# Nonperturbative Decoupling of Heavy Quarks

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# Decoupling

## The Appelquist-Carazzone theorem states that

- In QFTs the effect of (potentially yet to be discovered) heavy particles on low energy processes can be accounted for by adjusting the couplings and the masses of the light fields in a theory without the heavy particles.
- The remaining effects of the heavy particles are suppressed by inverse powers of their mass.

Consequences

- New discoveries need high energy experiments
- The discovery of new heavy particles does not render the standard-model useless
- The standard model is most likely an effective theory
- In lattice QCD simulations: it is OK to leave out top, bottom and even charm (?) quarks

# Decoupling

[S.Weinberg (1980)]

Fundamental theory: (Euclidean) QCD with $N_\ell$ light quarks and one heavy quark of mass $m_h$

$$\mathcal{L} = \frac{1}{2g^2}\mathrm{tr}[F_{\mu\nu}F_{\mu\nu}] + \sum_{f=1}^{N_\ell} \bar{\psi}_f[\not{D} + m_f]\psi_f + \bar{\psi}_h[\not{D} + m_h]\psi_h$$

Effective theory

$$\mathcal{L}^{\mathrm{eff}} = \frac{1}{2g_{\mathrm{eff}}^2}\mathrm{tr}[F_{\mu\nu}F_{\mu\nu}] + \sum_{f=1}^{N_\ell} \bar{\psi}_f[\not{D} + m_f^{\mathrm{eff}}]\psi_f + \sum_i \frac{1}{m_h^i}\mathcal{L}_i$$

$\mathcal{L}_i$ contains all local dimension $4 + i$ operators built from the light fields, that are compatible with the symmetries of the fundamental theory.

# Decoupling in QCD

- Leading order effective theory: ordinary QCD with light fields only. But: coupling and masses need to be "matched"
- Next-to-leading order effective theory

$$\begin{aligned} \mathcal{L}_1 &= 0 \\ \mathcal{L}_2 &= \sum_{i=1} \omega_i \Phi_i \end{aligned}$$

with local operators

$$\begin{aligned} \Phi_1 &= \mathrm{tr}[D_\mu F_{\nu\rho} D_\mu F_{\nu\rho}] \\ \Phi_2 &= \mathrm{tr}[D_\mu F_{\mu\rho} D_\nu F_{\nu\rho}] \\ \Phi_3 &= \bar{\psi}\gamma_\nu D_\mu F_{\mu\nu}\psi \\ &\dots \end{aligned}$$

To use an effective theory all its parameters need to be matched.
E.g. at leading order: Demand that $N_\ell + 1$ renormalized low energy hadronic quantities $\mathcal{S}_i$ are exactly the same in the two theories.
Every further such quantity is then $\mathcal{S}^{\mathrm{eff}} = \mathcal{S} + O(m_h^{-2})$

# Running Couplings

- Renormalized couplings $\alpha(\mu) \equiv \frac{\overline{g}^2(\mu)}{4\pi}$ depend on renormalization scale $\mu$
- Dependence is described by $\beta$ function
  $\mu \frac{\partial \overline{g}}{\partial \mu} = \beta(\overline{g})$
- In perturbation theory
  $\beta(g) \sim -g^3(b_0 + b_1 g^2 + b_2 g^4 + \ldots)$
- Integration of RG equation introduces the dimensionful $\Lambda$ parameter

$$\Lambda/\mu = \underbrace{(b_0\overline{g}^2)^{-b_1/(2{b_0}^2)}\, e^{-1/(2b_0\overline{g}^2)}\, \exp\left\{-\int_0^{\overline{g}}\left[\frac{1}{\beta(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{{b_0}^2 x}\right] dx\right\}}_{\equiv\phi(\overline{g})}$$

- Two values of the coupling, $\overline{g}_1$, $\overline{g}_2$ correspond to scale ratio
  $\frac{\mu_1}{\mu_2} = \frac{\phi(\overline{g}_2)}{\phi(\overline{g}_1)}$
- Special case: scale ratio of $\mu_1/\mu_2 = 2 \Rightarrow$ step-scaling function
  $\sigma(\overline{g}_1^2) = \overline{g}_2^2$
  Contains the same information as $\beta(g)$, but better suited for numerical methods

# Mass Dependent and Mass Independent Couplings

- Decoupling also applies to physical (mass dependent) couplings, e.g.
  [W. Bernreuther, W. Wetzel, Nucl.Phys.B 197 (1982)]
  $\alpha^{\text{eff}}_{\text{MO}}(\mu) \approx \alpha_{\text{MO}}(\mu, M)$, if $\mu \ll M$
- But not to mass-independent couplings, like $\alpha_{\overline{\text{MS}}}$

## Perturbative Decoupling

[W. Bernreuther, W. Wetzel, Nucl.Phys.B 197 (1982)], see also yesterday's talk by Peter Marquard!

