Quantum Solution of Classical Turbulence

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Abstract

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Using the loop equation, we reduce the problem of decaying turbulence in the $3+1$ dimensional Navier-Stokes equation to the quantum mechanics of N Fermi particles on a ring in one dimension, interacting with an Euler ensemble of random fractions $\frac{p}{q}$ with denominator $q < N$. We find the solution of this system in the statistical limit $N \to \infty$ and compute the energy spectrum, dissipation rate, and velocity correlation function in decaying turbulence without approximations and fitted parameters. We find the whole spectrum of critical indexes, some of which are real, but others are complex numbers related to zeros of the Riemann ζ function. Grid turbulence experimental data and the recent large-scale DNS verify our predictions for the energy decay curve and the energy spectrum. **All scaling laws -K41, multifractal and Heisenberg – are ruled out.**

Part 1

Duality of Fluid Mechanics

Theories and models

Our generation of theoretical physicists is accustomed to **building and solving models** based on **existing physics theories**.

These basic theories –**Classical Mechanics, Statistical Physics, Relativity, Quantum Mechanics, Quantum Field Theory**– were all built for us in previous centuries.

Statistical Theory of Turbulence? Still missing.

The phenomenological models like K41 of multifractal scaling laws all fall short of the micsposcopic theory we are looking for.

This theory must be built from the Navier-Stokes equations, like Gibbs Statistics built from Newton mechanics or the quantization of field theory by Dirac-Feynman sum over classical histories.

Turbulent statistics must emerge spontaneously from the NS internal symmetry, without ad hoc stochastic forces.

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Introduction

In this talk, we present a new perspective on fluid mechanics, leading to such a statistical theory of turbulence.

By employing the loop equation, we reformulate fluid mechanics in arbitrary spatial dimensions as a singular one-dimensional problem.

This transformation is based on the concept of rough initial conditions in the Cauchy problem for the Navier-Stokes equation.

These rough initial conditions, which arise from thermal fluctuations, are inherent in any physical fluid. Consequently, physical fluid dynamics can be viewed as the evolution of a statistical distribution.

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Loop Average and dimension reduction

The loop average is a particular case of the Hopf functional with the source $\vec{J}(\vec{r})$ concentrated on a fixed loop in space

$$\vec{J}_C(\vec{r}) = \frac{\imath\gamma}{\nu} \oint d\vec{C}(\theta)\delta\left(\vec{r} - \vec{C}(\theta)\right) \quad (1)$$

The loop average (over initial and boundary conditions) is defined as

$$\Psi[\gamma, C] = \left\langle \exp\left(\int_{\vec{r}\in\mathbb{R}_d} \vec{J}_C(\vec{r}) \cdot \vec{v}(\vec{r})\right)\right\rangle =$$

$$\left\langle \exp\left(\frac{\imath\gamma}{\nu}\Gamma_C\right)\right\rangle; \quad (2)$$

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Loop equation as QM in Loop space

We derived a closed functional equation for the loop average in incompressible Navier-Stokes equation**M93, M23PR**

$$\imath\nu\partial_t\Psi[\gamma,C] = \left\langle \gamma \oint d\vec{C}(\theta)\cdot\left(-\nu\vec{\nabla}\times\vec{\omega}+\vec{v}\times\vec{\omega}\right)\exp\left(\frac{\imath\gamma}{\nu}\Gamma_C\right)\right\rangle = \oint d\vec{C}(\theta)\cdot\vec{L}\left[\frac{\delta}{\delta\vec{C}(.)}\right]\Psi[\gamma,C] \quad (4)$$

The operator $\vec{L}\left[\frac{\delta}{\delta\vec{C}(.)}\right]$ only depends on the functional derivative, but does not depend on the coordinate $\vec{C}(.)$ in loop space.

This independence (translation invariance) is the key to the solution. External forces would break it.

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This equation is equivalent to the Schrödinger equation in loop space with Hamiltonian $\hat{H}_C = \oint d\vec{C}(\theta) \cdot \vec{L}\left[\frac{\delta}{\delta \vec{C}(.)}\right]$.

A plane wave in loop space solves this Schrödinger equation

$$\Psi[\gamma, C] = \left\langle \exp\left(\frac{\imath\gamma}{\nu} \oint d\vec{C}(\theta) \cdot \vec{P}(t,\theta)\right)\right\rangle; \tag{5}$$

$$\imath\gamma\partial_t \vec{P} = \vec{L}\left[-\imath\frac{\gamma}{\nu}\partial_\theta \vec{P}(t,\theta)\right]; \tag{6}$$

$$\nu\partial_t \vec{P} = -\gamma^2(\Delta\vec{P})^2\vec{P} + \Delta\vec{P}\left(\gamma^2\vec{P}\cdot\Delta\vec{P} + \imath\gamma\left(\frac{(\vec{P}\cdot\Delta\vec{P})^2}{\Delta\vec{P}^2} - \vec{P}^2\right)\right); \tag{7}$$

with $\Delta\vec{P} = \vec{P}(\theta+0) - \vec{P}(\theta-0), \vec{P} = \frac{\vec{P}(\theta+0)+\vec{P}(\theta-0)}{2}$.

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Initially, the sum goes over the distribution of the initial data for $\vec{P}(0,\theta)$ (which we consider below).

In the same way as in the Newton's mechanics the trajectory eventually covers the energy surface (ergodicity) we expect this trajectory asymptotically cover some universal manifold (decaying rurbulence trajectory).

