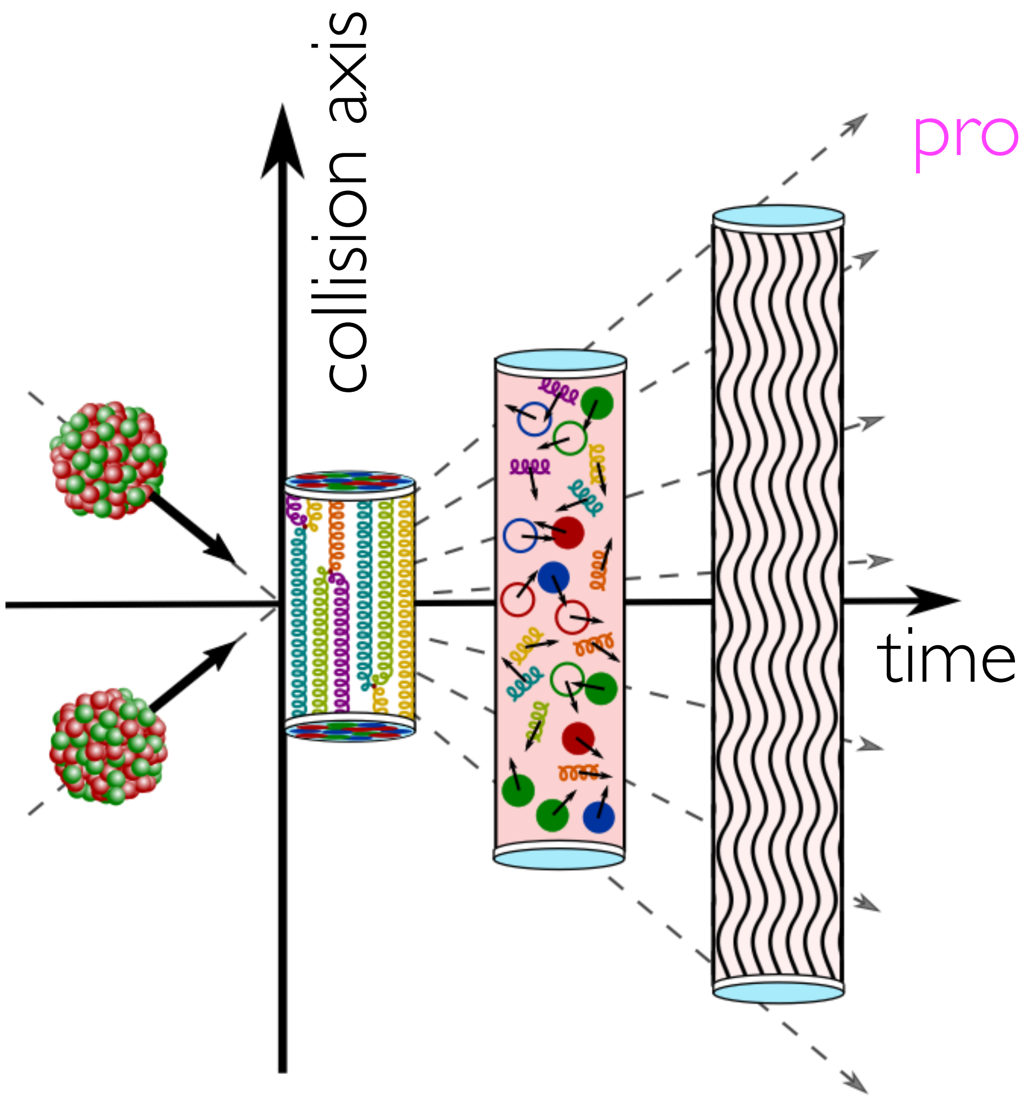
successful phenomenological description is based on relativistic hydro describing the post collision system from

time = 0.5-1 fm/c (> 90% of the lifetime)

2005.12299
with Berges, Mazeliauskas & Venugopalan

Why relativistic hydro at the xQCD school?

ultrarelativistic
heavy-ion collisions
at RHIC and LHC



2005.12299
with Berges, Mazeliauskas & Venugopalan

had we had a complete ab initio theoretical description of this intrinsically nonequilibrium process, as lattice QCD is to $\mu_B \approx 0$ thermodynamics, we would have likely not put so much effort into understanding theory of relativistic hydro

but we do not

Nonequilibrium frameworks on the market

Ab initio in QCD:

Strong classical (color) fields: weak coupling, large occupancies

Relativistic kinetic theory of QCD: weak coupling, not too large occupancies

Ab initio in cousins of QCD:

Holography (AdS/CFT): strong coupling, large number of degrees of freedom

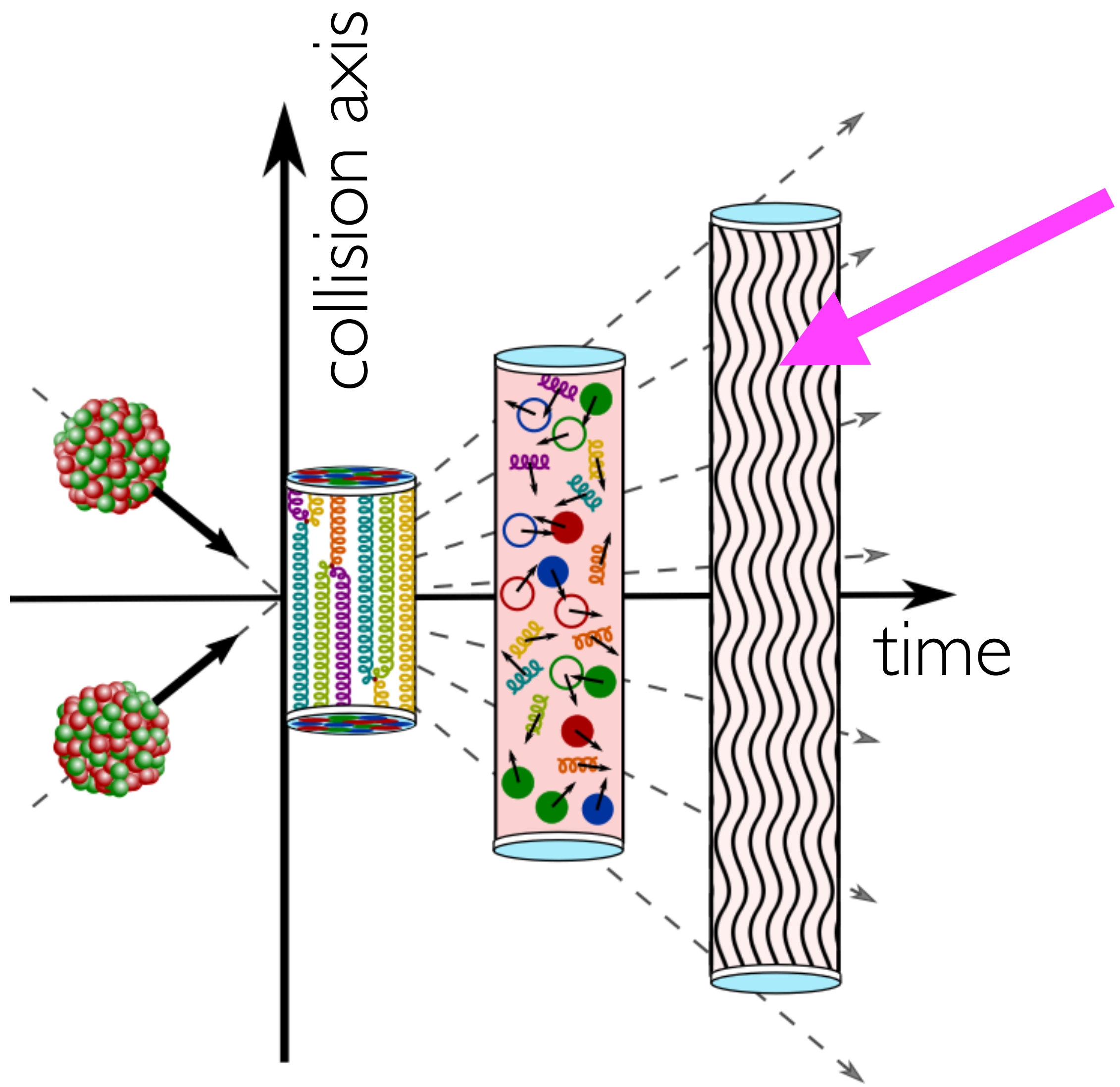
Theoretical and phenomenological models:

Partial differential equations (MIS, HJSW, BDNK)

Relativistic kinetic theory with hand-picked collision kernels

Why we talk about relativistic hydro here?

ultrarelativistic
heavy-ion collisions
at RHIC and LHC



2005.12299
with Berges, Mazeliauskas & Venugopalan

successful phenomenological description
is based on relativistic hydro describing
the post collision system from

time = 0.5-1 fm/c (> 90% of the lifetime)

Which frameworks for sure have a hydro regime?

Ab initio in QCD:

Strong classical (color) fields: weak coupling, large occupancies

Relativistic kinetic theory of QCD: weak coupling, not too large occupancies

Ab initio in cousins of QCD:

Holography (AdS/CFT): strong coupling, large number of degrees of freedom

Theoretical and phenomenological models:

Partial differential equations (ideal fluids, MIS, HJSW, BDNK)

Relativistic kinetic theory with hand-picked collision kernels

Key questions behind these lectures

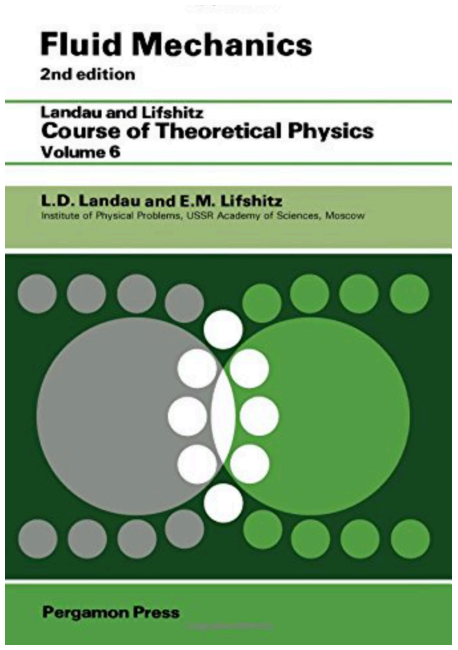
What is the relativistic hydro regime?

How does it emerge from nonequilibrium frameworks?

How is it and how can it be modelled in nuclear collisions?

First take on relativistic hydro

(a la Landau and Lifschitz on steroids)



Basics of thermalization dynamics

2005.12299 with Berges, Mazeliauskas & Venugopalan

initial nonequilibrium state $\hat{\rho}_{\rm ini}$ at $t = 0$

$\hat{\rho}(t) = \hat{U}^\dagger \hat{\rho}_{\rm ini} \hat{U}$

unitary evolution $\hat{U} = e^{-i\hat{H}t}$ time t

An observable $\hat{O}$ thermalizes if from some time onwards for physically relevant times $\mathbf{tr}\left[\hat{\rho}(t)\,\hat{O}\right] \approx \mathbf{tr}\left[\hat{\rho}_\beta\,\hat{O}\right]$ with thermal $\hat{\rho}_\beta \sim e^{-\beta\hat{H}}$: $\mathbf{tr}\left[\hat{\rho}_\beta\,\hat{H}\right] = \mathbf{tr}\left[\hat{\rho}_{\rm ini}\,\hat{H}\right]$

