Lessons from Quantum Strings for Quantum Gravity

Yuri Makeenko (ITEP)

Based on:

- Y. M. arXiv:2407.01136
- Y. M. Phys. Lett. B 845 (2023) 138170 [arXiv:2308.05030]
- Y. M. JHEP 09 (2023) 086 [arXiv:2307.06295]
- Y. M. JHEP 05 (2023) 085 [arXiv:2302.01954]
- Y. M. JHEP 01 (2023) 110 [arXiv:2212.02241]
- Y. M. IJMPA 38 (2023) 2350010 [arXiv:2204.10205]
- J. Ambjørn, Y. M. Phys. Lett. B 756 (2016) 142 [arXiv:1601.00540]
- J. Ambjørn, Y. M. Phys. Rev. D 93 (2016) 066007 [arXiv:1510.03390]

Talk at International Conference on QCD Vacuum Structure and Confinement, Naxos, August 26–30, 2024

Two no-go theorems for string existence

inherited from 1980's

- Non-perturbative lattice regularization (by dynamical triangulation) scales to a continuum string for $d \leq 1$ but does not for $d > 1$ (same for hypercubic latticization of Nambu-Goto string in $d > 2$)
  Durhuus, Fröhlich, Jonsson (1984), Ambjørn, Durhuus (1987)

- Knizhnik-Polyakov-Zamolodchikov (1988), David (1988), Distler-Kawai (1989)
  string susceptibility index of (closed) Polyakov's string is not real for $1 < d < 25$

$$\gamma_{\text{str}} = (1-h)\frac{d-25-\sqrt{(d-1)(d-25)}}{12} + 2$$

genus h

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The presented solutions rely on subtleties in Quantum Field Theory enjoying diffeomorphism invariance: Strings(!) and Gravity(?) $\Longrightarrow$

1) Continuum limit is not as in Quantum Field Theory: Lilliputian
2) Nambu-Goto and Polyakov strings differ by higher-derivative terms $\sim \Lambda^{-2}$ in emergent action which revive quantumly

Content of the talk

- Nambu-Goto versus Polyakov strings

- Mean-field ground state of regularized bosonic string
  - instability of the classical vacuum for $d > 2$
  - the Lilliputian scaling limit

- Generalized conformal anomaly
  - path-integrating over X^μ and ghosts
  - tracelessness of improved energy-momentum tensor
  - equivalence with four-derivative Liouville action
  - Salieri's check at one loop

- Exact solution and minimal models
  - singular products and universality of higher-derivative actions
  - BPZ null vectors and Kac's spectrum
  - Namb-Goto string in d=4 as (4,3) minimal model

2. Mean-field vs. classical ground state of bosonic string

Nambu-Goto and Polyakov strings

Nambu-Goto string (imaginary Lagrange multiplier λ^{ab}) independent metric tensor g_{ab}

$$K_0 \int \mathrm{d}^2\omega \sqrt{\det \partial_a X \cdot \partial_b X} = K_0 \int \mathrm{d}^2\omega \sqrt{g} + \frac{K_0}{2} \int \mathrm{d}^2\omega \, \lambda^{ab} \left(\partial_a X \cdot \partial_b X - g_{ab}\right)$$

Ground state $\lambda^{ab} = \bar{\lambda}\sqrt{g} g^{ab}$ classically $\bar{\lambda} = 1$ $\Longrightarrow$ Polyakov string

$$\mathcal{S} = \frac{K_0}{2} \int \mathrm{d}^2\omega \sqrt{g} g^{ab} \partial_a X \cdot \partial_b X$$

Closed bosonic string winding once around compactified dimension of circumference β, propagating (Euclidean) time L with topology of cylinder or torus (bagel). No tachyon if β is large enough

Gaussian path integral over X_{q}^{μ} by splitting $X^\mu = X_{\mathrm{cl}}^{\mu} + X_{\mathrm{q}}^{\mu}$: $\Longrightarrow$ Emergent (or effective)

$$\begin{aligned} S[\varphi, \lambda^{ab}] \; &= \; K_0 \int \mathrm{d}^2\omega \sqrt{g} + \frac{K_0}{2} \int \mathrm{d}^2\omega \, \lambda^{ab} \left(\partial_a X_{\mathrm{cl}} \cdot \partial_b X_{\mathrm{cl}} - g_{ab}\right) \\ &+ \frac{d}{2} \mathrm{tr} \log \left(-\frac{1}{\sqrt{g}} \partial_a \lambda^{ab} \partial_b \right) + \text{ ghosts} \end{aligned}$$

2D determinants regularized by ultraviolet cutoff Λ

Mean-field ground state

Ambjørn, Y.M. (2016)

Minimum of effective action is reached at (quantum ground state)

$$\bar{\lambda} = \frac{1}{2}\left(1+\frac{\Lambda^2}{K_0}+\sqrt{\left(1+\frac{\Lambda^2}{K_0}\right)^2-\frac{2d\Lambda^2}{K_0}}\right)$$

$$\hat{g}_{ab} = \bar{\rho}\,\hat{g}_{ab}, \qquad \bar{\rho} = \frac{\bar{\lambda}}{\sqrt{(1+\frac{\Lambda^2}{K_0})^2-\frac{2d\Lambda^2}{K_0}}}$$

$$S_{\rm mf} = K_0\bar{\lambda}L\sqrt{\beta^2-\frac{\pi(d-2)}{3K_0\bar{\lambda}}} \qquad \text{(Alvarez-Arvis)}$$

Variational mean field (like Peierls (1930s)) becomes exact at large d.
Like $O(N)$ sigma-model at large N where Lagrange multiplier does not fluctuate (summing the bubble graphs).
Square root is well-defined for $d \geq 2$ if $K_0 >$ critical value $=$ continuum

$$K_* = \left(d-1+\sqrt{d^2-2d}\right)\Lambda^2 \implies \bar{\lambda}_* = \frac{1}{2}\left(d-\sqrt{d^2-2d}\right) < 1$$

Instability of classical ground state

Energy of zero-point fluctuations (one-loop) Brink, Nielsen (1973)

$$E_{1l} = \left[K_0 - \frac{(d-2)}{2}\Lambda^2\right]\beta - \frac{\pi(d-2)}{6\beta}$$

bulk term Casimir energy

is usually made finite by introducing the renormalized string tension

$$K_R = K_0 - \frac{(d-2)}{2}\Lambda^2$$

It is assumed to works order by order about the classical ground state.