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Cauchy problem

The Cauchy problem, notoriously difficult for nonlinear PDE, can be solved analytically for the loop equation.

Let us describe this solution. Assuming the momentum loop equation is satisfied, we have certain conditions for the initial data $\vec{P}_0(\theta) = \vec{P}(\theta, 0)$.

This data is distributed with some unknown distribution $W[P]$ to be determined from the equation

$$\Psi_0(\gamma, C) = \Psi(\gamma, C)_{t=0} = \int [DP_0] W[\vec{P}_0] \exp\left(\frac{\imath\gamma}{\nu} \oint d\vec{C} \cdot \vec{P}_0\right);$$

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This is nothing but a functional Fourier transform, which can be inverted as follows

$$W[P_0] = \int [DC]\delta(\vec{C}(0))\Psi_0(\gamma, C)\exp\left(\frac{-\imath\gamma}{\nu}\oint d\vec{C}\cdot\vec{P}_0\right);$$

The definition of the parametric-invariant functional measure in this Fourier integral was discussed in detail in the old work **M93, M23PR**.

The periodic vector functions $\vec{C}(\theta), \vec{P}(\theta)$ are represented by the Fourier series, after which the measure becomes a limit of the multiple integrals over all the Fourier coefficients.

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In the previous work **M93, M23PR**, we completely solve the Cauchy problem for an interesting example – the exact fixed point of the loop equation corresponding to a global random rotation.

In a physically justified case of Gaussian thermal noise $\vec{\xi}(\vec{r})$ added to the initial velocity field $\vec{v}_0(\vec{r})$, we can advance solving the Cauchy problem for a generic initial velocity field.

Averaging the initial loop functional over Gaussian noise, we find

$$\Psi_0(\gamma, C) = \exp\left(\frac{\imath\gamma}{\nu}\oint_C d\vec{r}\cdot\vec{v}_0(\vec{r})\right)$$
$$\exp\left(-\frac{\gamma^2}{2\nu^2}\oint_C\oint_C d\vec{r}\cdot\left\langle\vec{\xi}(\vec{r})\otimes\vec{\xi}(\vec{r}\,')\right\rangle\cdot d\vec{r}\,'\right)$$

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Cauchy problem

This Gaussian noise is correlated at small distances r_0, related to the molecular structure of the fluid, which leads to the following estimate

$$\oint_C \oint_C d\vec{r} \cdot \left\langle \vec{\xi}(\vec{r}) \otimes \vec{\xi}(\vec{r}') \right\rangle \cdot d\vec{r}' \to \frac{|C|\sigma^2}{r_0^2};$$

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The constant velocity $v_0(\vec{C}(\theta))$ drops from the closed-loop integral, which brings the exponential to the ordinary momentum term in the Action. This path integral is computed by fixing the gauge for the parametric invariance $\theta \Rightarrow f(\theta)$, which is studied in the modern QFT, say in **PolyakovGFS**. The result is the following Gaussian distribution

$$W[P] \propto \int_0^\infty \frac{dT}{T} \exp\left(-\int_0^T dt \frac{m_0^2 + \vec{P}(t)^2}{2}\right); \quad (13)$$

$$\vec{P}(T) = \vec{P}(0); \quad (14)$$

The periodicity condition $\vec{P}(T) = \vec{P}(0)$ is essential in this distribution. Fourier coefficients $\vec{p}(n)$ can parametrize this periodic trajectory

$$\vec{P}(t) = \sum_{n=-\infty; n \neq 0}^{\infty} \vec{p}(n) \exp\left(\frac{2\pi \imath n t}{T}\right); \quad (15)$$

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As we argued, the unavoidable thermal noise in any physical fluid makes such a limit the correct definition.

We presented an explicit solution to the Cauchy problem for the loop equation. This solution reduces the Navier-Stokes equation in arbitrary space dimension to a one-dimensional momentum loop equation with a given distribution of initial values.

The Cauchy problem solution involves averaging over the Gaussian noise in initial data, but the turbulent asymptotic solution we are looking for corresponds to trajectory uniformly covering some unknown manifold (fixed point).

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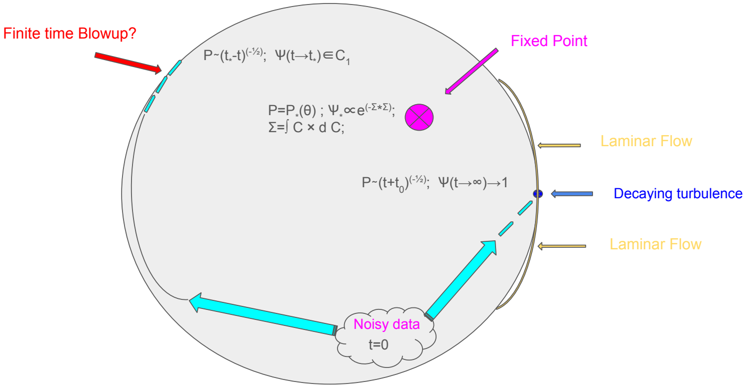
Asymptotic regimes of the loop equation

Loop Functional Evolution



Part II

Decaying turbulence solved

Exact solution for decaying turbulence

This singular nonlinear equation discouraged us back in the nineties.