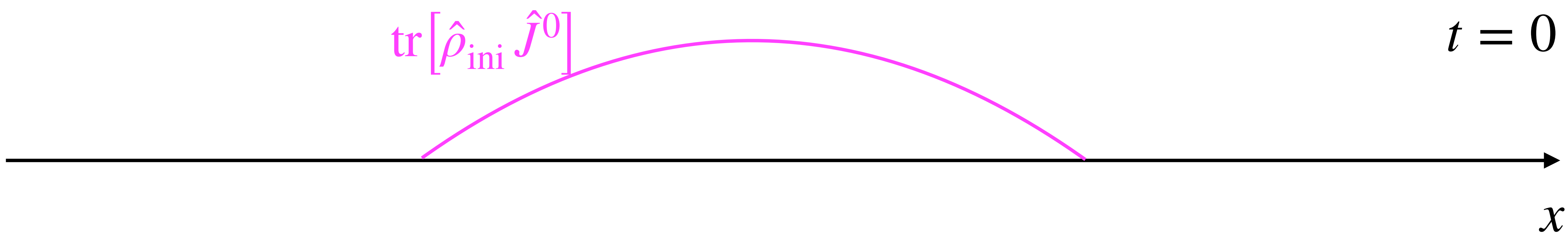
Not obvious thermalization will occur, but we expect from explicit ab initio studies that in generic interacting systems sufficiently simple observables do

Not all observables can thermalize, as unitary time evolution cannot turn a $\hat{\rho}_{\rm ini} \neq \hat{\rho}_\beta$ into $\hat{\rho}_\beta$

Where is hydro?

Let us focus from now on on local observables defined at a spacetime point

Imagine such an observable represents a conserved charge $\hat{J}^0$



Thermalization:

Conserved charge cannot just disappear: its excess needs to be transported throughout a system and this can happens at most with the speed of light

Relativistic hydro $\approx$ intermediate to late time dynamics of conserved charges

Equilibrium $T^{\mu\nu}$

The conserved charges in any QFT: energy and momentum described by $\hat{T}_{\mu\nu}$

In equilibrium its expectation value is given by

pressure whose dependence on ϵ follows from statistical mechanics

$$T^{\mu\nu} = \mathrm{diag}\big(\epsilon, P(\epsilon), P(\epsilon), P(\epsilon)\big)^{\mu\nu}$$

energy density characterizing a 1-parameter family of equilibria ($\epsilon \leftrightarrow \beta$)

Q2U: is it the most general form of $T^{\mu\nu}$ in equilibrium?

Thinking time 1:

Q2U: is it the most general form of $T^{\mu\nu}$ in equilibrium?

Equilibrium $T^{\mu\nu}$ redux

Q2U: is it the most general form of $T^{\mu\nu}$ in equilibrium?

No, equilibria can be still boosted and the boost symmetry is broken by $\epsilon > 0$:

constant boost 4-velocity $u^\mu u_\mu = -1$

$$T^{\mu\nu} = \epsilon\, u^\mu u^\nu + P(\epsilon)\left(g^{\mu\nu} + u^\mu u^\nu\right)$$

Minkowski metric $g_{\mu\nu} = \mathrm{diag}\left(-1,1,1,1\right)_{\mu\nu}$

Setting $u^\mu \partial_\mu = \partial_t$ we recover the previous result $T^{\mu\nu} = \mathrm{diag}\left(\epsilon, P(\epsilon), P(\epsilon), P(\epsilon)\right)^{\mu\nu}$

Punchline: equilibria are characterized by (at least) 4 parameters: ϵ (1) and u^μ (3)

The parameters are in 1-1 correspondence with conserved qties

Towards relativistic hydro: the ideal limit

Key idea (1): ϵ and u^μ constant $\longrightarrow$ ϵ and u^μ become functions of x^α

Key idea (2): $\hat{T}^{\mu\nu}$ is conserved and $\nabla_\mu T^{\mu\nu} = 0$ are 4 equations for 4 variables

Features:

Enormous reduction of complexity with respect to $\rho(t)$

- General $T^{\mu\nu}$ has 10* indep. entries and $T^{\mu\nu} = \epsilon\, u^\mu u^\nu + P(\epsilon)\left(g^{\mu\nu} + u^\mu u^\nu\right)$ has 4
- Knowing $\epsilon(\vec{x}), u^i(\vec{x})$ on $t = 0$ specifies $T^{\mu\nu}$ at arbitrarily later time

Time reversal invariance of EOMs results in no dissipation: $\nabla_\mu \left[\beta(\epsilon)\left(\epsilon + P(\epsilon)\right)u^\mu\right] = 0$

This is why $\nabla_\mu \left[\epsilon\, u^\mu u^\nu + P(\epsilon)\left(g^{\mu\nu} + u^\mu u^\nu\right)\right] = 0$ is called perfect or ideal fluid

Relativistic hydrodynamics as an EFT

Outside global equilibrium $T^{\mu\nu} = \epsilon\, u^\mu u^\nu + P(\epsilon)\left(g^{\mu\nu} + u^\mu u^\nu\right)$ cannot be exact

There are corrections: $T^{\mu\nu} = \epsilon\, u^\mu u^\nu + P(\epsilon)\left(g^{\mu\nu} + u^\mu u^\nu\right) + \Pi^{\mu\nu}$

Key idea of relativistic hydrodynamics as a classical effective field theory:

$\Pi^{\mu\nu}$ is defined as a systematic expansion in derivatives of ϵ and u^μ

0712.2451 by Baier, Romatschke, Son, Starinets and Stephanov

$\Pi^{\mu\nu}$ up to first order

$$T^{\mu\nu} = \epsilon\, u^\mu u^\nu + P(\epsilon)\left(g^{\mu\nu} + u^\mu u^\nu\right) + \Pi^{\mu\nu}$$

10* indep. components — 4 variables — should have 6* variables

A choice: Landau frame condition $u_\mu \Pi^{\mu\nu} = 0$

A consequence: a definition of ϵ and u^μ given $T^{\mu\nu}$ as an eigenproblem $T^\mu{}_\nu u^\nu = -\epsilon u^\mu$

EFT perspective: $\Pi^{\mu\nu} =$ (all terms with 1 derivative) + (all terms with 2 derivative) + …