- Establish the relation between $\overline{\text{MS}}$ and MO couplings

$$\alpha_{\text{MO}} = \alpha_{\overline{\text{MS}}} + \alpha^2_{\overline{\text{MS}}} B_1(\overline{m}^2/\mu^2)/\pi + \dots$$

- Equate $\alpha^{\text{eff}}_{\text{MO}}(\mu) = \alpha_{\text{MO}}(\mu, M)$
  This holds only if $\alpha^{\text{eff}}_{\overline{\text{MS}}}(\mu)$ and $\alpha_{\overline{\text{MS}}}(\mu)$ are related
- Extract this relation

$$\alpha^{\text{eff}}_{\overline{\text{MS}}} = \alpha_{\overline{\text{MS}}} + \alpha^2_{\overline{\text{MS}}} c_1\left(\ln \frac{\overline{m}^2}{\mu^2}\right) + \dots$$

# Perturbative Decoupling

- $\overline{g}_f$, renormalized coupling, $N_\mathrm{f}$ quarks, mass-independent scheme
- $\overline{g}_\ell$, renormalized coupling in effective theory, $N_\ell$ light quarks

Perturbation theory:

$$\overline{g}_\ell^2(\mu/\Lambda_\ell) = \overline{g}_\mathrm{f}^2(\mu/\Lambda_\mathrm{f}) + c_1(\mu/\overline{m}(\mu))\, \overline{g}_\mathrm{f}^4(\mu/\Lambda_\mathrm{f}) + \ldots$$

- $\overline{m}(\mu)$ renormalized heavy quark mass $\Leftrightarrow$ $M$ RGI mass
- Convenient choice of scheme and scale:
  $\overline{\mathrm{MS}}$-scheme with $\mu = m_*$ such that $\overline{m}(m_*) = m_*$
  - $c_1 = 0$
  - $log(\mu/\overline{m})$ vanish
    $\Rightarrow c_2, \ldots, c_4$ are pure numbers
  - $c_2, \ldots, c_4$ known for arbitrary $N_\mathrm{f} - N_\ell$

    [K.Chetyrkin, J.H.Kühn, C.Sturm, Nucl.Phys.B744 (2006)]

    [B.A.Kniehl, A.V.Kotikov, A.I.Onishchenko, O.L.Veretin, PRL 97 (2006)]

    [Y. Schröder and M. Steinhauser, JHEP 01, 051 (2006)]

    [A.G.Grozin, M.Hoeschele, J.Hoff, M.Steinhauser, JHEP 1109 (2011)]

    [M. Gerlach, F. Herren, and M. Steinhauser, JHEP 11, 141 (2018)]

# Application: $\alpha_s(m_z)$ from Lattice QCD

Equivalently: relation between $\Lambda$ parameters

$$P_{\ell,\mathrm{f}}(M/\Lambda^{(\mathrm{f})}) \equiv \Lambda^{(\ell)}/\Lambda^{(\mathrm{f})} = \frac{\phi^{(N_\ell)}(\overline{g}_\ell)}{\phi^{(N_\mathrm{f})}(\overline{g}_\mathrm{f})}$$

Application: compute $\Lambda^{(3)}$ in some convenient scheme, then

$$\Lambda^{(3)} \xrightarrow{\text{exact}} \Lambda^{(3)}_{\overline{MS}} \xrightarrow{P_{3,4}(M_c/\Lambda^{(4)})} \Lambda^{(4)}_{\overline{MS}} \xrightarrow{P_{4,5}(M_b/\Lambda^{(5)})} \Lambda^{(5)}_{\overline{MS}} \xrightarrow{\text{5-loop PT}} \alpha_s(m_z)$$

But: $\mu = \overline{m}_c$ is not a high scale. Does PT work well?

[M.Bruno et al Phys.Rev.Lett. 119 (2017)]

$\Lambda^{(3)}_{\overline{\mathrm{MS}}} = 341(12)$ MeV $\rightarrow$

| n (loops) | $\alpha_s(m_z)$ | $\alpha_n - \alpha_{n-1}$ |
|---|---|---|
| 1 | 0.11699 | |
| 2 | 0.11827 | 0.00128 |
| 3 | 0.11846 | 0.00019 |
| 4 | 0.11852 | 0.00006 |

# Goals

We try to give quantitative answers to the following questions

- How big are the leading power corrections $O(1/M^2)$?
- How well is perturbation theory suited, to "add" heavy quarks e.g. in a determination of $\alpha_s(m_Z)$
- How precise are lattice QCD charm physics results, without a charm sea quark?

# Decoupling Lab

We would like to compare $N_f = 2+1+1$ QCD with $N_f = 2+1$ QCD, but

- Tough multiscale problem:
  - $a \ll M_c^{-1}$
  - $M_c \gg M_{u,d}$
  - $m_\pi^{-1} \ll L$

  $\rightarrow$ enormous $L/a$ necessary

## Toy Model

We study decoupling in a QCD with 2 heavy quarks and 0 light quarks

- Leading effective theory = Yang Mills theory
- Finite size effects: no light pions, no problems
- Continuum extrapolations: very controlled, $a$ down to $\approx 0.02$ fm
- Affordable simulations

# Action

- Wilson's plaquette gauge action
- Doublet of $O(a)$ improved Wilson fermions
  $c_{sw}$: [K.Jansen, R.Sommer (1998)]
- And/or doublet of twisted mass fermions
  [R.Frezzotti, P.Grassi, S.Sint, P.Weisz (2000)]
- Open boundary conditions in time
  [M.Lüscher, S.Schaefer (Jun, 2012)]
  ((Anti-)periodic for some coarser lattices)