Surprisingly, an infinite family of analytic solutions was found.

$$\vec{P}(t,\theta) = \sqrt{\frac{\nu}{2(t+t_0)}}\hat{\Omega}\cdot\frac{\vec{F}(\theta)}{\gamma};\ \hat{\Omega}\in O(3); \tag{18}$$

$$\vec{F}_k = \frac{\left\{\cos(\alpha_k), \sin(\alpha_k), i\cos\left(\frac{\beta}{2}\right)\right\}}{2\sin\left(\frac{\beta}{2}\right)}; \tag{19}$$

$$\theta_k = \frac{2\pi k}{N};\ \beta = \frac{2\pi p}{q};\ N\to\infty; \tag{20}$$

$$\alpha_{k+1} = \alpha_k + \sigma_k\beta;\ \sigma_k = \pm 1,\ \beta\sum\sigma_k = 2\pi p r; \tag{21}$$

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Classical flow and quantum mechanics

The integer $1 \leq p < q$ is bound by being co-prime with q, and $2 < q < N, (N-q) \mod 2 = 0$.

The fractions $\frac{p}{q}$ with fixed denominator are counted by Euler totient function $\varphi(q)$

$$\varphi(q) = \sum_{\substack{p=1 \\ (p,q)}}^{q-1} 1 = q \prod_{p|q} \left(1 - \frac{1}{p}\right);$$

We suggested to call this manifold the Euler ensemble, for his discoveries to fluid dynamics merging with his discoveries in Number Theory.

What could the number theory have in common with the turbulent flow?

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Every distinct state in a quantum system enters the partition function with unit weight, which we also accept in our quantum solution of the nonlinear classical system.

This quantum statistical counting is not necessary for solving the loop equation; it is an extra conjecture that may be proven from the underlying Navier-Stokes equation, like an ergodic hypothesis in reversible Newton mechanics (accepted in Physics but mathematically proven only on a few special cases).

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The number $n(M)$ of positive sigmas $\sum_{l=1}^{M} \delta[\sigma_l - 1]$ coincides with the occupation number of these Fermi particles.

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This relation leads to the representation of the Markov transition matrices

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$$\Psi(\gamma, C) = \frac{\left\langle \hat{W}[C] \right\rangle_{\hat{\Omega}, \mathcal{E}(N)}}{\left\langle \hat{W}[0] \right\rangle_{\hat{\Omega}, \mathcal{E}}(N)};$$

$$\hat{W}[C] = \mathrm{tr}\left(\hat{Z}(qr) \exp\left(\frac{i\gamma\hat{\Gamma}}{\nu} \right) \prod_{M=1}^{N} \hat{Q}(M) \right),$$

$$\hat{\Gamma} = \sum_l \Delta\vec{C}_l \cdot \hat{\Omega} \cdot \vec{\mathcal{P}}_l(t);$$

$$\hat{Z}(s) = \oint \frac{d\omega}{2\pi} \exp\left(\imath\omega \left(\sum_l \hat{\sigma}_l - s \right) \right);$$

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$$\Delta\vec{C}_l = \vec{C}\left(\frac{l+1}{N}\right) - \vec{C}\left(\frac{l}{N}\right),$$

$$\vec{\mathcal{P}}_l(t) = \sqrt{\frac{\nu}{2\,(t+t_0)}}\frac{\vec{\mathcal{F}}_l}{\gamma}, \quad \hat{\Omega} \in O(3),$$

$$\vec{\mathcal{F}}_l = \frac{\{\cos\left(\hat{\alpha}_l\right), \sin\left(\hat{\alpha}_l\right), 0\}}{2\sin\left(\frac{\beta}{2}\right)},$$

$$\mathcal{E}(N): \quad p, q, r \in \mathbb{Z} \quad -N \leq qr \leq N,$$

$$\text{with} \quad 0 < p < q < N, \; \mathbf{gcd}(p,q) = 1,$$

$$\hat{\alpha}_l = \beta \sum_{k=1}^{l-1}(2\hat{\nu}_k - 1);$$

Duality

The equivalence of a strong coupling phase of the fluctuating vector field to the weak coupling phase of quantum geometry is a well-known duality phenomenon in gauge theory (the ADS/CFT correspondence).

The duality does not imply a direct relation between the dynamical variables of the two theories– only statistical averages are related.

The quantum Fermions of our dual theory is no less real than the turbulent vorticity field: arguably, more real, as the fluctuations of vorticity go to infinity, whereas fluctuations of our Fermion density go to zero in turbulent limit.

We are not just computing correlation functions of decaying turbulence; we are revealing its hidden true identity.

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Continuum Limit

Brownian motion on a circle $\alpha_k \Rightarrow \alpha(\xi = k/N)$. In continuum limit:

$$\Psi[C] = \frac{\sum\limits_{p<q;(p,q)} \int\limits_{\Omega \in O(3)} d\Omega \int [D\alpha] \exp\left(\imath S[\alpha]\right)}{\sum\limits_{p<q;(p,q)} |O(3)| \int [D\alpha]};$$

$$S[\alpha] = \frac{\int_0^1 d\xi \,\textbf{Im}\,\left(\mathcal{C}'_\Omega(\xi) \exp\left(\imath\alpha(\xi)\right)\right)}{2\sin(\pi p/q)\sqrt{2\nu(t+t_0)}};$$

$$\mathcal{C}_\Omega(\theta) = \vec{C}(\theta) \cdot \hat{\Omega} \cdot \{\imath, 1, 0\};$$

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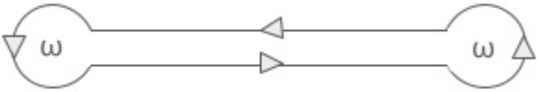
Correlation function and loop functional

The simplest observable quantities we can extract from the loop functional are the vorticity correlation functions **M23PR**, corresponding to the loop C backtracking between two points in space $\vec{r}_1 = 0, \vec{r}_2 = \vec{r}$.