Such an expansion is called a hydrodynamic constitutive relation due to dofs reduction: 10* component object written in terms of 4 functions and their derivs

One typically truncates the expansion at 1 or 2 order

Explicit construction of $\Pi^{\mu\nu}$ up to 1 derivative

$$T^{\mu\nu} = \epsilon\, u^\mu u^\nu + P(\epsilon)\,\underbrace{\left(g^{\mu\nu} + u^\mu u^\nu\right)}_{\equiv\,\Delta^{\mu\nu}} + \Pi^{\mu\nu}$$

$$\Pi^{\mu\nu} = -\,\eta(\epsilon)\,\nabla^{\langle\mu}u^{\nu\rangle} - \zeta(\epsilon)\Delta^{\mu\nu}\,\nabla_\alpha u^\alpha$$

where

$$\nabla^{\langle\mu}u^{\nu\rangle} \equiv \frac{1}{2}\Delta^{\mu\alpha}\Delta^{\nu\gamma}\left(\nabla_\alpha u_\gamma + \nabla_\gamma u_\alpha\right) - \text{trace}$$

Comments:

At each order one needs to construct all 2-tensors with a given number of derivs

Conservation equations give rise to redundancies (tensors equivalent on-shell)

Scalar coeffs are fixed by the microscopics and are called transport coeffs

Here $\eta(\epsilon)$ is the shear viscosity and $\zeta(\epsilon)$ is the bulk viscosity (we choose $\zeta(\epsilon) = 0$)

A priori there is no reason to stop at first order in derivatives

One notion of dissipation

Upon including first order corrections:

$$\nabla_\mu \left[\beta(\epsilon)\left(\epsilon + P(\epsilon)\right)u^\mu\right] \approx \frac{1}{2}\eta(\epsilon)\,\beta(\epsilon)\,\nabla_{\langle\mu}u_{\nu\rangle}\nabla^{\langle\mu}u^{\nu\rangle} + \zeta(\epsilon)\beta(\epsilon)\left(\partial_\mu u^\mu\right)^2 \geq 0$$

Generically there will be entropy production and never entropy decrease as long as $\eta(\epsilon) > 0$ and $\zeta(\epsilon) \geq 0$

Lecture 1 take homes

Background: relativistic hydro

- reduction in number of dofs to conserved currents dynamics
- reasonably behaved simple equations of motion enforcing such a reduction with time

Take home 1a: applicability of hydro = hydro constitutive relations approx. hold

Take home 1b: towards evolution equations for relativistic hydro

Take home 1c: first indication hydro dofs cannot exist alone

Thinking time II:

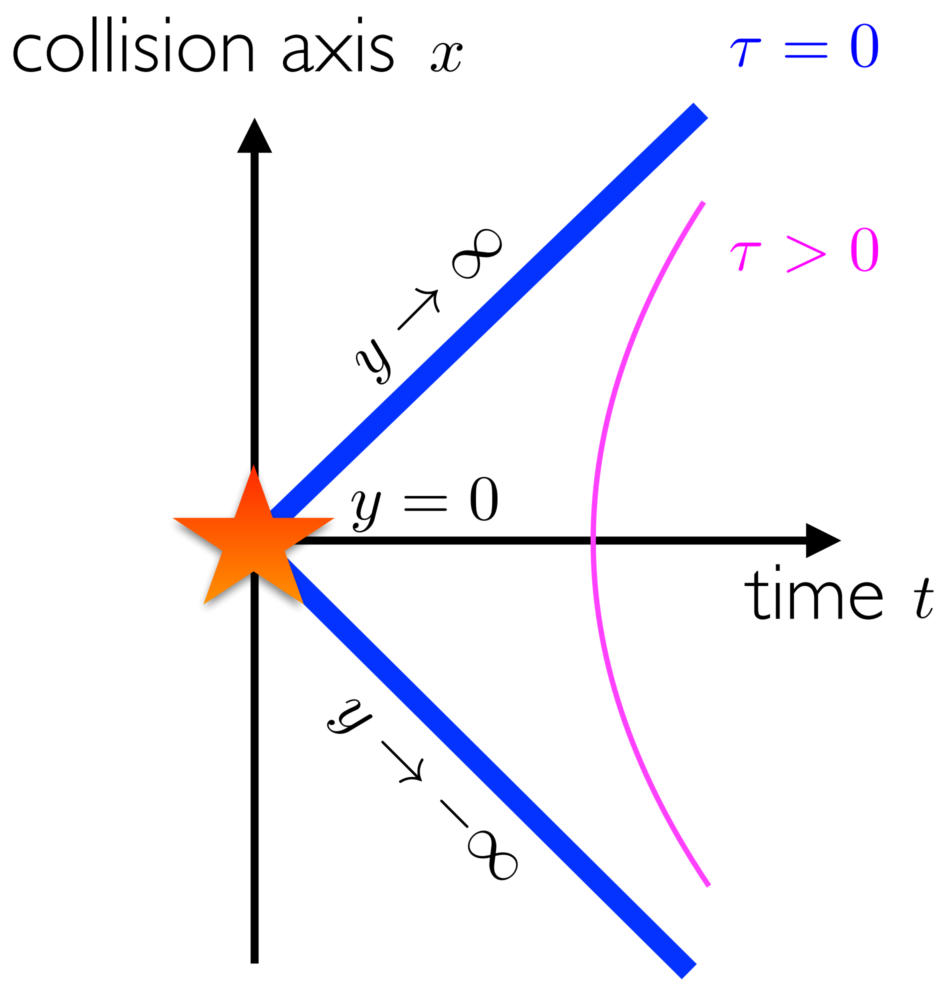
Q2U: what happens to hydro when there is no spatial dependence?

Relativistic hydro at work
in terms of hydro constitutive relations

Thinking time III:

Q2U: what is the simplest model of nuclear dynamics with a hydro tail?