However this does not work for the mean-field energy

$$E_{\rm mf} = K_0\bar{\lambda}\sqrt{\beta^2 - \frac{\pi(d-2)}{3K_0\bar{\lambda}}}$$

which never vanishes with changing K_0 (except for $\beta = \beta_{\rm min}$). Thus the one-loop correction simply lowers for $d > 2$ the energy of the classical ground state which may indicate its instability.

Who's right me or textbooks?

Instability of classical ground state (cont.)

Adding a source term like in QFT

$$S_{\text{src}} = \frac{K_0}{2} \int \mathrm{d}^2\omega\, j^{ab} g_{ab}$$

defining the field

$$\rho_{ab}(j) = -\frac{2}{K_0} \frac{\delta}{\delta j^{ab}} \log Z$$

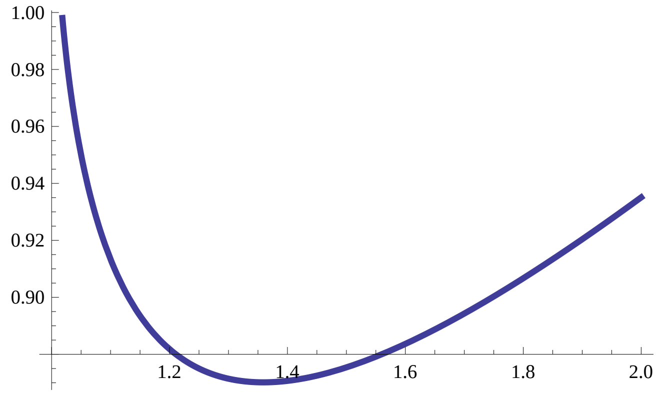
Minimizing for constant $j^{ab} = j\delta^{ab}$ we find   Ambjørn, Y.M. (2017)

"Effective potential" given by the Legendre transformation

$$\Gamma(\bar\rho) = -\frac{1}{K_0 L \beta} \log Z - j(\bar\rho)\bar\rho$$

In the mean-field approximation

$$\Gamma(\bar\rho) = \left(1 + \frac{\Lambda^2}{K_0}\right)\bar\rho - \sqrt{\frac{2d\Lambda^2}{K_0}\bar\rho(\bar\rho - 1)}$$



Classical vacuum $\bar\rho = 1$ is unstable and stable minimum occurs at

$$\bar\rho(0) = \bar\rho_{\text{m.f.}} \qquad \text{if } K_0 > K_* \text{ (same value as before)}$$

3. Two scaling regimes: Gulliver's vs. Lilliputian

Particle-like scaling limit (Gulliver's)

The ground state energy (Alvarez-Arvis)

$$E_0(\beta) = K_0 \bar{\lambda} \sqrt{\beta^2 - \frac{\pi(d-2)}{3K_0\bar{\lambda}}}$$

does not scale because $K_0 > K_* \sim \Lambda^2$ for $\bar{\lambda}$ to be real ($> \bar{\lambda}_*$). Choosing

$$\beta^2 = \beta^2_{\mathrm{min}} \approx \frac{\pi(d-2)}{3K_*\bar{\lambda}_*}, \qquad \bar{\lambda}_* = \frac{1}{2}\left(d - \sqrt{d^2 - 2d}\right)$$

only $E_0(\beta_{\mathrm{min}})$ can scale to finite – particle-like continuum limit similar to lattice regularizations, where

only the lowest mass scales to finite, excitations scale to infinity

Durhuus, Fröhlich, Jonsson (1984), Ambjørn, Durhuus (1987)

Particle-like scaling limit (Gulliver's)

The ground state energy (Alvarez-Arvis)

$$E_N(\beta) = K_0\bar{\lambda}\sqrt{\beta^2 + \frac{1}{K_0\bar{\lambda}}\left(-\frac{\pi(d-2)}{3} + 8N\right)}$$

does not scale because $K_0 > K_* \sim \Lambda^2$ for $\bar{\lambda}$ to be real ($> \bar{\lambda}_*$). Choosing

$$\beta^2 = \beta^2_{\text{min}} \approx \frac{\pi(d-2)}{3K_*\bar{\lambda}_*}, \qquad \bar{\lambda}_* = \frac{1}{2}\left(d - \sqrt{d^2 - 2d}\right)$$

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Lilliputian string-like scaling limit

Renormalized units of length $\Longrightarrow$ finite effective action

$$L_R = \sqrt{\frac{K_R}{K_*}}\, L, \quad \beta_R = \sqrt{\frac{K_R}{K_*}}\, \beta, \qquad S_{\text{mf}} = K_R L_R \sqrt{\beta_R^2 - \frac{\pi(d-2)}{3K_R}}$$

Renormalized string tension K_R scales to finite if

$$K_0 \to K_* + \frac{K_R^2}{K_*}, \qquad K_* = \left(d - 1 + \sqrt{d^2 - 2d}\right) \Lambda^2$$

reproducing the Alvarez-Arvis spectrum of continuum string

The average area is also finite

$$\langle Area \rangle = L_R \frac{\left(\beta_R^2 - \frac{\pi(d-2)}{6K_R}\right)}{\sqrt{\beta_R^2 - \frac{\pi(d-2)}{3K_R}}}$$

$\Rightarrow$ minimal area for large β_R and diverges if $\beta_R^2 \to \pi(d-2)/3K_R$

The Lilliputian world

Like for the zeta-function regularization except for nonlinearities, but

$$\text{length} \propto \frac{\sqrt{K_R}}{\Lambda} \text{length}_R$$

in target space is of order of the cutoff ($\Longrightarrow$ Lilliputian)

Nevertheless, the cutoff at the worldsheet is much smaller

$\boxed{\Delta\omega = 1/(\Lambda \sqrt[4]{g})}$ and fixes maximal number of modes

$$n_{\text{max}} \sim \Lambda \sqrt[4]{g} \text{ length} \propto \Lambda \text{ length}_{\text{R}}$$

is very large like in semiclassical expansion by Brink-Nielsen (1973)

- Continuum because infinitely smaller distances can be probed (classical music can be played on the Lilliputian strings)

- Gulliver's tools are too coarse to resolve the Lilliputian world (this is why lattice string regularizations of 1980's never reproduce canonical quantization)

The KING of BROBDINGNAG, and GULLIVER.

—Vide. *Swift's Gulliver*: Voyage to Brobdingnag.