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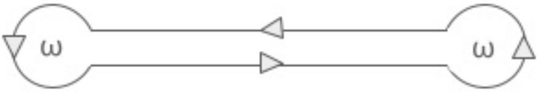

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The correlation function reduces to the following average:

$$\left\langle \vec{\omega}(\vec{0}) \cdot \vec{\omega}(\vec{r}) \right\rangle = \frac{\left\langle \sum_{0 \le n < m < N} \vec{\omega}_m \cdot \vec{\omega}_n \exp\left(\imath \vec{\rho} \cdot \left(\vec{S}_{n,m} - \vec{S}_{m,n}\right)\right) \right\rangle}{4(t+t_0)^2};$$

$$\vec{S}_{n,m} = \frac{\sum_{k=n}^{m-1} \vec{F}_k}{(m-n) \pmod N};$$

$$\vec{\omega}_m \cdot \vec{\omega}_n = \frac{-\sigma_m \sigma_n}{4} \cot^2\left(\frac{\beta}{2}\right);$$

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The averaging $\langle \dots \rangle$ in these formulas involves group integration $\int_{O(3)} d\hat{O}$ with $\vec{\rho} \Rightarrow \hat{O} \cdot \vec{\rho}$.

$$\left\langle H(\vec{\rho} \cdot \hat{\Omega} \cdot \vec{F}) \right\rangle_{O(3)} = {}^1\!/_2 \int_{-1}^{1} dz H\left(|\vec{\rho}||\vec{F}|z \right)$$

The rest is tedious but well-defined work, familiar to QFT experts, especially those involved with one-dimensional systems.

Our system has extra simplicity because in the turbulent limit $\nu \to \frac{\tilde{\nu}}{N^2}, N \to \infty$, we can use the WKB approximation to the path integral and asymptotic summation formulas of the number theory for Euler totients with large denominators $q \sim N \to \infty$.

Experimental verification

Part III

This equation is equivalent to the Schrödinger equation in loop space with Hamiltonian $\hat{H}_C = \oint d\vec{C}(\theta) \cdot \vec{L}\left[\frac{\delta}{\delta \vec{C}(.)}\right]$.

A plane wave in loop space solves this Schrödinger equation

$$\Psi[\gamma, C] = \left\langle \exp\left(\frac{\imath\gamma}{\nu}\oint d\vec{C}(\theta) \cdot \vec{P}(t,\theta)\right)\right\rangle; \tag{22}$$

$$\imath\gamma\partial_t\vec{P} = \vec{L}\left[-\imath\frac{\gamma}{\nu}\partial_\theta\vec{P}(t,\theta)\right]; \tag{23}$$

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Our solution

Remarkably, this nonlinear singular equation for $\vec{P}(\theta, t)$ can be solved analytically.

The mathematical details are too complex to present them here.

Here is the resulting formula for the second moment of velocity difference in decaying turbulence:

$$\left\langle \Delta \vec{v}^2 \right\rangle (r) = \frac{\tilde{\nu}^2}{\nu t} \int_{\epsilon - \imath\infty}^{\epsilon + \imath\infty} \frac{dp}{2\pi\imath} V(p) \left(\frac{|\vec{r}|}{\sqrt{\tilde{\nu} t}} \right)^p ; \tag{25}$$

$$V(p) = -\frac{f(-1-p)\zeta\left(\frac{13}{2} - p\right)\csc\left(\frac{\pi p}{2}\right)}{16\pi^2 (p+1)(2p-15)(2p-5)\zeta\left(\frac{15}{2} - p\right)} \tag{26}$$

Here $\Delta_1 = 0.157143, \Delta_2 = 0.43015$, and $f(z)$ is an entire function computed using Mellin integrals of elementary functions. $V(p)$ is meromorphic. ν is physical viscosity and turbulent viscosity $\tilde{\nu}$ is a free parameter of our solution.

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Spectrum of Poles and Scaling Dimensions

The spectrum p_n of poles $V(p)$ determines the scaling dimensions in the correlation function's expansion in $\left(|\vec{r}|/\sqrt{\tilde{\nu}t}\right)^{p_n}$.

indexes of velocity correlation
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$1/2(15 + 4n)$ if $n \in \mathbb{Z} \wedge n \geq 0$

Here $\pm t_n$ are Riemann zeros of $\zeta(1/2 + \imath t)$.

Imaginary parts of dimensions lead to quantum oscillations.