Bjorken flow: basics Bjorken 1982



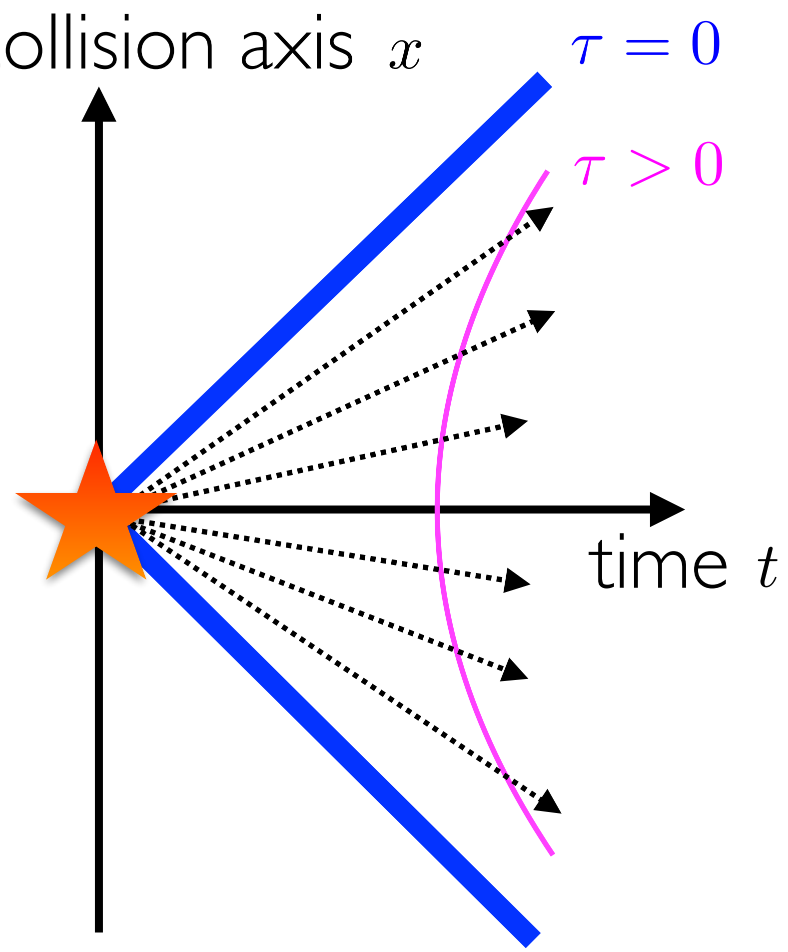
+ two transverse coordinates

Bjorken's simplification: physics is the same in all longitudinally boosted reference frame; this is Lorentzian analogue of rotational invariance.

analogue of the radius: $\tau = \sqrt{t^2 - x^2}$ analogue of the angle: $y = \text{arccosh}(t/x)$

Bjorken flow and relativistic hydrodynamics

e.g. 1707.02282 with Spaliński & Florkowski



Bjorken flow is a comoving flow in Minkowski:

$$u^\mu \partial_\mu = \partial_\tau \quad \text{and} \quad ds^2 = -d\tau^2 + \tau^2 dy^2 + d\mathbf{x}_\perp^2$$

$$\downarrow$$

$$\nabla^\mu u^\nu \sim \frac{1}{\tau} \quad \text{etc}$$

$$\epsilon \sim \beta^{-4}$$

It is an intrinsically nonlinear phenomenon

For conformal (no scale) fluids: $P = \frac{1}{3}\epsilon \quad , \quad \epsilon \sim \beta^{-4}$ and $\quad \eta \sim \beta^{-3}$