Pub^d June 26^th 1803 - by H. Humphrey 27 St. James's Street

4. Fluctuations about mean field

Generalized conformal anomaly

Path-integrating over X^μ (and the usual ghosts) in units $K_0 = 1$

$$S[g_{ab},\lambda^{ab}] = \int \sqrt{g} - \frac{1}{2}\int \lambda^{ab} g_{ab} + S_X[g_{ab},\lambda^{ab}],$$

$$\mathcal{S}_X = \frac{d}{96\pi}\int \left[-\frac{12\sqrt{g}}{\tau\sqrt{\det \lambda^{ab}}} + \sqrt{g}\, R\frac{1}{\Delta}R - (\beta\lambda^{ab}g_{ab}R + 2\lambda^{ab}\nabla_a\partial_b\frac{1}{\Delta}R)\right]$$

higher orders in Schwinger's proper-time ultraviolet cutoff τ dropped.
$\beta = 1$ for the Nambu-Goto string but kept arbitrary for generality

The action is derived from the DeWitt-Seeley expansion of

$$\mathcal{O} = (\sqrt{g})^{-1}\partial_a\lambda^{ab}\partial_b = h^{ab}\partial_a\partial_b + A^a\partial_a$$

$$\left\langle |\, \mathrm{e}^{\tau O} | \right\rangle = \frac{1}{4\pi\tau} + \frac{1}{4\pi}\left(\frac{1}{6}R + E\right) + \mathcal{O}(\tau)$$

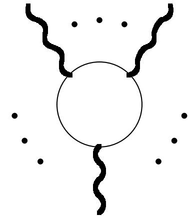
$$E = -\frac{1}{2}\left(\partial_a A^a - \partial_a\partial_b h^{ab} + \frac{1}{2}g_{ab}\Delta h^{ab}\right) \qquad \boxed{\text{inertial frame}}$$

O becomes the Laplacian for $\lambda^{ab} = \bar{\lambda}\sqrt{g}g^{ab}$ with constant $\bar{\lambda}$.
Alternatively, it was derived as Coleman-Weinberg's effective action for covariant Pauli-Villars' regularization

Coleman-Weinberg potential

Integrating out X_q^μ we get (a part of) the effective action

$$\frac{d}{2}\,\mathrm{tr}\ln\left[-\frac{1}{\rho}\partial_a\lambda^{ab}\partial_b\right]_{\mathrm{reg}} = \sum_n \frac{1}{n}$$



wavy lines correspond to fluctuations $\delta\lambda^{ab}$ or $\delta\rho$ about ground state.

Covariant Pauli-Villars regulator Y (preserves conformal invariance)

$$\mathcal{S}[Y] = \frac{K_0}{2}\int\left(\lambda^{ab}\,\partial_a Y\cdot\partial_b Y + \boxed{M^2\sqrt{g}}\,Y^2\right)$$

Actually two anticommuting Grassmann Y and $\bar{Y}$ of mass squared M^2 and one Z of mass squared $2M^2$ with normal statistics:

$$\mathrm{tr}\log O\big|_{\mathrm{reg}} = -\int_0^\infty \frac{\mathrm{d}\tau}{\tau}\,\mathrm{tr}\,\mathrm{e}^{\tau O}\left(1-\mathrm{e}^{-\tau M^2}\right)^2, \qquad \left\langle\left|\,\mathrm{e}^{\tau O}\right|\right\rangle = \frac{1}{4\pi\tau}+\ldots$$

Advantages over the proper-time regularization:

Feynman's diagrams apply for Pauli-Villars regularization

Gel'fand-Yaglom technique to compare with DeWitt-Seeley expansion

Conformal gauge and flat background

Emergent action becomes local in conformal gauge

$$g_{ab} = \hat{g}_{ab}\,\mathrm{e}^{\varphi}$$

where $\hat{g}_{ab}$ is background (or fiducial) metric tensor.
Usual ghosts and their usual contribution to effective action

Euclidean CFT: conformal coordinates z and $\bar{z}$ in flat background
$g_{zz} = g_{\bar{z}\bar{z}} = 0$, $g_{z\bar{z}} = g_{\bar{z}z} = 1/2$ (units $K_0 = 1$)

$$\mathcal{S}[\varphi, \lambda^{ab}] \;=\; \int \mathrm{e}^{\varphi}(1 - \lambda^{z\bar{z}}) + \frac{1}{24\pi}\int [-\frac{3d\,\mathrm{e}^{\varphi}}{\tau\sqrt{\det \lambda^{ab}}} + (d-26)\varphi\partial\bar{\partial}\varphi$$
$$+d\kappa(2(1+\beta)\lambda^{z\bar{z}}\partial\bar{\partial}\varphi + \lambda^{zz}\nabla\partial\varphi + \lambda^{\bar{z}\bar{z}}\bar{\nabla}\bar{\partial}\varphi)]$$

$\nabla = \partial - \partial\varphi$ is covariant derivative in conformal gauge so it describes a theory with interaction (no such interaction if only $\lambda^{z\bar{z}} = \lambda^{ab}\hat{g}_{ab}$)

Subtleties because of nonminimal interaction with background gravity

$$\sqrt{g}R = \sqrt{\hat{g}}\left(\hat{R} - \hat{\Delta}\varphi\right)$$

It vanishes only if the background curvature $\hat{R}$ vanishes

Improved energy-momentum tensor

Callan-Coleman-Jackiw (1970)

Symmetric minimal energy-momentum tensor (by applying $\delta/\delta\hat{g}^{ab}$)

$$\begin{aligned} T_{zz}^{(\text{min})} &= \frac{(d-26)}{24}(\partial\varphi)^2 + \frac{d\kappa}{24}[2(1+\beta)\partial\lambda^{z\bar{z}}\partial\varphi \\ &\quad +\bar{\partial}\lambda^{\bar{z}\bar{z}}\partial\varphi - \partial\lambda^{\bar{z}\bar{z}}\bar{\partial}\varphi - 2\lambda^{\bar{z}\bar{z}}\partial\bar{\partial}\varphi + 2\lambda^{\bar{z}\bar{z}}\partial\varphi\bar{\partial}\varphi] \\ T_{z\bar{z}}^{(\text{min})} &= \mathrm{e}^{\varphi}(1-\lambda^{z\bar{z}}) - \frac{d\,\mathrm{e}^{\varphi}}{2\tau\sqrt{\det\lambda^{**}}} + \frac{d\kappa}{24}[\bar{\partial}\lambda^{\bar{z}\bar{z}}\bar{\partial}\varphi \\ &\quad +\lambda^{\bar{z}\bar{z}}\bar{\partial}^2\varphi + \partial\lambda^{zz}\partial\varphi + \lambda^{zz}\partial^2\varphi] \end{aligned}$$

is conserved obeying $\bar{\partial}T_{zz}^{(\text{min})} + \partial T_{\bar{z}z}^{(\text{min})} = 0$ but not traceless