# Mass Parameters

Choose mass parameters such that the RGI quark mass is in the set

$$M \in \{1.2M_c, M_c, M_c/2, M_c/4, M_c/8\}$$

With twisted mass fermions at maximal twist: $\overline{m} = Z_P^{-1}\mu_0$

- $m_0 = m_{\mathrm{crit}}$, Interpolation of data from

  [B. Blossier et al, JHEP 09 (2012)] , [P. Fritzsch et al, Nucl. Phys. B865 (2012)] ,

  [P. Fritzsch, N. Garron, J. Heitger, JHEP 01 (2016)]

- $a\mu_0 = \frac{M}{\Lambda} Z_P \frac{\overline{m}}{M} \Lambda L_1 \frac{a}{L_1}$
  - $M_c/\Lambda \approx 4.87$
  - $Z_P(L_1^{-1}) = 0.5184(53)$
  - $M/\overline{m} = 1.308(16)$
  - $\Lambda L_1 = 0.649(45)$
  - Relation $L_1/a$ vs $g_0^2$ known

  [P. Fritzsch et al, Nucl. Phys. B865 (2012)] , [M. Della Morte et al, Nucl. Phys. B729 (2005)] ,

  [M. Della Morte et al, Nucl. Phys. B713 (2005)]

With Wilson fermions: $\overline{m} = Z_m(m_0 - m_{\mathrm{crit}}) + O(a)$

- More difficult to set $m_0$ correctly, but better studied, details:

  [Athenodorou et al, Nucl.Phys.B 943 (2019)]

# Gradient Flow

- We need: low-energy ”hadronic” quantities, that are purely gluonic
- Most natural candidates: glueball masses

But: not precise enough for our purposes. Instead we use artificial scales based on the

## Gradient Flow

Gradient flow $\sim$ (covariant) diffusion in “flow time” $t$

[M.F. Atiyah, R. Bott, Phil.Trans.Roy.Soc.Lond. A308 (1982)], see also Robert Harlander’s talk yesterday

$$\begin{aligned} \partial_t B_\mu(t,x) &= D_\nu G_{\nu\mu}(t,x), \qquad B_\mu(0,x) = A_\mu(x) \\ D_\mu &= \partial_\mu + [B_\mu, \cdot] \\ G_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu] \end{aligned}$$

- Correlators of $B$ at $t > 0$ need no renormalization

  [M. Lüscher, JHEP 1008 (2010)]

  [M. Lüscher and P. Weisz, JHEP 1102 (2011)]

# Gradient Flow Scales

- Most common use: definition of scales, e.g. $t_0$, $w_0$
  [M. Lüscher (2010)], [S. Borsanyi et al. (2012)]

$$
\begin{aligned}
\frac{t^2}{4}\left\langle G^a_{\mu\nu}(t,x)G^a_{\mu\nu}(t,x)\right\rangle\Big|_{t=t_0} &= 0.3 \\
t\frac{d}{dt}\frac{t^2}{4}\left\langle G^a_{\mu\nu}(t,x)G^a_{\mu\nu}(t,x)\right\rangle\Big|_{t=w_0^2} &= 0.3
\end{aligned}
$$

- Choice 0.3: $\sqrt{t_0}$, $w_0$ are hadronic scales. Other values $c$ also possible.
  $c < 0.3 \rightarrow$ higher energy scale

# Lattice Gradient Flow

$A_\mu \to SU(3)$ gauge links $U_\mu$, $B_\mu(t) \to V_\mu(t)$
Different discretizations are possible

- For the flow equations
  - Wilson flow

$$a^2[\partial_t V_\mu] V_\mu^\dagger = -g_0^2 \partial_{x,\mu} S_W[V]$$

[M. Lüscher, JHEP 1008 (2010)]

  - Zeuthen flow (Symanzik improved flow)

$$a^2[\partial_t V_\mu] V_\mu^\dagger = -g_0^2 \left(1 + \frac{a^2}{12} \nabla_\mu^* \nabla_\mu\right) \partial_{x,\mu} S_{LW}[V]$$

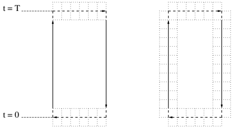
[A.Ramos, S.Sint, Eur.Phys.J.C 76 (2016)]

- For the action density $G_{\mu\nu} G_{\mu\nu}$
  - Plaquette definition, errors $O(a^2)$
  - Symmetric clover definition, errors $O(a^2)$
  - Improved combinations of both, errors $O(a^4)$
  - Lüscher-Weisz action density, errors $O(a^4)$

- Two point functions are formed with operators

$$\mathcal{O}(x_0, r) = \sum_{\boldsymbol{x}} \bar{\phi}(x) \underbrace{\left[\prod_{i=0}^{r-1} U_k(x + i\hat{k})\right]}_{\text{possibly smeared}} \phi(x + r\hat{k})$$