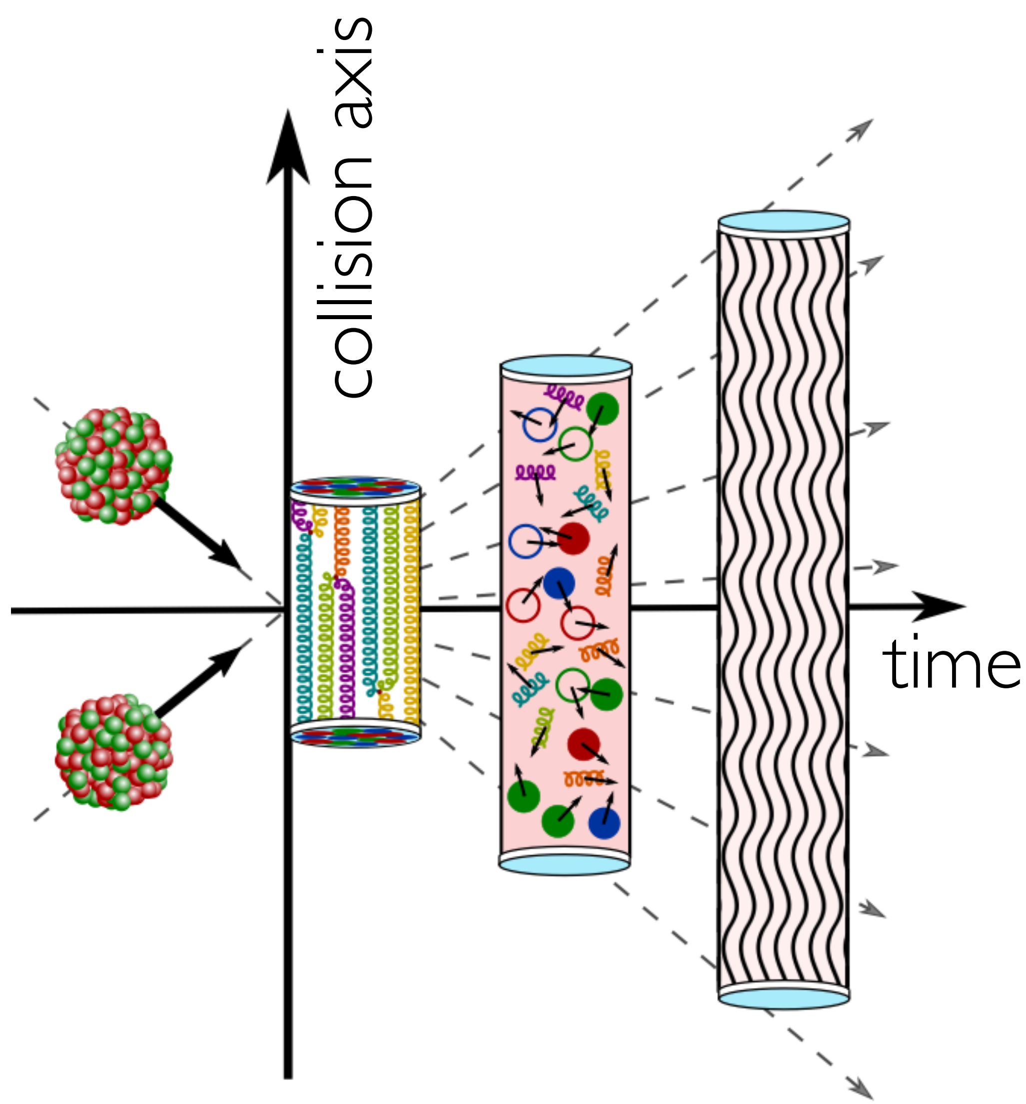
$$\downarrow$$

$$\mathcal{A} \equiv \frac{\pi^\perp_\perp - \pi^y_y}{\mathcal{E}/3} = 8\frac{\eta}{s}\boxed{\frac{1}{\tau T(\tau)}} + \mathcal{O}(\nabla^2)$$