IEMT is given by the sum $T_{ab} = T_{ab}^{(\text{min})} + T_{ab}^{(\text{add})}$

$$\begin{aligned} T_{zz}^{(\text{add})} &= -\frac{(d-26)}{12}\partial^2\varphi - \frac{d\kappa}{24}\left[2(1+\beta)\partial^2\lambda^{z\bar{z}} + \partial\bar{\partial}\lambda^{\bar{z}\bar{z}} + \partial(\lambda^{\bar{z}\bar{z}}\bar{\partial}\varphi)\right] \\ &\quad -\frac{d\kappa}{24}\left[\frac{1}{\bar{\partial}}\left(\partial^3\lambda^{zz} + \partial^2(\lambda^{zz}\partial\varphi)\right)\right] \end{aligned}$$

nonlocal term!

as a price for $\bar{\partial}T_{zz} = 0$ and $T_{z\bar{z}} = 0$.

Non-local term gives classically an addition to Virasoro algebra

$$\delta_\xi T_{zz} = \xi'''\frac{1}{2b^2} + 2\xi' T_{zz} + \xi\partial T_{zz} - \xi''\frac{1}{\bar{\partial}}\partial\nabla\lambda^{zz}$$

Improved energy-momentum tensor (cont.)

Conservation and tracelessness of classical IEMT follows from

$$\frac{1}{\pi}\bar\partial T_{zz} = \partial\varphi\frac{\delta\mathcal{S}}{\delta\varphi} - \partial\frac{\delta\mathcal{S}}{\delta\varphi} - \lambda^{\bar z\bar z}\partial\frac{\delta\mathcal{S}}{\delta\lambda^{\bar z\bar z}} + \partial\lambda^{z\bar z}\frac{\delta\mathcal{S}}{\delta\lambda^{z\bar z}} + \partial(\lambda^{zz}\frac{\delta\mathcal{S}}{\delta\lambda^{zz}}) + \partial\lambda^{zz}\frac{\delta\mathcal{S}}{\delta\lambda^{zz}}$$

General property of improved energy-momentum tensor:

$$T^a_a \equiv \hat g^{ab}\frac{\delta\mathcal{S}}{\delta\hat g^{ab}} = -\frac{\delta\mathcal{S}}{\delta\varphi}$$

i.e. trace of IEMT = the classical equation of motion for φ.
In quantum theory variations of $\mathcal{S}$ replaced by variational derivatives.
For generator of conformal transformation $\delta z = \xi(z)$ this yields*

$$\hat\delta_\xi = \frac{1}{\pi}\int \xi\bar\partial T_{zz} = \int\Big[(\xi' + \xi\partial\varphi)\frac{\delta}{\delta\varphi} + (\xi'\lambda^{\bar z\bar z} + \xi\partial\lambda^{\bar z\bar z})\frac{\delta}{\delta\lambda^{\bar z\bar z}} + \xi\partial\lambda^{z\bar z}\frac{\delta}{\delta\lambda^{z\bar z}} + (-\xi'\lambda^{zz} + \xi\partial\lambda^{zz})\frac{\delta}{\delta\lambda^{zz}}\Big]$$

Classically it produces the right transformation laws of φ and λ^{ab} with components $\lambda^{\bar z\bar z}$, $\lambda^{z\bar z}$, λ^{zz} of conformal weights 1, 0, −1, respectively

*Note $\delta_\xi\lambda^{ab} = -(\partial_c\xi^a)\lambda^{bc} - (\partial_c\xi^b)\lambda^{ac} + (\partial_c\xi^c)\lambda^{ab} + \xi^c\partial_c\lambda^{ab}$ under diffeomorphisms

Improved energy-momentum tensor (cont. 2)

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Equivalence with four-derivative Liouville action

Path integral over $\delta\lambda^{ab}$ has a saddle point justified by small τ at

$$\delta\lambda^{ab} = \sqrt{g}\tau\left(g^{ac}g^{bd}\nabla_c\partial_d\varphi + \frac{(\beta-1)}{4}g^{ab}\Delta\varphi\right)\frac{\kappa}{3} + \mathcal{O}(\tau^2)$$

Thus we arrive at four-derivative Liouville action

$$\mathcal{S}[\varphi] = \frac{1}{16\pi b_0^2}\int\sqrt{\hat{g}}[\hat{g}^{ab}\partial_a\varphi\partial_b\varphi + \varepsilon\,\mathrm{e}^{-\varphi}\hat{\Delta}\varphi\left(\hat{\Delta}\varphi - G\hat{g}^{ab}\,\partial_a\varphi\partial_b\varphi\right)]$$

with $G = -1/3$ for the Nambu-Goto string

$$b_0^2 = \frac{6}{26-d}, \qquad G = -\frac{1}{1+(1+\beta)^2/2}, \qquad \varepsilon = -\frac{2d\kappa^2\bar{\lambda}^3}{3G(26-d)}\tau$$

which was exactly solved previously Y.M. (2023)

Classically higher-derivative terms vanish for smooth $\varepsilon R \ll 1$.
Quantumly quartic derivative provides UV cutoff but also interaction with coupling $\varepsilon \Rightarrow$ uncertainties $\varepsilon \times \varepsilon^{-1}$ which revive $\Longrightarrow$ anomalies.
Yet higher terms which are primary scalars like R^n do not change – universality. $g^{ab}\,\partial_a\varphi\partial_b\varphi$ is not primary

Smallness of ε is compensated by change of the metric (shift of φ)

5. CFT á la KPZ-DDK

Review of KPZ-DDK

Knizhnik-Polyakov-Zamolodchikov (1988), David (1988), Distler-Kawai (1989)

Liouville action in fiducial (or background) metric $\hat{g}_{ab}$

$$S_L = \frac{1}{8\pi b^2} \int \sqrt{\hat{g}} \left(\frac{1}{2} \hat{g}^{ab} \partial_a \varphi \partial_b \varphi + q \hat{R} \varphi \right) + \mu^2 \int \sqrt{\hat{g}}\, \mathrm{e}^{\varphi}$$

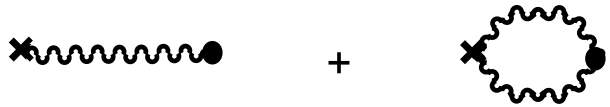
with "renormalized" parameters of effective action

$$b^2 = b_0^2 + \mathcal{O}(b_0^4), \quad q = 1 + \mathcal{O}(b_0^2), \quad b_0^2 = \frac{6}{26 - d}$$