  Where $\phi$ is an infinitely heavy quark
- $\langle \mathcal{O}(t, r)\mathcal{O}^{\dagger}(0, r)\rangle \sim e^{-V(r)t}$,
  $V(r)$ is the static quark potential
- Integrating out the static quarks $\rightarrow$ Wilson loops
- Sommer scale $r_0$: distance at which
  $r^2\, F(r) = 1.65$, [R. Sommer, Nucl.Phys.B 411 (1994)]
- Numerical setup as in
  [M.Donnellan, F.Knechtli, B.Leder, R.Sommer (2011)]
  - Operator basis with different HYP smearing levels
  - GEVP for $aV(r)$
  - Improved distance: $F(r_I) = \frac{V(r+a)-V(r)}{a}$
    $r_I = r + a/2 + O(a^2)$ such that (perturbative) lattice artifacts are absent



# Power Corrections

In pure gauge theory, the only scale is given by $\Lambda^{(0)}$
$\rightarrow$ every dimension 1 quantity

$$\mathcal{S}^{\mathrm{eff}} = \text{number} \times \Lambda^{(0)}$$

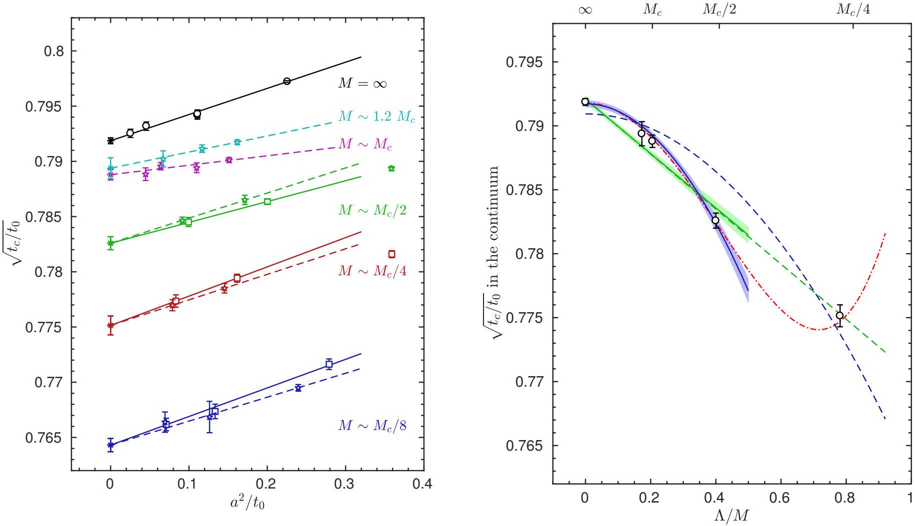
A ratio in the $N_\mathrm{f} = 2$ theory

$$\frac{\mathcal{S}_1}{\mathcal{S}_2} = \frac{\mathcal{S}_1^{\mathrm{eff}} + O(M^{-2})}{\mathcal{S}_2^{\mathrm{eff}} + O(M^{-2})} = \underbrace{\text{number}}_{\text{indep. of } \Lambda} + O(M^{-2})$$

$\rightarrow$ No matching necessary to isolate power corrections

- We look at ratios of artificial scales
    - $\sqrt{t_c/t_0}$, $c = 0.2$
    - $\sqrt{t_0}/w_0$
    - $r_0/\sqrt{t_0}$

# Power Corrections




$\Rightarrow$ power corrections $\approx$ 0.2% with one charm quark

[F.Knechtli , T.K., B.Leder, G.Moir, Phys.Lett. B774 (2017)]

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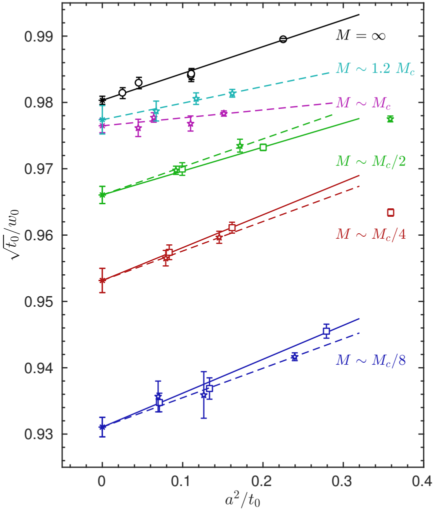


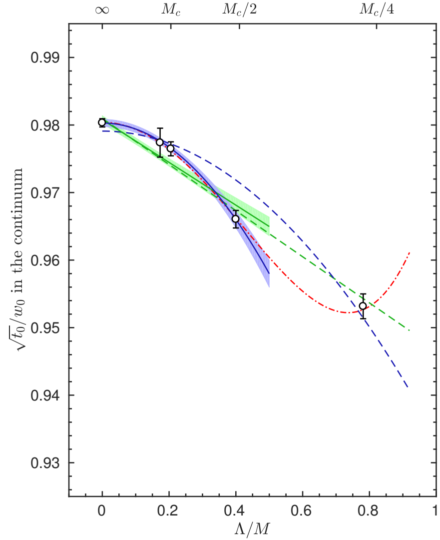


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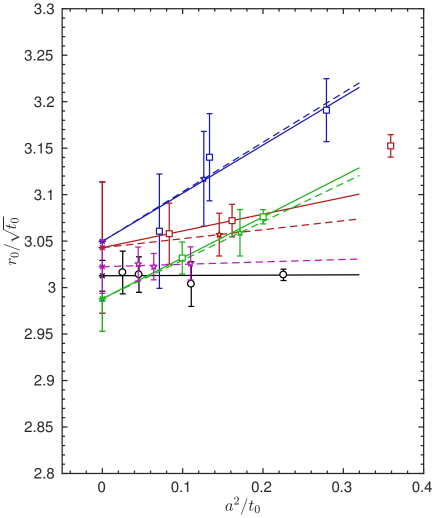


