EFTs on the Jet Bundle

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based on *Craig, YL [2307.15742]*

Field theory is redundant

Physical observables		math descriptions.
Scattering amplitudes	*are independent of*	field redefinitions.
Geometric meanings		coordinates.

The scalar manifold

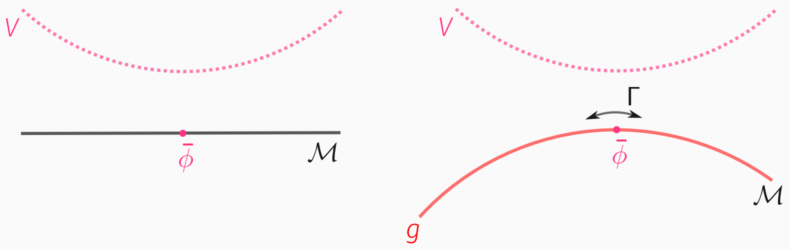
$$\mathcal{L} = V(\phi^z) + g_{ij}(\phi^z)\,(\partial_\mu\phi^i\partial^\mu\phi^j)$$

Under non-derivative field redefinitions $\phi^i(\tilde{\phi}^j)$:

$$\tilde{V} = V\,, \quad \tilde{g}_{kl} = g_{ij}\,\frac{\partial\phi^i}{\partial\tilde{\phi}^k}\frac{\partial\phi^j}{\partial\tilde{\phi}^l}$$

V and g_{ij} are tensors on the *field space* $\mathcal{M}$.

Riemannian geometry



A special point on $\mathcal{M}$ is the *vacuum* $\bar{\phi}^i = \mathrm{argmin}\, V$.

Riemannian geometry from g_{ij}:

- the Levi-Civita connection $\Gamma^i{}_{jk}$;
- a geometric invariant — the curvature $R^i{}_{jkl}$.

Why geometry is useful

Standard Model $\subsetneq$ Standard Model EFT $\subsetneq$ Higgs EFT

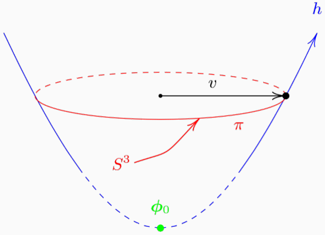
Distinction blurred: field redefs transform away non-analyticities.
Basis-independent criteria:

SM if $\mathcal{M}$ is flat;

SMEFT if $\exists O(4)$ fixed point on $\mathcal{M}$ where $R_{;\ldots}$ and $V_{;\ldots} < \infty$;

HEFT otherwise.

$;$ = covariant derivative. Up to $\mathcal{O}(\partial^2)$.

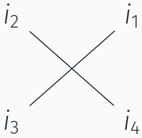


Alonso, Jenkins, Manohar [1511.00724, 1605.03602]. *Cohen, Craig, Lu, Sutherland* [2008.08597].

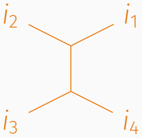
Scattering amplitudes

Tree level:

$$\mathcal{A}_4 =$$



$$+$$



$$+ \cdots$$

$$\propto \bar{V}_{;\cdots\cdot} - \frac{2}{3} \sum_{a<b} s_{ab} \, \bar{R}_{\cdots\cdot} - \bar{V}_{;\cdots} \frac{\bar{g}^{\cdot\cdot}}{2(s_{12} - m_{\cdot}^2)} \bar{V}_{;\cdots} - \ldots$$

s_{ab} = Mandelstam. $\bar{V} = V(\bar{\phi})$, etc.

Cohen, Craig, Lu, Sutherland [2108.03240].

Riemannian geometry is not complete

Add higher-order operators, e.g.:

$$\mathcal{L} = V(\phi^z) + g_{ij}(\phi^z)\,(\partial_\mu\phi^i\partial^\mu\phi^j) + h_i(\phi^z)\,(\partial_\mu\partial^\mu\phi^i) \\ + c_{ijkl}(\phi^z)\,(\partial_\mu\phi^i\partial^\mu\phi^j)(\partial_\nu\phi^k\partial^\nu\phi^l)$$

Then g_{ij} is no longer a tensor!

$$\tilde{g}_{kl} = g_{ij}\,\frac{\partial\phi^i}{\partial\tilde{\phi}^k}\frac{\partial\phi^j}{\partial\tilde{\phi}^l} + h_i\frac{\partial^2\phi^i}{\partial\tilde{\phi}^k\partial\tilde{\phi}^l}$$

- $\mathcal{M}$ knows $\mathcal{L}$ only up to first derivatives on fields.
- Must identify covariant pieces by hand via integration by parts.