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Energy Spectrum and its Singularities

The singularities of the Mellin transform $h(p)$ for the energy spectrum are defined as follows:

$$E(k,t) = \frac{\tilde{\nu}^{5/2}}{\nu\sqrt{t}} \int_{-\epsilon-\imath\infty}^{-\epsilon+\imath\infty} \frac{dp}{2\pi\imath} h(p) \left(k\sqrt{\tilde{\nu} t}\right)^{p};$$

These singularities are given by the following table of simple poles:

indexes of energy spectrum
n if $n \in \mathbb{Z} \wedge n \geq 0$
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These singularities are given by the following table of simple poles:

indexes of energy spectrum
n if $n \in \mathbb{Z} \wedge n \geq 0$
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Mellin Transform for Energy

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$$e(q) = 2\pi k_0^2 \tilde{\nu}^2 \frac{h(2q-1)}{q(q+1)}$$

The table of complex poles of this function is:

indexes of energy decay
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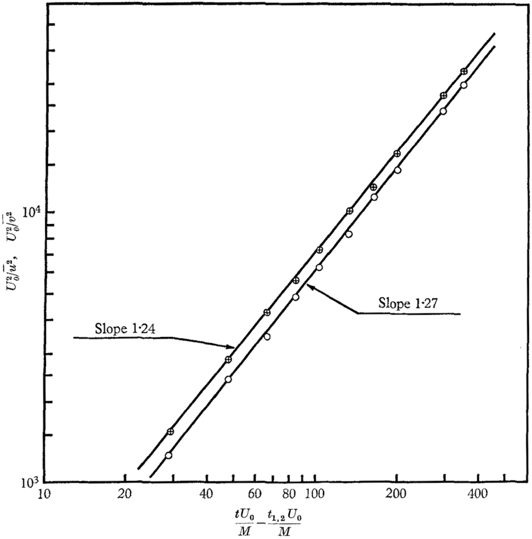
Energy decay in grid turbulence experiments



Reynolds number decay in DNS

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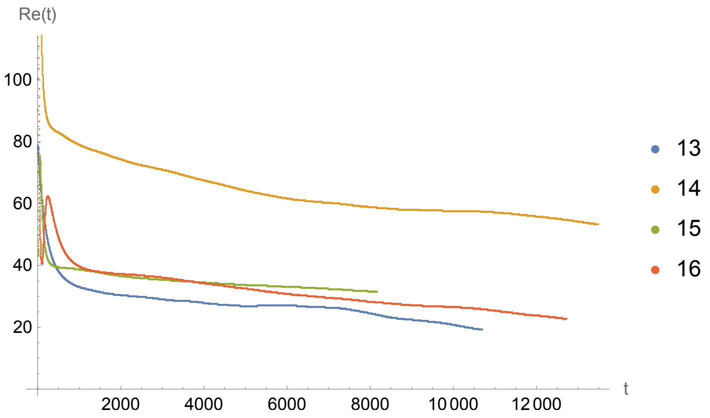

The time decay of Reynolds numbers for each of the four samples from **SreeniDecaying, GDSM24**

Verifying effective length and energy decay

The next test is the effective length scale, which we define as

$$L(t) = \frac{\int E(k,t) k dk}{\int E(k,t) k^2 dk}; \tag{27}$$

The log-log plot of $L(t)$ is shown in Fig.40. In the turbulent region, it perfectly matches our theory $L(t) \propto \sqrt{t}$.

We plot $E(t)$ as a function of $L(t)$

$$E(t) = \int E(k,t) dk; \tag{28}$$

$$\log E(t) \approx a + f(b + \log L(t)) \tag{29}$$

The parameters a, b were fitted by nonlinear regression using "NonlinearModelFit" in *Mathematica*®. The resulting log-log plot is shown in(Fig. 41).

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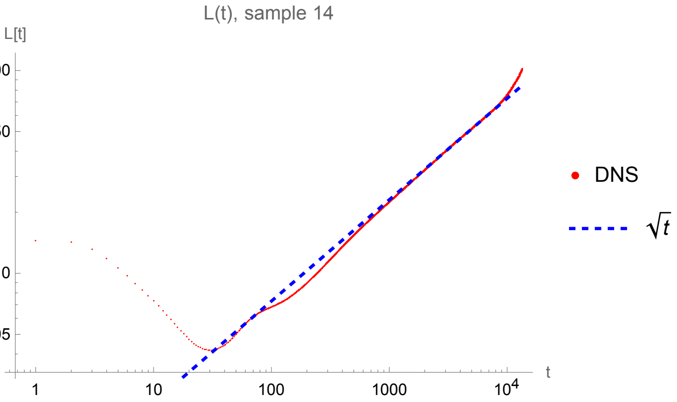

The log-log plot of the effective length $L(t)$ for the sample 14 from **SreeniDecaying, GDSM24**. The turbulent part, $1000 < t < 8000$, closely fits our theory $L(t) \propto \sqrt{t}$.

Verifying effective length and energy decay

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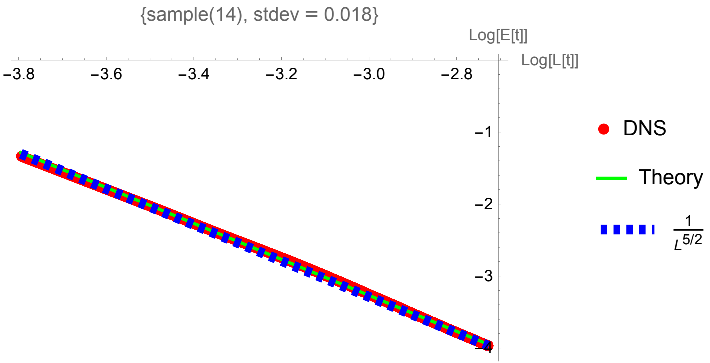

The log-log plot of the decaying energy as a function of decaying length scale $L(t)$. Red dots are the DNS data, the green curve is an exact theoretical curve, and the dashed blue line is its asymptotic limit $E(t) \propto L(t)^{-5/2}$ corresponding to $E \propto t^{-5/4}$.