Energy-momentum pseudotensor

$$T_{zz}^{(\varphi)} = -\frac{1}{4b^2} \left(\partial_z \varphi \partial_z \varphi - 2q \partial_z^2 \varphi \right) \qquad \sqrt{g} R = \sqrt{\hat{g}} \, (q \hat{R} - \hat{\Delta} \varphi)$$

Background independence:



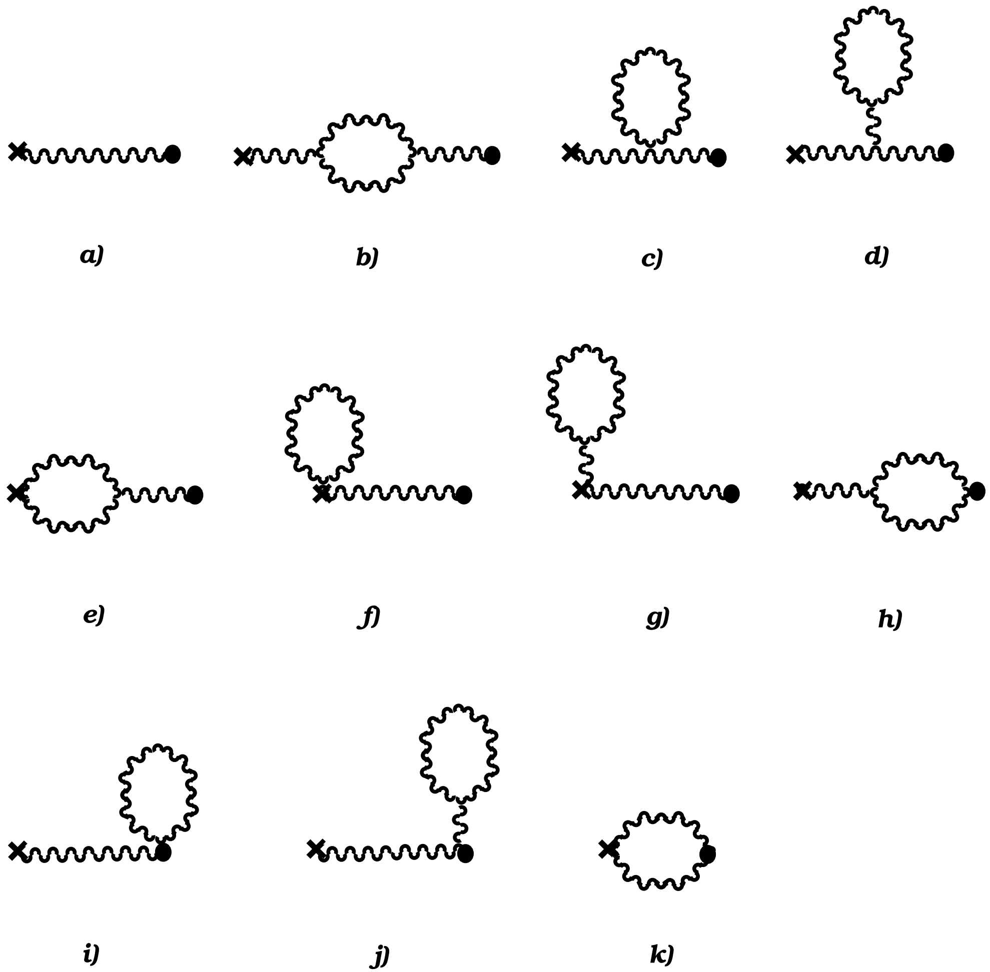
total central charge $\qquad c = d - 26 + 6\frac{q^2}{b^2} + 1 = 0$

conformal weight $\qquad \Delta(\mathrm{e}^{\varphi}) = q - b^2 = 1$

$$\Longrightarrow \qquad b = \sqrt{\frac{25 - d}{24}} - \sqrt{\frac{1 - d}{24}}, \qquad q = 1 + b^2$$

KPZ-DDK for the four-derivative Liouville action

One-loop operator products $T_{zz}(z)\,e^{\varphi(0)}$ and $T_{zz}(z)T_{zz}(0)$



Conformal weight of $e^{\varphi(0)}$: $\quad 1 = q - b^2.$

In central charge of φ nonlocal term revives: $\quad c^{(\varphi)} = \frac{6q^2}{b^2} + 1 + 6G$

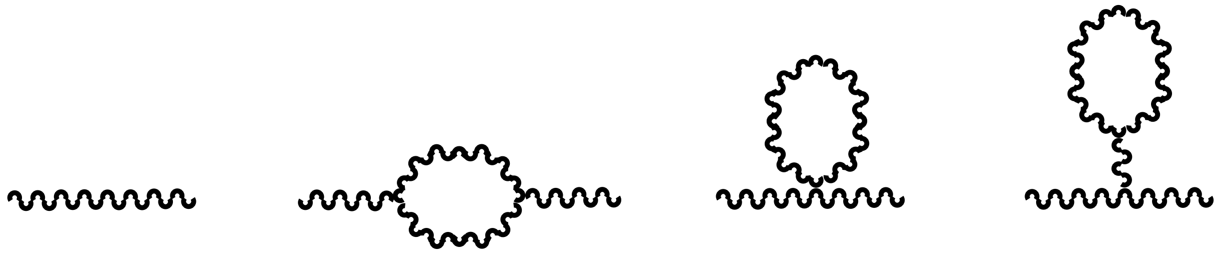
6. Algebraic check of DDK

Salieri:

"I checked the harmony with algebra.
Then finally proficient in the science,
I risked the rare delights of creativity."