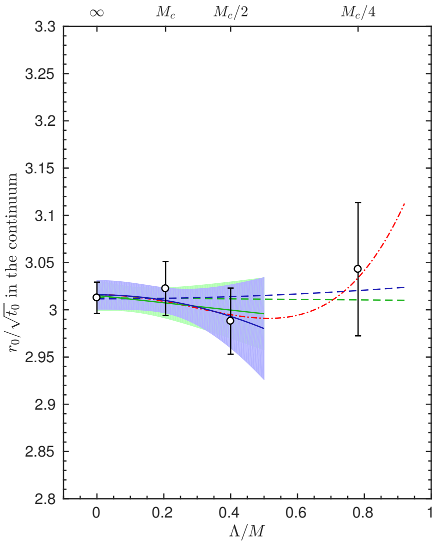


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# Accuracy of Perturbation Theory

Reminder: $\Lambda^{(\ell)} = P_{\ell,f}(M/\Lambda^{(f)})\Lambda^{(f)}$
On the lattice we have direct access to the

## Mass Scaling Function

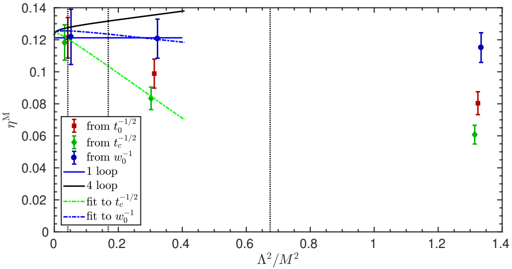
$$\eta^M(M) = \frac{M}{P}\,\frac{\partial P}{\partial M}\bigg|_{\Lambda^{(f)}}$$

If all quarks get decoupled, $\eta^M$ is "universal". For quantity $\mathcal{S}$

$$\begin{aligned}
\frac{M}{\mathcal{S}}\,\frac{\partial \mathcal{S}}{\partial M} &= \frac{M}{\mathcal{S}^{\rm eff}}\,\frac{\partial \mathcal{S}^{\rm eff}}{\partial M} + O(M^{-2}) \\
&= \frac{M}{c\Lambda^{(0)}}\,\frac{c\partial\Lambda^{(0)}}{\partial M} + O(M^{-2}) \\
&= \frac{M}{P_{0,f}\Lambda^{(f)}}\,\frac{\Lambda^{(f)}\partial P_{0,f}}{\partial M} + O(M^{-2}) \\
&= \eta^M + O(M^{-2})
\end{aligned}$$

# Accuracy of Perturbation Theory

We need the mass dependence of some quantity $\mathcal{S}$ in the fundamental theory.
$\rightarrow$ determine it for several gradient-flow quantities in the continuum limit
[A.Athenodorou, J.Finkenrath, F.Knechtli, T.K., B.Leder, M.Marinković, R.Sommer (2019)]




$\rightarrow$ At $M = M_c$, no deviation from PT resolvable.

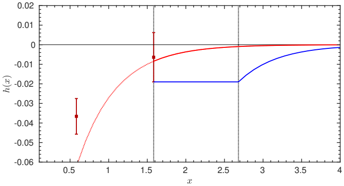
# Accuracy of Perturbation Theory

Perturbation theory for $\eta^M$ works well. How to quantify the effect on $P$ itself?

- Investigation of: $\log(P_{0,2}(M/\Lambda)) - \log(P_{0,2}(M/\Lambda))|_{PT} = -\int\limits_{\log(M/\Lambda)}^{\infty} h(x)\,dx$

  $h(x) = \eta^M(x) - \eta^M(x)|_{\mathrm{PT}}$

  [A.Athenodorou, J.Finkenrath, F.Knechtli, T.K., B.Leder, M.Marinković, R.Sommer (2019)]




$\rightarrow$ Non-perturbative effect in $P_{2,0}(M_c/\Lambda)$ is 0.4% – 2%

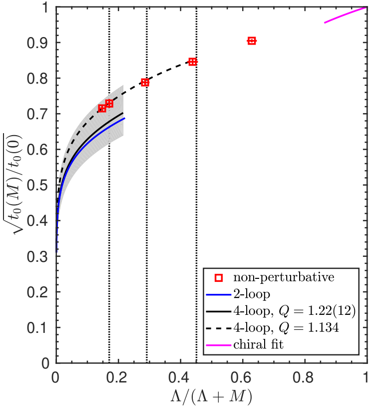
# Factorization formula

[A.Athenodorou, J.Finkenrath, F.Knechtli, T.K., B.Leder, M.Marinković, R.Sommer (2019)]