The energy spectrum: DNS vs theory and K41

We have a scaling law $E(k,t) = \frac{H(k\sqrt{t+t_0})}{\sqrt{t+t_0}}$, which means that the two-dimensional array of the data for $E(k,t)$ must collapse at one-dimensional subset.

We already saw the consequence of that collapse in the scaling law for $L(t)$. However, the low k part of the spectrum is discrete, which introduces lattice artifacts.

We found the following method to avoid choosing the range of discrete wavelengths and suppress statistical errors and lattice artifacts.

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Mellin transform and decay index spectrum

First, we consider the Mellin transform of the energy spectrum, normalized at $p = 0$ together with its derivative

$$M(p,t) = \frac{e^{-\alpha p} \int dk E(k,t)(kL(t))^p}{\int dk E(k,t)}; \tag{30}$$

$$\alpha = \left\langle \frac{\int dk E(k,t) \log[(kL(t))]}{\int dk E(k,t)} \right\rangle_t; \tag{31}$$

$$M(0,t) = 1; \tag{32}$$

$$\langle \partial_p M(p=0,t) \rangle_t = 0; \tag{33}$$

The numerical integration (summation) over the energy spectrum suppresses the data noise and the lattice artifacts when the Mellin parameter p is not too large.

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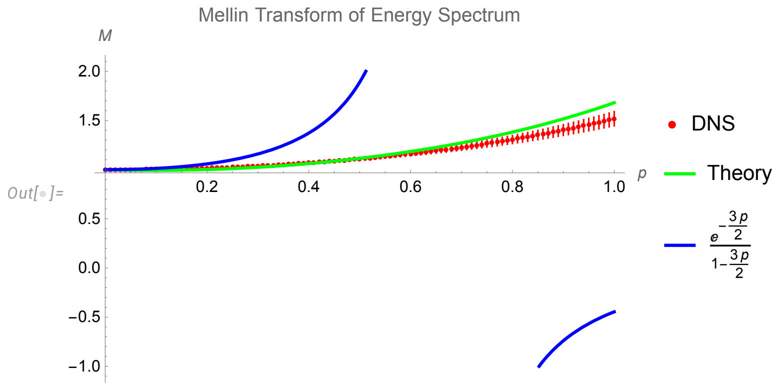

The Mellin integral (30) for the DNS (red dots with error bars), our theory (green curve) and K41 pole term, shown as a blue curve. Obviously, there is no such pole in DNS.

Verifying second velocity moment by the DNS data

Now, consider the second moment of velocity, related to the energy spectrum by Fourier transform

$$\left\langle \Delta \vec{v}^2 \right\rangle (r) = \int \frac{d^3 k}{4\pi^3} \frac{\left(1 - e^{\imath \vec{k}\cdot\vec{r}}\right) E(k,t)}{4\pi \vec{k}^2}; \tag{34}$$

There is a sharper test, namely the effective index $\xi_2(r,t)$ defined as a log-log derivative of this second moment

$$\xi_2(r,t) = r\partial_r \log \left\langle \Delta v^2 \right\rangle (r) = \frac{\int_0^\infty dk \left(-\cos(kr) + \frac{\sin(kr)}{kr}\right) E(k,t)}{\int_0^\infty dk \left(1 - \frac{\sin(kr)}{kr}\right) E(k,t)}; \tag{35a}$$

$$\left\langle \xi_2(xL(t),t) \right\rangle_t = f(x); \tag{35b}$$

This universal function $f(x)$ is numerically well defined as the integration over the spectrum suppresses the DNS noise.

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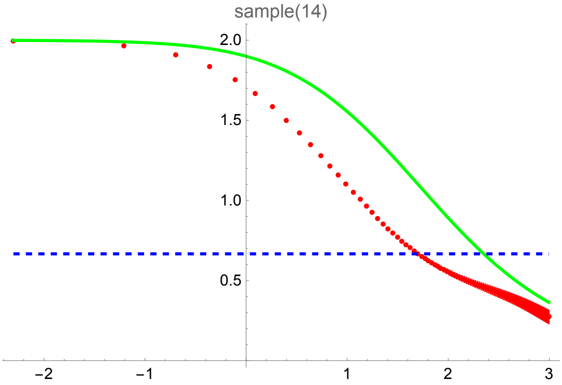

The plot of $f(x)$ in (35) as a function of $\log x$. The red dots with error bars are the DNS data. The green curve is the theoretical index without adjustment of the coordinate scale, and the dashed blue line is the K41 constant value $\frac{2}{3}$.

Verifying second velocity moment by the DNS data

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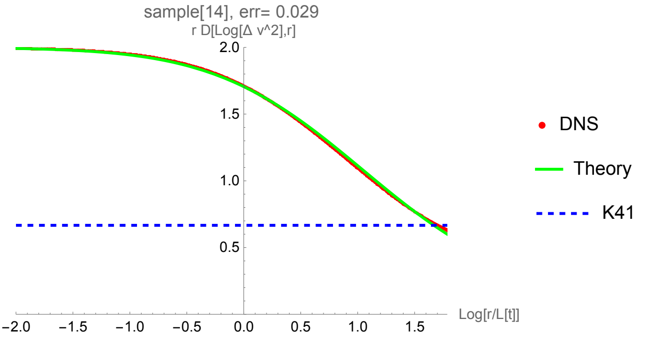

The plot of the effective index $f(x)$ in (35) as a function of $\log x$ in the turbulent range. Red dots with error bars are the DNS data. The K41 $\frac{2}{3}$ law (blue dashed line) is off the charts, our theory (green curve) fits perfectly.