A. Pushkin, *Mozart and Salieri*

One-loop propagator



a) *b)* *c)* *d)*

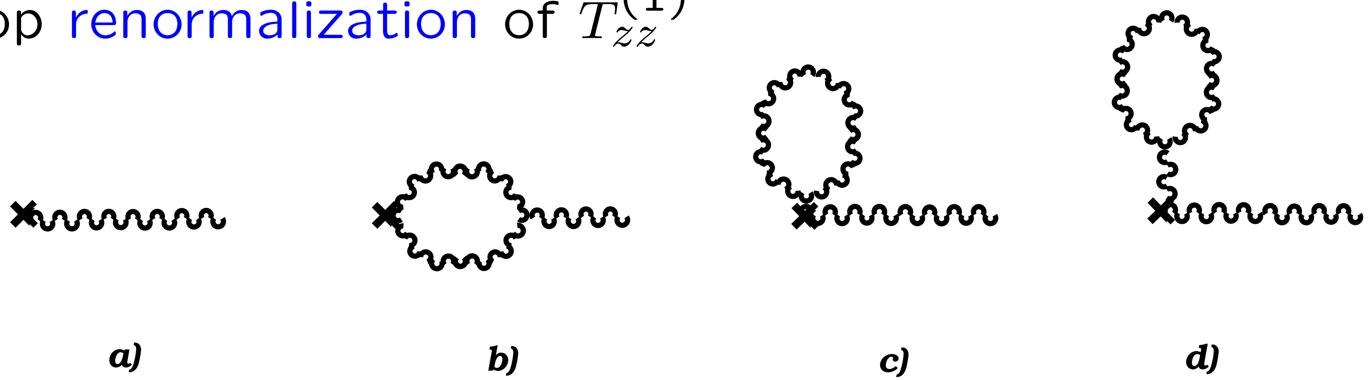
$$
\begin{aligned}
b) \;=\; & -\frac{1}{4}\int \frac{\mathrm{d}^2 k}{(2\pi)^2}\Bigg\{ \frac{\varepsilon^2 k^2 (k-p)^2}{(1+\varepsilon k^2)[1+\varepsilon(k-p)^2]} \\
& -2\frac{(\varepsilon k^2 (k-p)^2 - M^2)^2}{(k^2+M^2+\varepsilon k^4)[(k-p)^2+M^2+\varepsilon(k-p)^4]} \\
& +\frac{(\varepsilon k^2 (k-p)^2 - 2M^2)^2}{(k^2+2M^2+\varepsilon k^4)[(k-p)^2+2M^2+\varepsilon(k-p)^4]}\Bigg\}|\varphi(p)|^2 \\
\to\; & b)|_{\text{reg div}} - \frac{p^2}{96\pi}|\varphi(p)|^2
\end{aligned}
$$

One-loop renormalization of b^2 where $A(\varepsilon M^2) \sim \varepsilon M^2 =$ tadpole $d)$

$$
\frac{1}{b^2} = \frac{1}{b_0^2} - \left(\frac{1}{6} - 4 + A + 2G\int \mathrm{d}k^2 \frac{\varepsilon}{(1+\varepsilon k^2)} - \frac{1}{2}GA\right) + \mathcal{O}(b_0^2)
$$

One-loop renormalization of T_{zz}

One-loop renormalization of $T_{zz}^{(1)}$



$$\frac{q}{b^2} = \frac{1}{b_0^2} - \frac{1}{6} + 2 - \frac{1}{2}A - \frac{1}{2}G - G\int \mathrm{d}k^2 \frac{\varepsilon}{(1+\varepsilon k^2)} + \frac{1}{4}GA$$

or multiplying by b^2

$$\frac{q^2}{b^2} = \left(\frac{q}{b^2}\right)^2 \times b^2 = \frac{1}{b_0^2} - \frac{1}{6} - G + \mathcal{O}(b_0^2)$$

This precisely confirms the above shift of the central charge by $6G$ obtained by conformal field theory technique of DDK.

Tremendous cancellation due to diffeomorphism invariance proving (intelligent) one loop to be exact: (like Duistermaat-Heckman?)

$$-\frac{6}{b_0^2} + \frac{6q^2}{b^2} + 1 + 6Gq = 0, \qquad 1 = q - b^2$$

7. Method of singular products as pragmatic mixture of CFT and QFT

Conformal transformation revisited

Generator of conformal transformation for nonquadratic e-m tensor

$$\hat{\delta}_\xi \equiv \int_{D_1} \left(\xi' \frac{\delta}{\delta\varphi} + \xi \partial\varphi \frac{\delta}{\delta\varphi} \right) \overset{\text{w.s.}}{=} \int_{C_1} \frac{\mathrm{d}z}{2\pi \mathrm{i}} \xi(z) T_{zz}(z)$$

where D_1 includes singularities of $\xi(z)$ and C_1 bounds D_1.
Equivalence of two forms is proved by integrating the total derivative

$$\bar{\partial} T_{zz} = -\pi \partial \frac{\delta S}{\delta\varphi} + \pi \partial\varphi \frac{\delta S}{\delta\varphi}$$

and using the (quantum) equation of motion

$$\frac{\delta S}{\delta\varphi} \overset{\text{w.s.}}{=} \frac{\delta}{\delta\varphi}$$

Actually, the form of $\hat{\delta}_\xi$ in the middle is primary.
It takes into account a tremendous cancellation of the diagrams, while there are subtleties associated with singular products

Conformal transformation revisited (cont.)

After averaging over Pauli-Villars' regulators

$$\left\langle \hat{\delta}_\xi X(\omega_i) \right\rangle = \left\langle \int_{D_1} \mathrm{d}^2 z \left(q\xi'(z) \frac{\delta}{\delta\varphi(z)} + \xi(z)\partial\varphi(z) \frac{\delta}{\delta\varphi(z)} \right) X(\omega_i) \right\rangle$$

It is easy to reproduce $\delta_\xi \, \mathrm{e}^{\alpha\varphi(\omega)}$ via the singular products

$$\begin{aligned} \hat{\delta}_\xi \, \mathrm{e}^{\alpha\varphi(\omega)} &= q\alpha\xi'(\omega)\, \mathrm{e}^{\alpha\varphi(\omega)} + \int_{D_1} \mathrm{d}^2 z \, \alpha\xi(z)\partial\varphi(z)\, \mathrm{e}^{\alpha\varphi(\omega)} \delta^{(2)}(z-\omega) \\ &\overset{\text{w.s.}}{=} q\alpha\xi'(\omega)\, \mathrm{e}^{\alpha\varphi(\omega)} + \int_{D_1} \mathrm{d}^2 z \, \alpha\xi(z) \langle \partial\varphi(z)\, \mathrm{e}^{\alpha\varphi(\omega)} \rangle \delta^{(2)}(z-\omega) \\ &\quad + \alpha\xi(\omega)\partial\varphi(\omega)\, \mathrm{e}^{\alpha\varphi(\omega)} \\ &= (q\alpha - b^2\alpha^2)\xi'(\omega)\, \mathrm{e}^{\alpha\varphi(\omega)} + \xi(\omega)\partial\, \mathrm{e}^{\alpha\varphi(\omega)} \end{aligned}$$

$-b^2\alpha^2$ comes from the singular product

$$\int_{D_1} \mathrm{d}^2 z \, \xi(z) \langle \partial\varphi(z)\varphi(\omega) \rangle \delta^{(2)}(z-\omega) = -b^2\xi'(\omega)$$