Ratio of quantity $\mathcal{S}$ at mass $M$ to its chiral value. Decoupling implies

$$\frac{\mathcal{S}(M)}{\mathcal{S}(0)} = \underbrace{\frac{\mathcal{S}^{\rm eff}/\Lambda^{(\ell)}}{\mathcal{S}(0)/\Lambda^{(f)}}}_{Q^{\mathcal{S}}_{\ell,\rm f}} \times \underbrace{\frac{\Lambda^{(\ell)}}{\Lambda^{(f)}}}_{P_{\ell,\rm f}(M/\Lambda_{\rm f})} + O(M^{-2})$$

- $Q$: independent of $M$, non-perturbative
- $P$: independent of $\mathcal{S}$, perturbative at high enough $M$
- Test PT in this setup
  - Apply to $\mathcal{S} = t_0^{-1/2}$
  - $Q = \frac{[\sqrt{t_0(0)}\Lambda]_{N_{\rm f}=2}}{[\sqrt{t_0}\Lambda]_{N_{\rm f}=0}}$
  - Error dominated by $\Lambda^{(2)}$
    $\rightarrow$ strategy for determining $\Lambda^{(2)}$?



# Partial Quenching

Meson correlators in our $N_\mathrm{f} = 2$ theory

$$\begin{aligned} & \langle \bar{c}_1(x)\Gamma c_2(x)\ \bar{c}_2(y)\Gamma c_1(y)\rangle \\ = & \left\langle \mathrm{tr}\left[\Gamma D^{-1}(x,y)\Gamma D^{-1}(y,x)\right]\right\rangle_{\mathrm{gauge}} \end{aligned}$$

(Partial) quenching: use bottom expression in a theory without sea quarks.

- $\Gamma$ selects channel
- Matching: demand $[\sqrt{t_0}]_{N_\mathrm{f}=2} = [\sqrt{t_0}]_{N_\mathrm{f}=0}$
- Valence quark mass: demand $[\sqrt{t_0}m_P]_{N_\mathrm{f}=2} = [\sqrt{t_0}m_P]_{N_\mathrm{f}=0}$
  (sacrifice pseudo-scalar mass for mass-matching)

This is widely used: e.g. charmonium spectrum from $N_\mathrm{f} = 2 + 1$ QCD

(see Gregorio Herdoiza's talk yesterday for further applications)

But, how well does this work quantitatively?

# Numerical setup

[S. Calì, F.Knechtli, T.K., Eur.Phys.J.C 79 (2019)]

- Use only $M = M_c$ runs
- Use only twisted mass fermions (also in the quenched theory)
- Use open boundaries in time

## Operators

| State | $J^{PC}$ | Particle | Physical basis | Twisted basis |
|---|---|---|---|---|
| Scalar | $0^{++}$ | $\chi_{c0}$ | $S^{1,2} = \bar{\psi}\frac{\tau^{1,2}}{2}\psi$ | $\bar{\chi}\frac{\tau^{1,2}}{2}\chi$ |
| Pseudoscalar | $0^{-+}$ | $\eta_c$ | $P^{1,2} = \bar{\psi}\gamma_5\frac{\tau^{1,2}}{2}\psi$ | $\bar{\chi}\gamma_5\frac{\tau^{1,2}}{2}\chi$ |
| Vector | $1^{--}$ | $J/\psi$ | $V_i^{1,2} = \bar{\psi}\gamma_i\frac{\tau^{1,2}}{2}\psi$ | $\pm\bar{\chi}\gamma_i\gamma_5\frac{\tau^{2,1}}{2}\chi$ |
| Axial vector | $1^{++}$ | $\chi_{c1}$ | $A_i^{1,2} = \bar{\psi}\gamma_i\gamma_5\frac{\tau^{1,2}}{2}\psi$ | $\pm\bar{\chi}\gamma_i\frac{\tau^{2,1}}{2}\chi$ |
| Tensor | $1^{+-}$ | $h_c$ | $T_{ij}^{1,2} = \bar{\psi}\gamma_i\gamma_j\frac{\tau^{1,2}}{2}\psi$ | $\bar{\chi}\gamma_i\gamma_j\frac{\tau^{1,2}}{2}\chi$ |

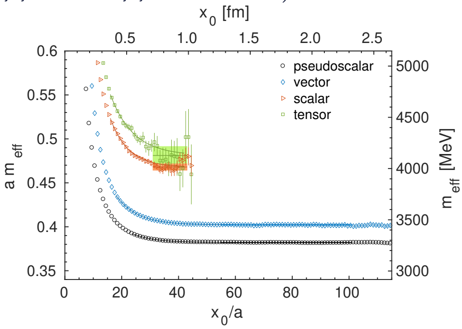
We call the corresponding ground state masses $m_S, m_P, m_V, m_A, m_T$