Conclusion

- We have unveiled a duality between decaying classical turbulence in 3+1 dimensions and the one-dimensional quantum theory of $N \to \infty$ Fermi particles on a ring.
- The density fluctuations of these Fermi particles disappear in the turbulent limit $\nu \sim 1/N^2 \to 0$, leading to the exact WKB (instanton) solution for this density.
- This establishes a new relation between classical nonlinear dynamics and quantum theory: the Fourier transform of classical probability adheres to the QM evolution with the interference of alternative histories.
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Details of Computations

Continuum limit of small Euler ensemble

Certain scaling variables have nontrivial distributions governed by the number theory. As we shall see, rather than β, we would need an asymptotic distribution of two scaling variables:

$$X(p,q) = \frac{1}{q^2} \cot^2\left(\frac{\pi p}{q}\right) \to \frac{1}{\pi^2 p^2} \quad (36)$$

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These distributions can be derived from the number theory

$$\mathbb{P}(q < yN) = \Phi\left(\lfloor Ny \rfloor\right); \tag{37}$$

$$\Phi(n) = \sum_{k=1}^{n} \varphi(k) \to \frac{3}{\pi^2} n^2; \tag{38}$$

$$f_y(y) = \sum_{q=2}^{\infty} \delta\left(y - \frac{q}{N}\right) \varphi(q) \tag{39}$$

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The βN variable can be related to y and $X(p,q)$

$$\beta N = 2N \operatorname{arcCot}(q\sqrt{X}) \to \frac{2}{y\sqrt{X}}; \tag{40}$$

$$f_X(X) = (1-\alpha)\delta(X) + \pi X\sqrt{X}\Phi\left(\left\lfloor \frac{1}{\pi\sqrt{X}} \right\rfloor\right); \tag{41}$$

$$\alpha = \pi\int_0^\infty X\sqrt{X}dX\Phi\left(\left\lfloor \frac{1}{\pi\sqrt{X}} \right\rfloor\right) = \frac{2\pi}{5}\sum_1^\infty \frac{\varphi(k)}{(\pi k)^5} = \frac{1}{225\zeta(5)} \tag{42}$$

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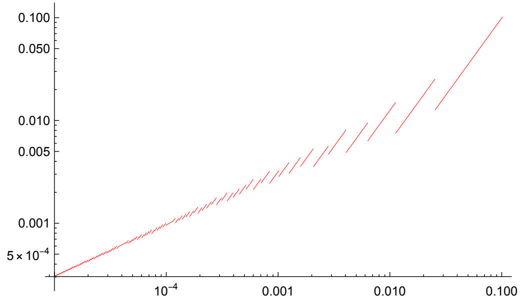
$$\pi X^{3/2} \sum_{q=1}^{\left\lfloor \frac{1}{\pi \sqrt{X}} \right\rfloor} \phi(q)$$



Gaussian fluctuations around instanton

The classical equation can be solved in elementary functions

$$\alpha(\xi) = A\cos(K(\xi-\xi_2)) + B\sin(K(\xi-\xi_2)); \tag{43}$$

$$K = \left\{\sqrt{\frac{r}{\Delta}}, \sqrt{\frac{r}{\Delta-1}}\right\}; \tag{44}$$

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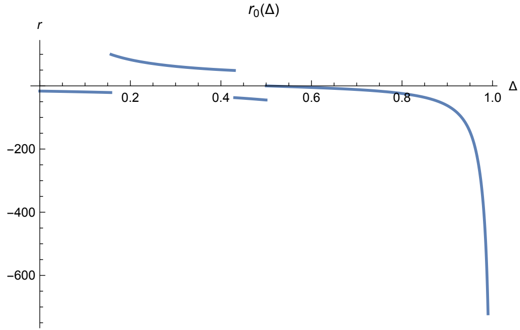

Gaussian fluctuations around instanton

$$f(\omega_n, \Delta) = 0; \tag{48}$$

$$f(\omega, \Delta) = (\Delta - 1)\Delta\sqrt{\omega}\sin\left(\frac{\sqrt{\omega}}{2}\right)((\Delta - 1)\Delta\omega + r)+$$
$$r\cos\left(\frac{1}{2}(1 - 2\Delta)\sqrt{\omega}\right) - r\cos\left(\frac{\sqrt{\omega}}{2}\right); \tag{49}$$

$$r = r_0(\Delta); \tag{50}$$

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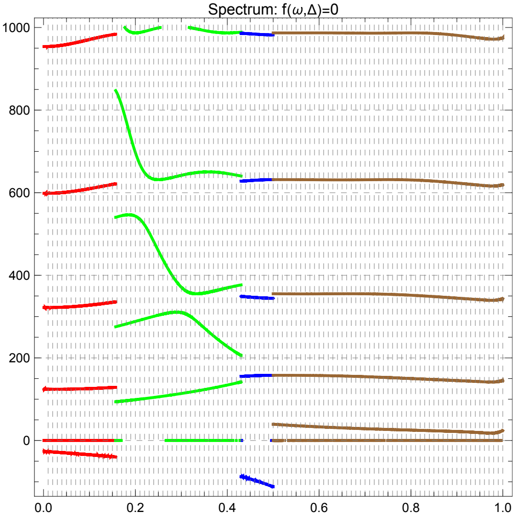