The formula for $\delta_\xi \, \mathrm{e}^{\alpha\varphi(\omega)}$ is EXACT for normal-ordered $\mathrm{e}^{\alpha\varphi(\omega)}$ $\Longrightarrow$ the first DDK equation does not change

$$1 = q\alpha - b^2\alpha^2$$

List of singular products

The simplest singular product

$$\frac{1}{b^2}\int \mathrm{d}^2 z\, \xi(z)\, \langle \partial^n \varphi(z)\varphi(0)\rangle\, \delta^{(2)}(z) = (-1)^n \frac{2}{n(n+1)} \partial^n \xi(0)$$

arises already in a free CFT by the formulas

$$\delta^{(2)}(z) = \bar\partial \frac{1}{\pi z}, \qquad \frac{1}{z^n}\bar\partial \frac{1}{z} = (-1)^n \frac{1}{(n+1)!} \partial^n \bar\partial \frac{1}{z}$$

It can be alternatively derived introducing the regularization by ε

$$G_\varepsilon(k) = \frac{1}{k^2(1+\varepsilon k^2)}, \qquad \delta^{(2)}_\varepsilon(k) = \frac{1}{(1+\varepsilon k^2)}$$

We then have

$$8\pi \int \mathrm{d}^2 z\, \xi(z) \partial^n G_\varepsilon(z) \delta^{(2)}_\varepsilon(z) = (-1)^n \frac{2}{n(n+1)} \partial^n \xi(0)$$

$$8\pi \int \mathrm{d}^2 z\, \xi(z)[-4\varepsilon \partial^{n+1} \bar\partial G_\varepsilon(z)] \delta^{(2)}_\varepsilon(z) = (-1)^n \frac{2}{(n+1)} \partial^n \xi(0)$$

Computation of the central charge

Y.M. (2023)

Central charge $c^{(\varphi)}$ of φ can be computed for normal-ordered T_{zz} as

$$\left\langle \hat{\delta}_\xi T_{zz}(\omega) \right\rangle = \frac{c^{(\varphi)}}{12} \xi'''(\omega)$$

For quadratic part of T_{zz}

$$\left\langle \hat{\delta}_\xi T^{(2)}_{zz}(\omega) \right\rangle = \frac{1}{2b^2} \int \mathrm{d}^2 z \langle q^2 \xi'''(z) + \xi'(z) \partial^2 \varphi(z) \varphi(\omega) + \xi(z) \partial^3 \varphi(z) \varphi(\omega) \rangle$$

$$\times \delta^{(2)}(z-\omega) = \frac{\xi'''(\omega)}{2} \left(\frac{q^2}{b^2} + \frac{1}{3} - \frac{1}{6} \right) = \xi'''(\omega) \left(\frac{q^2}{2b^2} + \frac{1}{12} \right)$$

Here $1/12$ gives the usual quantum addition 1 to the central charge.
DDK formula for the central charge is reproduced for quadratic action.
Propagator is exact $\Longrightarrow$ this is why b^2 cancels

Computation of the central charge (cont.)

Computation for quartic part is lengthy but doable with Mathematica

$$\left\langle \hat{\delta}_\xi T^{(4)}_{zz}(\omega)\right\rangle_{G=0} = \frac{1}{b^2}\int d^2 z \left\langle [2q\alpha\varepsilon\xi'''(z)\partial\bar{\partial}\varphi(z) + (4q\alpha-2)\varepsilon\xi''(z)\partial^2\bar{\partial}\varphi(z) \right.$$

$$\left. -6\varepsilon\xi'(z)\partial^3\bar{\partial}\varphi(z) - 4\varepsilon\xi(z)\partial^4\bar{\partial}\varphi(z)]\varphi(\omega)\right\rangle \delta^{(2)}_\varepsilon(z-\omega)$$

$$= \frac{\xi'''(\omega)}{4}\left(-2\cdot 2q\alpha + (4q\alpha-2)\cdot 1 + 6\frac{2}{3} - 4\frac{1}{2}\right) = 0$$

Central charge of φ equals 1 at $G=0$ as for quadratic action. Computations is similar to one loop but higher loops are taken into account by b^2, q and α $\Longrightarrow$ why I call it "intelligent" one (ione) loop

Contribution from the G-term comes solely from the nonlocal part

$$\left\langle \hat{\delta}_\xi T^{(4)}_{zz}(\omega)\right\rangle_{G} = -\frac{2}{b^2} G q\varepsilon \int d^2 z \left\langle [\xi'''(z)\partial\bar{\partial}\varphi(z) + \xi''(z)\partial^2\bar{\partial}\varphi(z)]\varphi(\omega)\right\rangle$$

$$\times \delta^{(2)}_\varepsilon(z-\omega) = \frac{1}{2} G q \xi'''(\omega)$$

The vanishing of total central charge results in the modified second DDK equation

$$\frac{6q^2}{b^2} + 1 + 6Gq = \frac{6}{b_0^2}$$

8. Relation to minimal models

Exact solution for four-derivative action

Solution to two modified DDK equations

$$
\begin{aligned}
b^2 &= \frac{13 - d - 6G - \sqrt{(d-d_+)(d-d_-)}}{12(1+G)} \\
q &= 1 + b^2 \\
d_\pm &= 13 - 6G \pm 12\sqrt{1+G}
\end{aligned}
$$

where $d = 26 - 6/b_0^2$ to comply with the Liouville action.
The KPZ barriers of the Liouville theory are shifted to $d_\pm$

The string susceptibility equals

$$
\gamma_{\text{str}} = (h-1)\frac{q}{b^2} + 2 = (h-1)\frac{25 - d - 6G + \sqrt{(d-d_+)(d-d_-)}}{12} + 2
$$

It is real for $d < d_-$ with $d_- > 1$ increasing from 1 at $G = 0$ to 19 at $G = -1$ for $-1 \leq G \leq 0$ required for stability as it follows from the identity (modulo boundary terms)

$$
\int \mathrm{e}^{-\varphi}\left[(\partial\bar\partial\varphi)^2 - G\partial\varphi\bar\partial\varphi\partial\bar\partial\varphi\right] = \int \mathrm{e}^{-\varphi}\left[(1+G)(\partial\bar\partial\varphi)^2 - G\nabla\partial\varphi\bar\nabla\bar\partial\varphi\right]
$$