The jet bundle

$$\mathcal{M} = \{(\phi^z)\}$$

$$\Big\downarrow \text{add spacetime manifold } \mathcal{T}$$

$$E = \{(x^\mu, \phi^z)\}$$

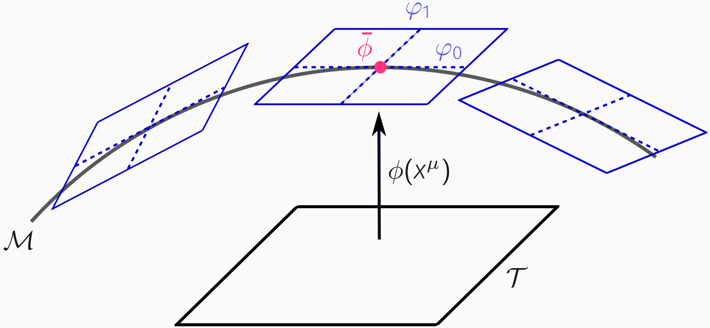
$$\Big\downarrow \text{add four-velocities } \varphi^z_\mu := \partial_\mu \varphi^z$$

$$J^1(\pi) = \{(x^\mu, \phi^z, \varphi^z_\mu)\}$$

$$\Big\downarrow \text{add } \phi^z_{\mu\nu} := \partial_\mu \partial_\nu \varphi^z$$

$$\vdots$$

The jet bundle



A spacetime-varying field $\phi^i(x^\mu)$ is a section of $\pi : E \to \mathcal{T}$.

A q-th order Lagrangian $\mathcal{L}(\phi^z, \varphi^z_\mu, \ldots)$ is a scalar function on $J^q(\pi)$.

Constructing a connection

Under $\phi^i(\tilde{\phi}^j)$, single derivatives on $J^q(\pi)$ transform by the chain rule:

$$\frac{\partial}{\partial \tilde{\varphi}^i_{\mu\ldots\nu}} = \frac{\partial \phi^j}{\partial \tilde{\phi}^i} \frac{\partial}{\partial \varphi^j_{\mu\ldots\nu}} + \frac{\partial \varphi^j_{\rho\ldots\sigma\tau}}{\partial \tilde{\varphi}^i_{\mu\ldots\nu}} \frac{\partial}{\partial \varphi^j_{\rho\ldots\sigma\tau}} + \ldots$$
$$= \text{covariant} + \text{higher-order derivatives}$$

Cancel using a non-linear connection N:

$$\frac{\delta}{\delta \varphi^i_{\mu\ldots\nu}} := \frac{\partial}{\partial \varphi^i_{\mu\ldots\nu}} - N^{j(\mu\ldots\nu)}_{(\rho\ldots\sigma\tau)i} \frac{\partial}{\partial \varphi^j_{\rho\ldots\sigma\tau}} - \ldots$$

Constructing a connection

For multiple derivatives, cancel with an N-linear connection F:

$$\text{horizontal:} \quad T^{\cdots}{}_{\cdots/l} := \frac{\delta}{\delta\phi^l} T^{\cdots}{}_{\cdots} + F^{\cdot}{}_{kl} T^{k\cdots}{}_{\cdots} - F^{k}{}_{\cdot l} T^{\cdots}{}_{k\cdots} + \dots$$

$$\text{vertical:} \quad T^{\cdots}{}_{\cdots}|_{l}^{\mu\dots\nu} := \frac{\delta}{\delta\varphi^l_{\mu\dots\nu}} T^{\cdots}{}_{\cdots}$$

A connection Γ on $\mathcal{M}$ extends to N and F on $J^q(\pi)$ as:

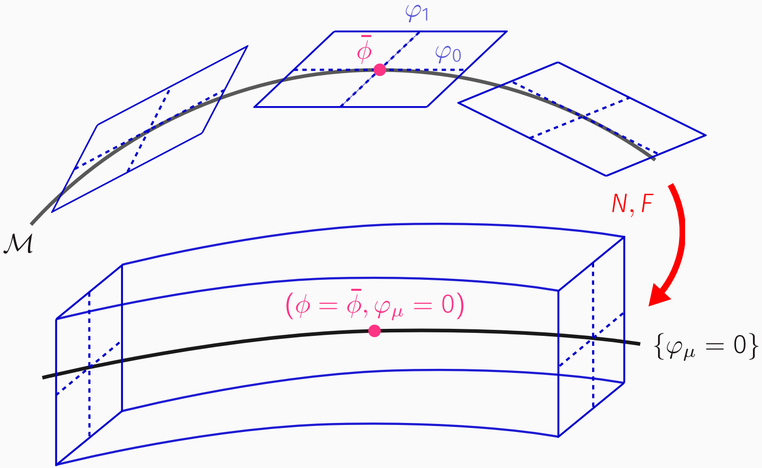
$$\frac{\delta}{\delta\phi^i} = \frac{\partial}{\partial\phi^i} - \left[\Gamma^{\cdot}{}_{i\cdot}\phi^{\cdot}_{\rho}\right]\frac{\partial}{\partial\varphi^{\cdot}_{\rho}} - \left[\left(\Gamma^{\cdot}{}_{i\cdot,\cdot} - \Gamma^{\cdot}{}_{\cdot\cdot}\Gamma^{\cdot}{}_{i\cdot}\right)\varphi^{\cdot}_{(\rho}\varphi^{\cdot}_{\sigma)} + \Gamma^{\cdot}{}_{i\cdot}\varphi^{\cdot}_{\rho\sigma}\right]\frac{\partial}{\partial\varphi^{\cdot}_{\rho\sigma}} - \dots$$

$$\frac{\delta}{\delta\varphi^i_{\mu}} = \frac{\partial}{\partial\varphi^i_{\mu}} - \left[2\Gamma^{\cdot}{}_{i\cdot}\varphi^{\cdot}_{(\rho}\delta^{\mu}_{\sigma)}\right]\frac{\partial}{\partial\varphi^{\cdot}_{\rho\sigma}} - \dots$$

$$\vdots$$

$$F^{i}{}_{jk} = \Gamma^{i}{}_{jk}$$

Constructing a connection



Covariant Wilson coefficients

$$\mathcal{L} = V(\phi^z) + g_{ij}(\phi^z)\,\eta^{\cdot\cdot}\varphi^i_{\cdot}\varphi^j_{\cdot} + h_i(\phi^z)\,\eta^{\cdot\cdot}\varphi^i_{\cdot\cdot} + c_{ijkl}(\phi^z)\,\eta^{\cdot\cdot}\eta^{\cdot\cdot}\varphi^i_{\cdot}\varphi^j_{\cdot}\varphi^k_{\cdot}\varphi^l_{\cdot}$$

On $\mathcal{M}$: manually pick out covariant tensors in $\mathcal{L}$.

On $J^q(\pi)$: systematically take h/v-covariant derivatives and evaluate on the null section.

$$V = (\mathcal{L})\Big|_{\varphi^z_{\cdots}=0}$$

$$g_{ij} - h_{(i,j)} = \frac{\eta_{\cdot\cdot}}{8}\left(\mathcal{L}|_{(i}^{\cdot}|_{j)}^{\cdot} - 2\mathcal{L}_{/(i}|_{j)}^{\cdot\cdot}\right)\Big|_{\varphi^z_{\cdots}=0}$$

$$c_{ijkl} = \frac{5\eta_{\cdot\cdot}\eta_{\cdot\cdot} - \eta_{\cdot\cdot}\eta_{\cdot\cdot} - \eta_{\cdot\cdot}\eta_{\cdot\cdot}}{576}\left(\mathcal{L}|_i^{\cdot}|_j^{\cdot}|_k^{\cdot}|_l^{\cdot}\right)\Big|_{\varphi^z_{\cdots}=0}$$

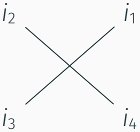
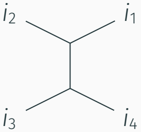
Invariance under total derivatives

Integration by parts is automatic at two derivatives.

$$g_{ij} - h_{(i,j)} = \frac{\eta_{\cdot\cdot}}{8} \left(\mathcal{L}|_{(i}{}^{\cdot}|_{j)}{}^{\cdot} - 2\mathcal{L}_{/(i}|_{j)}^{\cdot\cdot} \right) \Big|_{\varphi^{z}_{\dots}=0} =: G_{ij} \text{ on } J^{q}(\pi)$$

The canonical choice for Γ is the Levi-Civita connection of G.

Then since Γ determines N and F, the covariant jet bundle geometry is invariant under total derivatives.