Gaussian fluctuations around instanton

The functional determinant, resulting from the WKB approximation to the α path integral, would be related to the infinite product of positive eigenvalues $\epsilon_n = \tau^2 \omega_n$, which can be written using a contour integral

$$Q_\alpha(\Delta, \tau) = \prod_{\omega_n > 0} (\tau^2 \omega_n)^{-1/2} =$$

$$\exp\left(1/2 \partial_\alpha \mathbf{Im} \oint_\Gamma \frac{f'(\omega)}{f(\omega)} \frac{d\omega}{2\pi(\omega\tau^2)^\alpha} \right)\Bigg|_{\alpha \to 0}; \qquad (51)$$

and the integration contour Γ encircles anticlockwise the positive real poles of the meromorphic function $f'(\omega)/f(\omega)$. The integral converges at $\alpha > 1/2$ and should be analytically continued to $\alpha = 0$.

Gaussian fluctuations around instanton

We introduce another function

$$\Phi(\omega) = \frac{f(\omega)}{\cos(\sqrt{\omega}/2)}\omega^{-3/2} \tag{52}$$

At large $\omega = \imath y$ this function reaches finite limits

$$\Phi(\imath y) \to \imath \operatorname{sign} y(\Delta - 1)^2\Delta^2 + \frac{(\Delta - 1)\Delta r}{|y|} \tag{53}$$

Gaussian fluctuations around instanton

The logarithmic derivative of the original function differs from $\frac{\Phi'(\omega)}{\Phi(\omega)}$ by the following meromorphic function

$$\frac{f'(\omega)}{f(\omega)} - \frac{\Phi'(\omega)}{\Phi(\omega)} = -\frac{\tan\left(\frac{\sqrt{\omega}}{2}\right)}{4\sqrt{\omega}} + \frac{3}{2\omega} \tag{54}$$

This difference produces a calculable contribution to our integral. By summing residues of the poles of the tangent, we get

$$\oint_\Gamma \left(-\frac{\tan\left(\frac{\sqrt{\omega}}{2}\right)}{4\sqrt{\omega}} + \frac{3}{2\omega}\right) \frac{d\omega}{2\pi(\omega\tau^2)^\alpha} = \imath\left(1 - 2^{2\alpha}\right)(2\pi\tau)^{-2\alpha}\zeta(2\alpha) \tag{55}$$

Gaussian fluctuations around instanton

The derivative at $\alpha = 0$ yields a constant

$$^1/_2\partial_\alpha \mathbf{Im}\ \imath \left(1 - 2^{2\alpha}\right) (2\pi\tau)^{-2\alpha} \zeta(2\alpha) \to \frac{\log(2)}{2} \tag{56}$$

leading to an irrelevant renormalization of $Q(\alpha, \tau)$ by a factor $\sqrt{2}$.

The remaining integral with $f(\omega) \Rightarrow \Phi(\omega)$ already converges at **Re** $\alpha > -1$, so that we can set $\alpha = 0$ and rotate the integration contour Γ parallel to the imaginary axis at **Re** $\Gamma = \epsilon > 0$:

$$Q_\alpha(\Delta, \tau) = \exp\left(^1/_2 \mathbf{Im} \int_{\epsilon - \imath\infty}^{\epsilon + \imath\infty} \frac{\Phi'(\omega)}{\Phi(\omega)} \frac{\log(\omega\tau^2)\, d\omega}{2\pi}\right) \tag{57}$$

Gaussian fluctuations around instanton

The remarkable property of this functional determinant is the factorization of the τ dependence

$$Q_\alpha(\Delta, \tau) = \tau^{\mu(\Delta)} Q_\alpha(\Delta, 1); \tag{58}$$

$$\mu(\Delta) = \textbf{Im} \int_{\epsilon - \imath\infty}^{\epsilon + \imath\infty} \frac{\Phi'(\omega)}{\Phi(\omega)} \frac{d\omega}{2\pi} \tag{59}$$

The index $\mu(\Delta)$ has a topological origin and can be computed analytically.

$$\mu(\Delta) = \frac{\arg \Phi(\imath\infty) - \arg \Phi(-\imath\infty)}{2\pi} = {}^1\!/_2 \tag{60}$$

Gaussian fluctuations around instanton

$$\left\langle \vec{\omega}(\vec{0}) \cdot \vec{\omega}(\vec{k}) \right\rangle = \frac{\tilde{\nu}^{\frac{3}{2}} H\left(k\sqrt{\tilde{\nu} t}\right)}{\sqrt{t}}; \tag{61}$$

$$H(\kappa) = \frac{1}{\mathcal{Z}} \sum_{n=1}^{\infty} \varphi(n) \int_0^{1/n} d\tau \left(\tau^5/n^5 - \tau^{10}\right) \int_{\Delta_1}^{\Delta_2} d\Delta (1 - \Delta) G\left(\Delta, \tau, \kappa\right); \tag{62}$$

$$G\left(\Delta, \tau, \kappa\right) = Q_\alpha\left(\Delta, 1\right) \tau^{-\frac{5}{2}} \sqrt{r_0(\Delta)} \left(\tau\kappa \frac{J\left(\Delta\right)}{|S(\Delta)|} + \frac{2(r_0(\Delta) - 6)}{(12 + r_0(\Delta))}\right) \exp\left(-\frac{\tau\kappa L(\Delta)}{2\pi |S(\Delta)|}\right) \tag{63}$$