BPZ null-vectors and Kac's spectrum

Representations of Virasoro algebra are unitary for central charge

$$c = 1 - 6\frac{(p-q)^2}{pq}$$

with $q = p + 1$, $p \geq 2$. Like in usual Liouville theory the operators

$$V_\alpha = \mathrm{e}^{\alpha\varphi}, \qquad \alpha = \frac{1-n}{2} + \frac{1-m}{2b^2}$$

are the BPZ null-vectors for integer n and m obeying

$$\left(L_{-1}^2 + b^2 L_{-2}\right)\mathrm{e}^{-\varphi/2} = 0, \quad \left(L_{-1}^2 + b^{-2} L_{-2}\right)\mathrm{e}^{-b^{-2}\varphi/2} = 0, \quad \ldots$$

Their conformal weights

$$\Delta_\alpha = \alpha + (\alpha - \alpha^2)b^2$$

coincide with Kac's spectrum of the minimal models

$$\Delta_{m,n}(c) = \frac{c-1}{24} + \frac{1}{4}\left((m+n)\sqrt{\frac{1-c}{24}} + (m-n)\sqrt{\frac{25-c}{24}}\right)^2$$

$$c = 26 - d + G\frac{\left[25 - d - 6G + \sqrt{(d-d_+)(d-d_-)}\right]}{2(1+G)} = 1 + 6(b+b^{-1})^2$$

Minimal models from four-derivative action

To describe minimal models we choose like in usual Liouville theory

$$c = 25 + 6\frac{(p-q)^2}{pq} \quad \Longrightarrow \quad G = \frac{(1-d-6\frac{(p-q)^2}{pq})q}{6(q+p)}$$

with coprime $q > p$

If $G = 0$ this would imply

$$d = 1 - 6\frac{(p-q)^2}{pq}$$

for central charge of matter but now d is a free parameter obeying

$$1 - 6\frac{(p-q)^2}{pq} \leq d \leq 19 - 6\frac{p}{q} \quad \Longleftarrow \quad 0 \geq G \geq -1$$

Contrary to the Liouville theory now Kac's $c \neq c^{(\varphi)} = 26 - d$

The KPZ barriers shifted to $d_\pm$ which depend on $G \in [-1, 0]$.
For $G = -1/3$ (the Nambu-Goto string) $\Longrightarrow$ $d_- = 15 - 4\sqrt{6} \approx 5.2 > 4$

Minimal models from four-derivative action (cont.)

From the above formula for b^2

$$b^{-2} = \begin{cases} \frac{q}{p} & \text{perturbative branch} \\ -1 + \frac{(25-d)p}{6(q+p)} & \text{the other branch} \end{cases} \quad \text{for } d > 25 - 6\frac{(p+q)^2}{p^2}$$

Perturbative branch is as in the usual Liouville theory, but the second branch is no longer $p \leftrightarrow q$ with it. It is $b^{-2} = p/q$ for $d = 1 - 6\frac{(p-q)^2}{pq}$

There are no obstacles against $d = 4$ for $q = p + 1$ (unitary case)!

$$d_+ = d_- = 19 \quad \text{for} \quad d = d_{\rm C} = 13 - \frac{6}{p}$$

For $1 \leq d < d_{\rm C}$ ($d_{\rm C}$ is always >10) we have $d \leq d_-$ and $\gamma_{\rm str}$ is REAL. Remarkably, $G = -1/3$ is associated in $d = 4$ with $p = 3$, $q = p+1 = 4$ unitary minimal model like critical Ising model on a random lattice

The perturbative branch is as in the usual Liouville theory but the domain of applicability is now broader which may have applications of the four-derivative Liouville action in Statistical Mechanics á la Kogan-Mudry-Tsvelik (1996)

8. Why ione loop?

Operatorial central charge otherwise

Y.M. (2022)

Generator of conformal transformation

$$\widehat{\delta}_\xi \equiv \int_{C_1} \frac{\mathrm{d}z}{2\pi \mathrm{i}} \xi(z) T_{zz}(z) = \frac{1}{\pi} \int_{D_1} \xi \bar{\partial} T_{zz} \overset{\text{w.s.}}{=} \int_{D_1} \left(q \xi' \frac{\delta}{\delta \varphi} + \xi \partial \varphi \frac{\delta}{\delta \varphi} \right)$$

with the commutator (where $\zeta = \xi\eta' - \xi'\eta$ as it should)

$$\left\langle (\widehat{\delta}_\eta \widehat{\delta}_\xi - \widehat{\delta}_\xi \widehat{\delta}_\eta) X \right\rangle = \left\langle \widehat{\delta}_\zeta X \right\rangle + \int_{D_1} \mathrm{d}^2 z \int_{D_z} \mathrm{d}^2 \omega$$

$$\times \left\langle [q\xi'(z) + \xi(z)\partial\varphi(z)][q\eta'(\omega) + \eta(\omega)\partial\varphi(\omega)] \frac{\delta^2 S}{\delta\varphi(z)\delta\varphi(\omega)} X \right\rangle$$

$$= \left\langle \widehat{\delta}_\zeta X \right\rangle + \frac{1}{24} \oint_{C_1} \frac{\mathrm{d}z}{2\pi \mathrm{i}} [\xi'''(z)\eta(z) - \xi(z)\eta'''(z)] \langle c X \rangle$$

DDK is reproduced for quadratic action S

Still usual central charge c for higher-derivative action with $G = 0$ but field-dependent for $G \neq 0$. Usual Virasoro algebra at one loop with

$$c^{(\varphi)} = \frac{6q^2}{b^2} + 1 + 6G + \mathcal{O}(b_0^2)$$

Where is $SL(2,R)$ Kac-Moody algebra at higher loops?

Conclusion

- Classical (perturbative) ground state is stable only for $d < 2$.
  For $2 < d < 26$ the mean-field ground state is stable instead

- Lilliputian strings for $d > 2$ versus Gulliver's strings for $d \leq 2$

- Higher-derivative terms in the beyond Liouville action for φ revive, telling the Nambu-Goto and Polyakov strings apart

- 2D conformal invariance is maintained by fluctuations in spite of ε but the central charge of φ gets additional $6Gq$

- The Nambu-Goto string is described by (4,3) minimal model like the critical Ising model on a random lattice

- All that is specific to the theory with diffeomorphism invariance

Conclusion

- Classical (perturbative) ground state is stable only for $d < 2$. For $2 < d < 26$ the mean-field ground state is stable instead

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- All that is specific to the theory with diffeomorphism invariance

Final remark:
Large-d strings = bubble diagrams like $O(N)$ sigma model but
Large-d gravity = planar diagrams like Yang-Mills Strominger (1981)