$\mathcal{A}_4 =$ 
$+$ 
$+ \dots$

$$\propto \bar{\mathcal{L}}_{/\cdots\cdot} - \frac{2}{3}\sum_{a<b} s_{ab}\,\bar{\mathcal{R}}_{\cdots\cdot} - \bar{\mathcal{L}}_{/\cdots}\,\frac{\bar{G}^{\cdot\cdot}}{2(s_{12} - m_{\cdot}^2)}\,\bar{\mathcal{L}}_{/\cdots} - \dots$$

$$+ \frac{\cdots}{576}\left[4\sum_{a<b} m_a^2 m_b^2 - 2\sum_{a<b} s_{ab}(m_c^2 + m_d^2) + 2(s_{12}^2 + s_{13}^2 + s_{14}^2)\right]\bar{\mathcal{L}}_{|\because|\because|\because|\because}$$

$\mathcal{R}$ is the hh-curvature of G.

An overbar denotes evaluation at the vacuum $\{\phi^z = \bar{\phi}^z, \varphi^z_{\cdots} = 0\}$.

Summary and outlook

Generalized field space by adding all derivative degrees of freedom and extending notion of field redefinition covariance.

Invariance under total derivatives manifest in the two-derivative contribution and jet bundle geometry.

- *Loop effects*.
- Theories with *higher spin*.
- *Non-Riemannian geometry* and Lagrange spaces.
- *Derivative field redefinitions* on $J^\infty(\pi)$.

The non-linear connection

Given a connection Γ on $\mathcal{M}$, the covariant vectors and one-forms:

$$\frac{\delta}{\delta\phi^i_{(\ldots\mu_r)}} = \frac{\partial}{\partial\phi^i_{(\ldots\mu_r)}} - N^{j(\ldots\mu_r)}_{(\ldots\rho_{r+1})i}\frac{\partial}{\partial\phi^j_{(\ldots\rho_{r+1})}} - \ldots$$

$$\delta\phi^i_{(\ldots\mu_r)} = d\phi^i_{(\ldots\mu_r)} + M^{i(\ldots\rho_{r-1})}_{(\ldots\mu_r)j}d\phi^j_{(\ldots\rho_{r-1})} + \ldots + M^i_{(\ldots\mu_r)j}d\phi^j$$

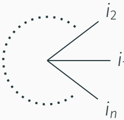
are given by the coefficients:

$$\begin{aligned}
M^i_{(\mu)j} &= \Gamma^i{}_{jk}\phi^k_{(\mu)} \\
M^{i(\ldots\rho_{r-1})}_{(\ldots\mu_r)j} &= r\, M^i_{(\mu_1|j}\delta^{(\ldots\rho_{r-1})}_{|\ldots\mu_r)} \\
M^{i(\ldots\rho_s)}_{(\ldots\mu_r)j} &= \frac{r}{r-s}\left[\frac{d}{dx^{(\mu_1|}}M^{i(\ldots\rho_s)}_{|\ldots\mu_r)j} + M^i_{(\mu_1|m}M^{m(\ldots\rho_s)}_{|\ldots\mu_r)j}\right] \\
N^{i(\ldots\rho_s)}_{(\ldots\mu_r)j} &= M^{i(\ldots\rho_s)}_{(\ldots\mu_r)j} - M^{i(\ldots\sigma_{r-1})}_{(\ldots\mu_r)k}N^{k(\ldots\rho_s)}_{(\ldots\sigma_{r-1})j} - \ldots - M^{i(\ldots\sigma_{s+1})}_{(\ldots\mu_r)k}N^{k(\ldots\rho_s)}_{(\ldots\sigma_{s+1})j}
\end{aligned}$$

Assembling amplitudes

Write down Feynman rules in terms of V, G and c on $\mathcal{M}$.

$$= \frac{i\,\bar{G}^{ij}}{2\,(p^2 - m_i^2)}$$



$$= i\left\{\bar{V}_{,\,\ldots} - \sum_{a<b} s_{ab}\,\bar{G}_{ab,\,\ldots} + (n-1)\sum_{a} p_a^2\,\bar{G}_{a(1,\,\ldots)} + 2\sum_{\substack{a<b\\a<c<d}} s_{ab}s_{cd}\,\bar{c}_{abcd,\,\ldots}\right.$$

$$\left. - 2\,(n-3)\sum_{a<b} s_{ab}\sum_{c\neq a,b} p_c^2\,\bar{c}_{abc(1,\,\ldots)} + 2\,(n-2)(n-3)\sum_{a<b} p_a^2 p_b^2\,\bar{c}_{ab(12,\,\ldots)}\right\}$$

$i_a \to a$. Ellipses represent omitted indices from i_1 to i_n.

Rewrite p_a using s_{ab} and $m_i^2 = -\bar{G}^{ii}\,\bar{V}_{,ii}\,/\,2$. As a function of s_{ab}, the amplitude comprises partial derivatives of V, G and c at $\bar{\phi}^I$.