$\equiv w^{-1}$

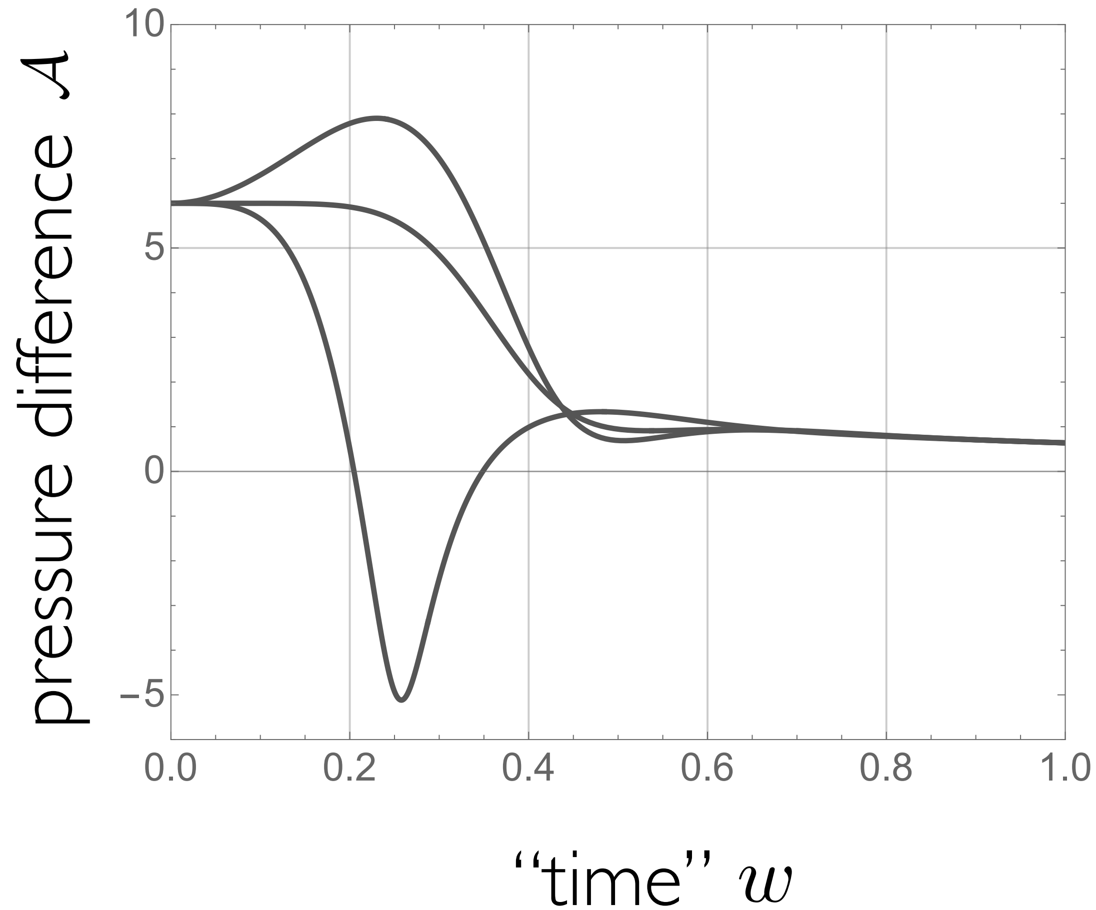
A holographic nuclear collision

heavy-ion collisions
at RHIC and LHC



2005.12299
with Berges, Mazeliauskas & Venugopalan

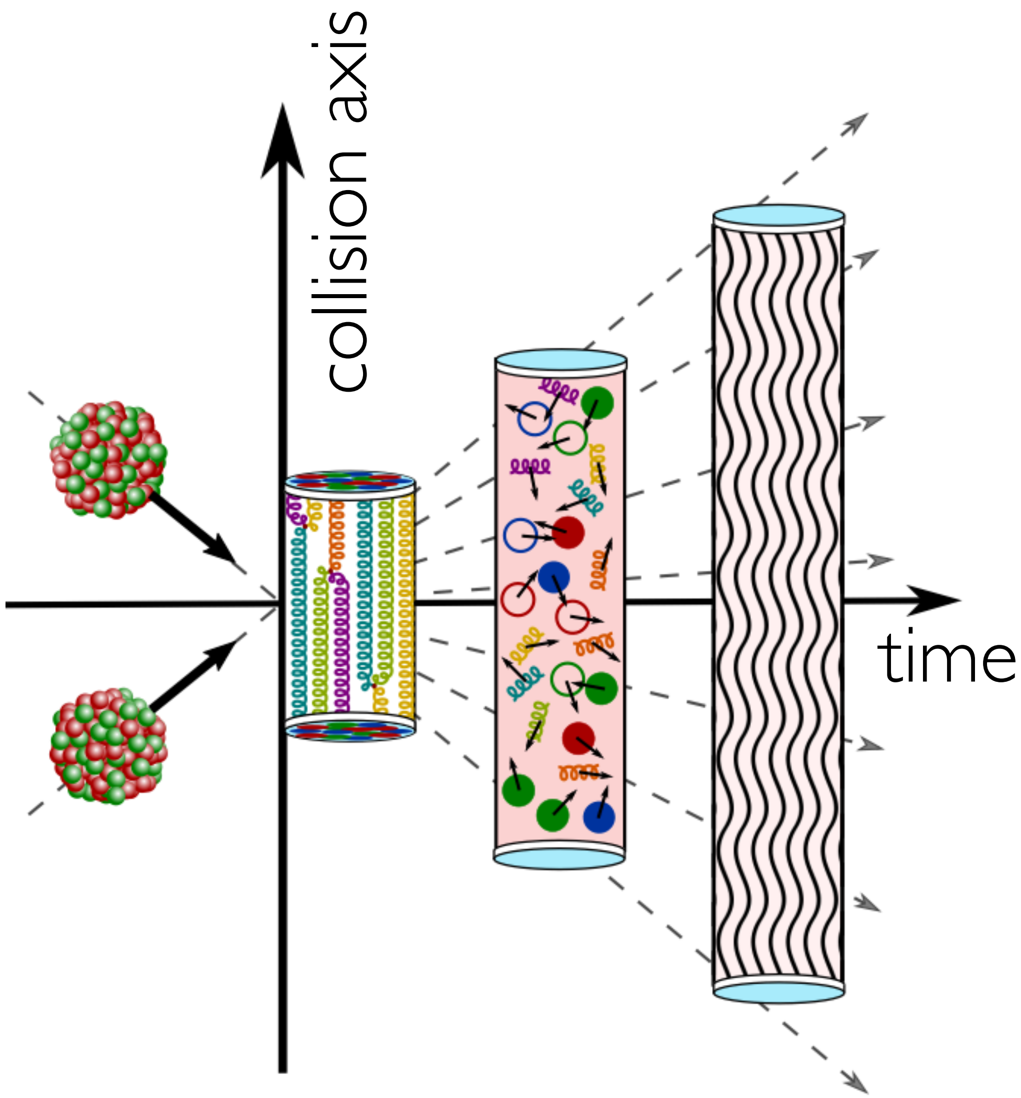
holographic Bjorken flow



1103.3452 with Janik & Witaszczyk

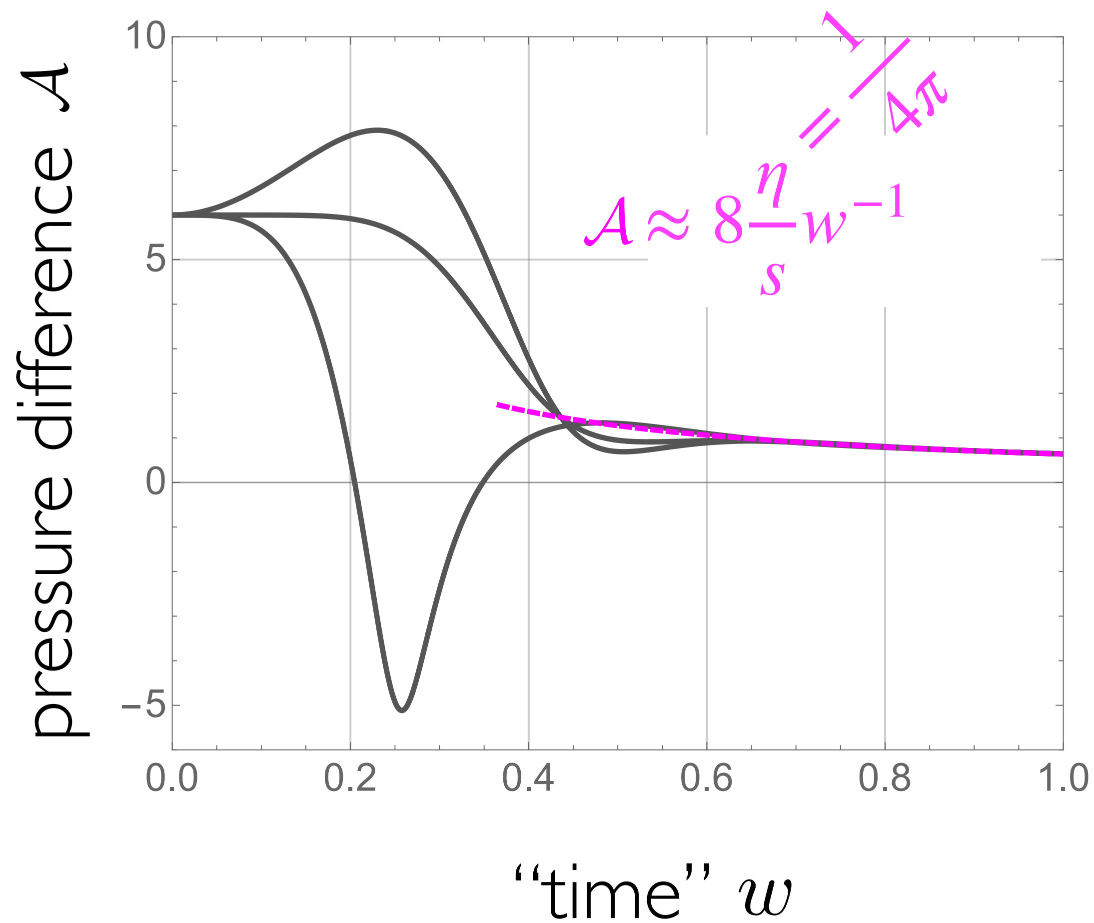
Hydro works = hydro constitutive relations work

heavy-ion collisions
at RHIC and LHC



2005.12299
with Berges, Mazeliauskas & Venugopalan

holographic Bjorken flow



1103.3452 with Janik & Witaszczyk

Lecture 1 take homes

Background: relativistic hydro → reduction in number of dofs to conserved currents dynamics