# Effective Masses

- Project operators to 0 momentum
- Correlate operators at $x_0$ and $y_0$, e.g. $f_{PP}(x_0, y_0) = \langle P(x_0)P^\dagger(y_0)\rangle$
- Average:

$$
\begin{aligned}
\bar{f}_{PP}(x_0 - a) &\equiv \frac{1}{2}\left(f_{PP}(x_0, a) + f_{PP}(T - x_0, T - a)\right) \\
\bar{f}_{VV}(x_0 - a) &\equiv \frac{1}{6}\sum_{k=1}^{3}\Big(f_{V_k V_k}(x_0, a) + f_{V_k V_k}(T - x_0, T - a)\Big) \\
\bar{f}_{SS}(x_0 - a) &\equiv \frac{1}{2}\left(f_{SS}(x_0, a) + f_{SS}(T - x_0, T - a)\right) \\
\bar{f}_{TT}(x_0 - a) &\equiv \frac{1}{6}\sum_{j>i}\Big(f_{T_{ij} T_{ij}}(x_0, a) + f_{T_{ij} T_{ij}}(T - x_0, T - a)\Big)
\end{aligned}
$$

- Effective mass:
  $a\, m_{\text{eff}}(x_0 + a/a) = \log\left(\frac{\bar{f}(x_0)}{\bar{f}(x_0+a)}\right)$
- Averaged in a plateau region



# Twisted Mass Derivatives

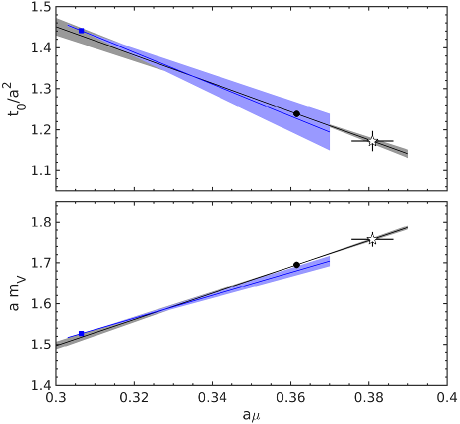
- $M_c/\Lambda$ as input to fix $a\mu_0$ was a bit imprecise, we can do better
- Choose $a\mu_0$ auch that $\sqrt{t_0}m_P = 1.807463$ exactly

But: simulations were already done
$\rightarrow$ compute twisted-mass derivatives

- $\frac{\mathrm{d}\langle A\rangle}{\mathrm{d}\mu_0} = -\left\langle \frac{\mathrm{d}S}{\mathrm{d}\mu_0} A \right\rangle + \left\langle \frac{\mathrm{d}S}{\mathrm{d}\mu_0} \right\rangle \langle A\rangle + \left\langle \frac{\mathrm{d}A}{\mathrm{d}\mu_0} \right\rangle$
- $\frac{\mathrm{d}\Phi(\langle A_1\rangle,\ldots,\langle A_N\rangle,\mu_0)}{\mathrm{d}\mu_0} = \frac{\partial\Phi}{\partial\mu_0} + \sum_{i=1}^{N} \frac{\partial\Phi}{\partial\langle A_i\rangle} \frac{\mathrm{d}\langle A_i\rangle}{\mathrm{d}\mu_0}$

And correct all quantities $\Phi$

- $\mu^\star = \mu_0 + (\sqrt{t_0}m_P - 1.807463)\left(\frac{\mathrm{d}\sqrt{t_0}m_P}{\mathrm{d}\mu_0}\right)^{-1}$
- $\Phi(\mu^\star) = \Phi(\mu_0) + (\mu^\star - \mu_0)\frac{\mathrm{d}\Phi}{\mathrm{d}\mu_0}$



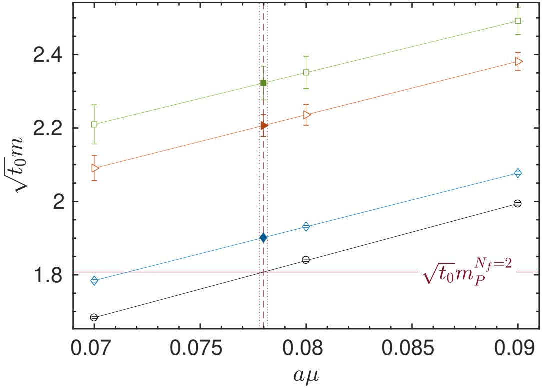
# Valence Quark Mass

Similarly for $N_f = 0$

- No $\mathrm{dS}/\mathrm{d}\mu_0$
- Measure quantities at 3 values of $a\mu_0$
- Find $a\mu^\star$ by linear interpolation

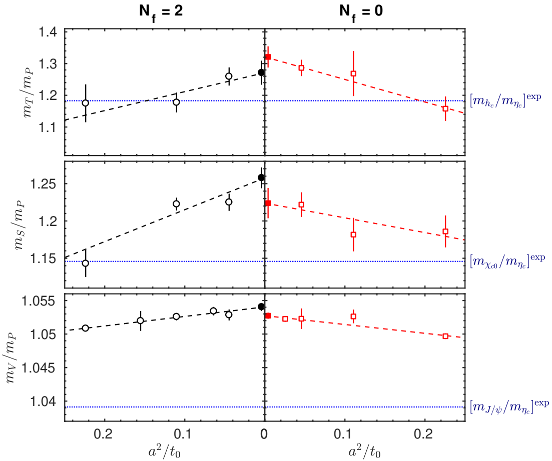
On the right:

- $m_P$ black circles
- $m_V$ blue diamonds
- $m_S$ orange triangles
- $m_T$ green squares



# Meson Masses

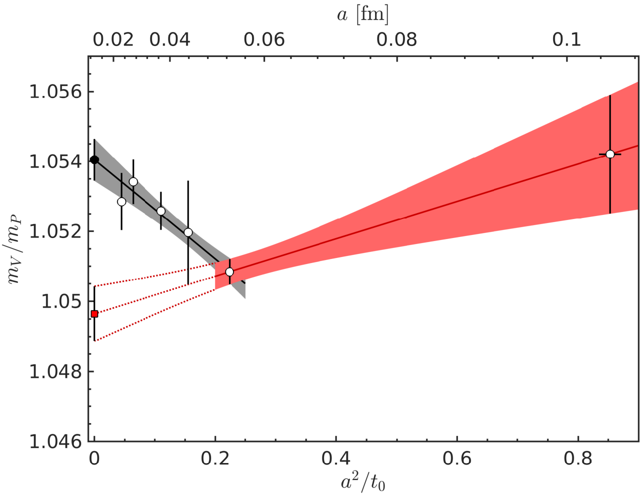
Main result: masses in the continuum limit




Sea quark effects: from 0.12(7)% to 2.7(1.6)%

# Lattice Artifacts

Side remark: Lattice artifacts in “charmonia” can be substantial



# Meson Decay Constants

[S. Calì, K.Eckert, J.Heitger, F.Knechtli, T.K. (2021)]

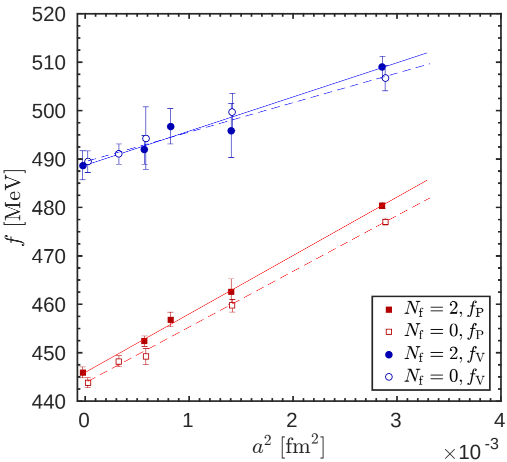
- Matching: As before
- Twisted mass (max twist) specific relations

$$f_P m_P^2 = 2\mu \, \langle 0|\bar{c}_1\gamma_5 c_2|P\rangle$$

$$f_V m_V = \frac{1}{3}\sum_{i=1}^{3}\langle 0|\bar{c}_1\gamma_i\gamma_5 c_2|V_i\rangle,$$

- Extract these matrix elements with open b.c. from ratios $\sqrt{\frac{a^3|f_{PP}(x_0,y_0)f_{PP}(T-x_0,y_0)|}{f_{PP}(T-y_0,y_0)}}$ Boundary-boundary correlator is challenging
- $f_P$ does not need any renormalization
- $f_V$ needs $Z_A$

  [M. Della Morte et al, JHEP 07 (2005) 007]

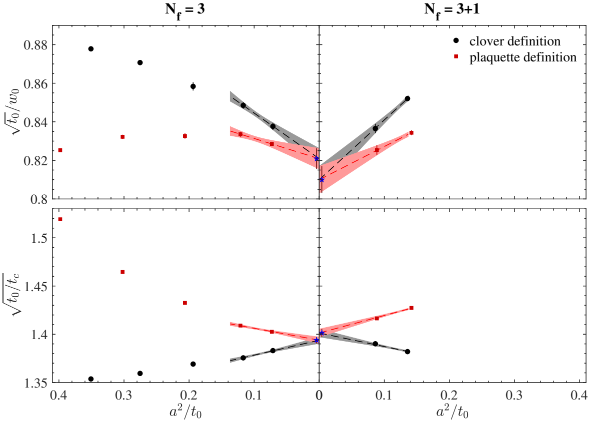



Effects: up to 0.48(34)%

# Realistic QCD

Decoupling of a charm quark in a realistic setup
$N_f = 3$ SU(3) symmetric point (CLS) vs $N_f = 3 + 1$

[R.Höllwieser, F.Knechtli, T.K., Eur.Phys.J.C 80 (2020)]



# Conclusions

## Scale Setting

- Example: determine $\sqrt{t_0} f_\pi$ in the continuum, divide by experimental $f_\pi^{\mathrm{exp}}$ $\to \sqrt{t_0}$ in fm
- Dimensionless combinations of low energy quantities were shown to be nearly identical with and without quarks of mass $M \approx M_c$
- $t_0$ from $N_\mathrm{f} = 2+1+1$ and $N_\mathrm{f} = 2+1$ should agree to a few permille

## $\alpha_S$

- 5 loop $\overline{MS}$ works well for decoupling already at $\mu = \overline{m}_c$

## Partial Quenching

- Surprizingly small effects of a dynamical charm quark
- Heavy quarks need fine lattices