Assembling amplitudes

In the $J^{\mathrm{q}}(\pi)$ coords induced by the normal coords on $\mathcal{M}$ of Γ at $\bar{\phi}^z$, non-covariant field space objects $\longrightarrow$ covariant jet bundle objects:

$$\begin{aligned}
\bar{V}_{,\ldots} &\longrightarrow \bar{\mathcal{L}}_{/(\ldots)} \\
\bar{G}_{ij,\ldots} &\longrightarrow \frac{\eta_{\cdot\cdot}}{8}\left(\bar{\mathcal{L}}|_{(i}^{\cdot}|_{j)/(\ldots)}^{\cdot} - 2\bar{\mathcal{L}}_{/(i}|_{j)/(\ldots)}^{\cdot\cdot}\right) \\
&\quad + \frac{n-3}{n-1}\left[\bar{\mathcal{R}}_{i(\cdot\cdot|j/|\cdot)} + \bar{\mathcal{R}}_{j(\cdot\cdot|i/|\cdot)}\right] + \mathcal{O}(\mathcal{R}^2) \\
\bar{c}_{ijkl,\ldots} &\longrightarrow \frac{5\eta_{\cdot\cdot}\eta_{\cdot\cdot} - \eta_{\cdot\cdot}\eta_{\cdot\cdot} - \eta_{\cdot\cdot}\eta_{\cdot\cdot}}{576}\left(\bar{\mathcal{L}}|_i^{\cdot}|_j^{\cdot}|_k^{\cdot}|_{l/(\ldots)}^{\cdot}\right) + \mathcal{O}(c\mathcal{R})
\end{aligned}$$

The tensorial replacement for the *total* amplitude must hold in all coordinates, since the amplitude is field-redefinition invariant.

Relation with Lagrange spaces

Lagrange spaces are jet bundles with $\dim \mathcal{T} = 1$ and order $q = 1$.

$$J^1(\pi) = \{(x, \phi^z, \varphi^z = d\phi^z/dx)\} = \mathbb{R} \times T\mathcal{M}$$

Special $\dim \mathcal{T} = 4$ Lagrangians fit into $T\mathcal{M}$ via $(\partial_\mu \phi^i \partial^\mu \phi^j) \to \varphi^i \varphi^j$:

$$\begin{aligned} \mathcal{L} &= V(\phi^z) + g_{ij}(\phi^z)\,(\partial_\mu \phi^i \partial^\mu \phi^j) + c_{(ijkl)}(\phi^z)\,(\partial_\mu \phi^i \partial^\mu \phi^j)(\partial_\nu \phi^k \partial^\nu \phi^l) \\ &\to V(\phi^z) + g_{ij}(\phi^z)\,\varphi^i \varphi^j + c_{ijkl}(\phi^z)\,\varphi^i \varphi^j \varphi^k \varphi^l \end{aligned}$$

Relation with Lagrange spaces

Lagrange spaces afford a non-Riemannian geometry that captures higher-derivative amplitudes and positivity/unitarity bounds.

metric	$g_{ij}(\phi^z, \varphi^z) = g_{ij}(\phi^z) + 6\, c_{ijkl}(\phi^z)\, \varphi^k \varphi^l$		
connection	$F^{\cdot}{}_{\cdot\cdot}(\phi^z, 0) = \Gamma^{\cdot}{}_{\cdot\cdot}(\phi^z)$	or	$\Gamma^{\cdot}{}_{\cdot\cdot}(\phi^z) + 3c^{\cdot\cdot}{}_{\cdot\cdot}(\phi^z) V_{,\cdot}(\phi^z)$
hh-curvature	$\mathcal{R}_{\cdot\cdot\cdot\cdot}(\bar{\phi}^z, 0) = R_{\cdot\cdot\cdot\cdot}(\bar{\phi}^z)$	or	$R_{\cdot\cdot\cdot\cdot}(\bar{\phi}^z) - 6c_{\cdot\cdot\cdot}{}^{\cdot}(\bar{\phi}^z) V_{,\cdot\cdot}(\bar{\phi}^z)$
$(v)hv$-torsion	$P^{\cdot}{}_{\cdot\cdot}(\phi^z, 0) = 0$	or	$-3c^{\cdot\cdot}{}_{\cdot\cdot}(\phi^z) V_{,\cdot}(\phi^z)$

	Lagrange spaces	Jet bundles
Dimensions	Finite	Countable
Intrinsic applicability	Special $\mathcal{L}$	Generic $\mathcal{L}$
Freedom in connection	Yes	No