On calculating the mass-gap in 4d Yang-Mills Theory

Paul Romatschke, CU Boulder

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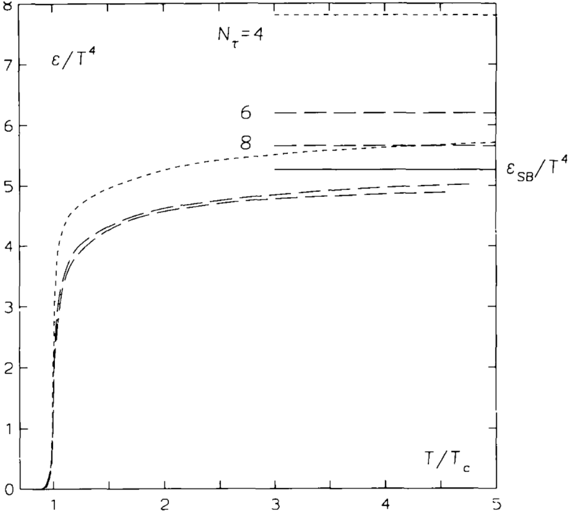
- Theory of the strong nuclear force

Numerical Studies



Boyd et al, Nucl. Phys. B (1996)

Numerical Studies

THE GLUEBALL MASS SPECTRUM IN QCD:
FIRST RESULTS OF A LATTICE MONTE CARLO CALCULATION

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Received 18 January 1982

J^{PC}	mass
0^{++}	1.7 GeV
2^{++}	2.4 GeV
0^{-+}	2.6 GeV
...	

Lightest glueballs are the scalar 0^{++} and tensor 2^{++} glueball

Theoretical Studies

- Known Lagrangian:

$$\mathcal{L} = \frac{1}{4g^2} F^a_{\mu\nu} F^a_{\mu\nu}\,, \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu$$

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We only know how to do Gaussian integrals!

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YM Mass Gap: Open Millennium Prize Problem

A potentially related system

Non-relativistic fermions

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$$\mathcal{L} = \sum_{s=\uparrow,\downarrow} \vec{\psi}_s^\dagger \left(\partial_\tau - \frac{\vec{\nabla}^2}{2m} - \mu \right) \vec{\psi}_s + \frac{4\pi a_s}{mN} \left(\vec{\psi}_\downarrow \cdot \vec{\psi}_\uparrow \right)^\dagger \left(\vec{\psi}_\downarrow \cdot \vec{\psi}_\uparrow \right)$$

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- $\mathcal{L}$ has quartic term just like Yang-Mills theory

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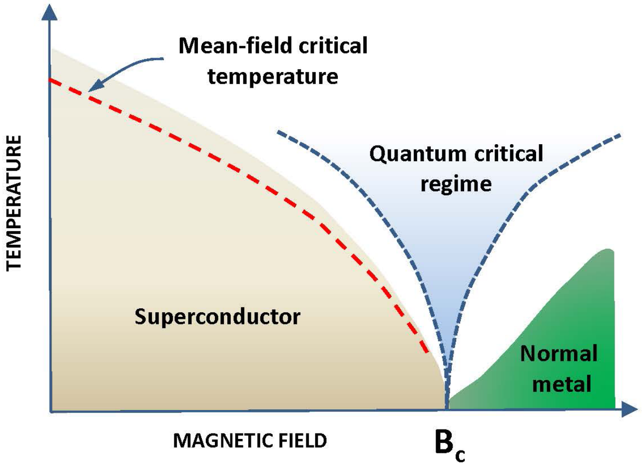

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- Shift the auxiliary field

$$\zeta \to \zeta - \frac{4\pi i a_s}{mN} \vec{\psi}_\downarrow \cdot \vec{\psi}_\uparrow$$

A potentially related system

- HS-transformed Lagrangian:

$$\mathcal{L} = \sum_{s=\uparrow,\downarrow} \vec{\psi}_s^\dagger \left(\partial_\tau - \frac{\vec{\nabla}^2}{2m} - \mu \right) \vec{\psi}_s - \frac{Nm\zeta\zeta^*}{4\pi a_s} + i\zeta^* \left(\vec{\psi}_\downarrow \cdot \vec{\psi}_\uparrow \right) - i\zeta \left(\vec{\psi}_\downarrow \cdot \vec{\psi}_\uparrow \right)^\dagger$$

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- Integrating out fermions can be done exactly

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- New partition function $Z = \int \mathcal{D}\zeta e^{-S}$ with /pause

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- Mathematically exact rewriting of original system (no approximation)

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- ζ_0 is the zero-frequency mode of the auxiliary field

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- Above a critical temperature $T > T_c$, $\zeta_0 = 0$ and the gap closes (ungapped fermions)
- All of these emerge rigorously from a mathematical rewriting of the large N fermion Lagrangian

Can we do something like this for Yang-Mills?

Hubbard-Stratonovic for Yang-Mills

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- For SU(2), $f^{abc} = \epsilon^{abc}$ and

$$f^{abc} f^{ade} = \delta^{bd}\delta^{ce} - \delta^{be}\delta^{cd}$$

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- Rewrite in terms of Lorentz scalar and (traceless) tensor:

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- Get

$$f^{abc} A^b_\mu A^c_\nu f^{ade} A^d_\mu A^e_\nu = \left(1 - \frac{1}{d}\right) \sigma_S^2 - \sigma^T_{\mu\nu} \sigma^T_{\mu\nu}\,.$$

Hubbard-Stratonovic for Yang-Mills

- The path-integral version of this requires two auxiliary fields for both S, T, e.g.

$$1 = \int \mathcal{D}\sigma_S \delta\left(\sigma_S - A^a_\mu A^a_\mu\right) = \int \mathcal{D}\sigma_S \mathcal{D}\zeta_S e^{i\int dx \zeta_S\left(\sigma_S - A^a_\mu A^a_\mu\right)}$$

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- HS formulation of Yang-Mills (mathematically exact) gives

$$4g^2\mathcal{L} = \left(\partial_{[\mu}A^a_{\nu]}\right)^2 + 2\partial_{[\mu}A^a_{\nu]} f^{abc} A^b_\mu A^c_\nu + 2i\zeta_S A^a_\mu A^a_\mu + \frac{\zeta_S^2}{1-\frac{1}{d}} + Tensor$$

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- Let's see what happens if we just drop it!

Hubbard-Stratonovic for Yang-Mills

Dropping cubic term in A

- $$4g^2\mathcal{L} = \left(\partial_{[\mu} A^a_{\nu]}\right)^2 + 2i\zeta_S A^a_\mu A^a_\mu + \frac{\zeta_S^2}{1-\frac{1}{d}} + Tensor$$

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- Dropping the cubic term gauge field Green's function becomes exact:

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- For fermions, we had $\zeta_0 \neq 0$; here this implies gluon $G_{\mu\nu}$ becomes invertible

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- For fermions, we had $\zeta_0 \neq 0$; here this implies gluon $G_{\mu\nu}$ becomes invertible
- No extra gauge fixing necessary! No Faddeev-Popov ghosts!

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- Transverse gluons get a mass gap!
- Completely analogous to superconducting gap for non-relativistic fermions!

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- The two lightest particles in low-temperature SU(2) are the scalar and tensor glueball, respectively

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- β-function is wrong (also wrong sign!)
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- How to convince the broader community that this is viable?