Background: relativistic hydro → reasonably behaved simple equations of motion enforcing such a reduction with time

Take home Ia: applicability of hydro = hydro constitutive relations approx. hold

Take home Ib: towards evolution equations for relativistic hydro

Take home Ic: first indication hydro dofs cannot exist alone

Towards eoms for relativistic hydro

Proving Landau and Lifschitz wrong

Take $\nabla_\mu\left(\epsilon\, u^\mu u^\nu + P(\epsilon)\left(g^{\mu\nu} + u^\mu u^\nu\right) + \Pi^{\mu\nu}\right) = 0$ with $\Pi^{\mu\nu} = -\,\eta(\epsilon)\,\nabla^{\langle\mu} u^{\nu\rangle}$

Let's consider a small perturbation of equilibrium

$$
\begin{aligned}
&\epsilon = \epsilon_0 = \text{const}\\
&u^0 = 1\\
&u^1 = 0\\
&u^2 = 0\\
&u^3(x^0 \equiv t, x^1 \equiv x) \equiv \psi \ll 1
\end{aligned}
$$

Linearized equations of motion give us the diffusion equation

$$\partial_t \psi - D\partial_x^2 \psi = 0 \quad \text{with} \quad D = \frac{\eta(\epsilon_0)}{\epsilon_0 + P(\epsilon_0)}$$

Thinking time IV:

Q2U: is the diffusion equation good, bad or neutral in this context?

Diffusion equation is acausal

Take* $\psi(t=0,x)=\delta(x)$

The exact solution is $\psi(t,x)=\frac{1}{\sqrt{4\pi Dt}}e^{-\frac{x^2}{4Dt}}$

At any $t>0$ the solution spreads past the future light cone $|x|\leq t$

Thinking time V:

Q2U: can you find a simple fix to cure it?

The telegraphers' equation

a wave equation

$$\tau_R \partial_t^2 \psi + \partial_t \psi - D \partial_x^2 \psi = 0$$

the original diffusion equation

Thinking time VI:

Q2U: is it causal?

Causal telegraphers' equation

a wave equation

$$\tau_R \partial_t^2 \psi + \partial_t \psi - D \partial_x^2 \psi = 0$$

the original diffusion equation

We still need to impose $\tau_R \geq \dfrac{1}{D}$ so that wavefront propagate (sub)luminally

Lecture 1 take homes

Background: relativistic hydro → reduction in number of dofs to conserved currents dynamics

→ reasonably behaved simple equations of motion enforcing such a reduction with time

Take home Ia: applicability of hydro = hydro constitutive relations approx. hold

Take home Ib: towards evolution equations for relativistic hydro

Take home Ic: first indication hydro dofs cannot exist alone

Thinking time VII:

Q2U: what is the key consequence of restoring causality?

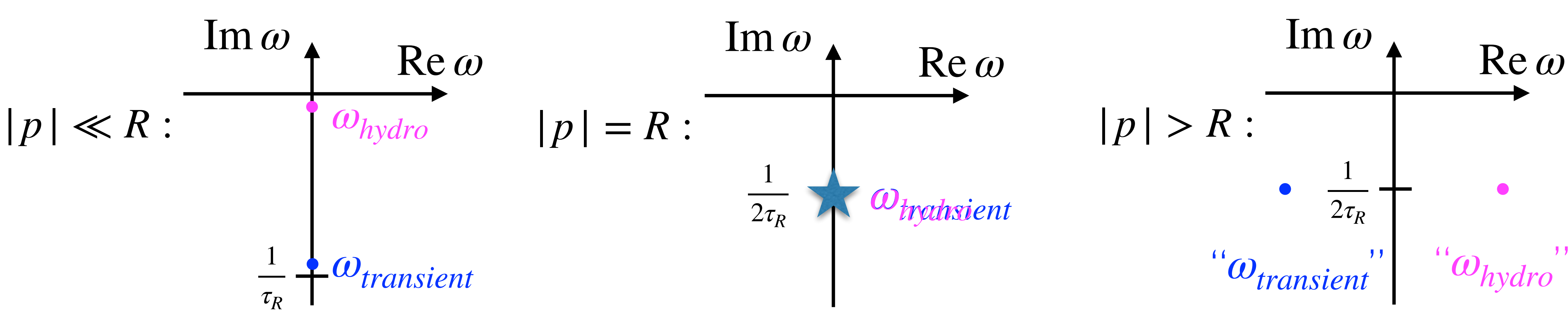
Causality introduces transients 1803.08058 by Withers

presented as in 2007.05524 with Serantes, Svensson, Spaliński & Withers

Let's look at modes $\psi \sim e^{-i\omega t + ipx}$ $\left(R \equiv \frac{1}{\sqrt{4\tau_R D}}\right)$:

$$\omega_{hydro} = -\frac{i}{2\tau_R} + \frac{i}{2\tau_R}\sqrt{1 - \frac{p^2}{R^2}} \approx -iDp^2$$

$$\omega_{transient} = -\frac{i}{2\tau_R} - \frac{i}{2\tau_R}\sqrt{1 - \frac{p^2}{R^2}} \approx -\frac{i}{\tau_R}$$



Lecture 1 take homes

Background: relativistic hydro

→ reduction in number of dofs to conserved currents dynamics

→ reasonably behaved simple equations of motion enforcing such a reduction with time

Take home 1a: applicability of hydro = hydro constitutive relations approx. hold

Take home 1b: towards evolution equations for relativistic hydro

Take home 1c: first indication hydro dofs cannot exist alone