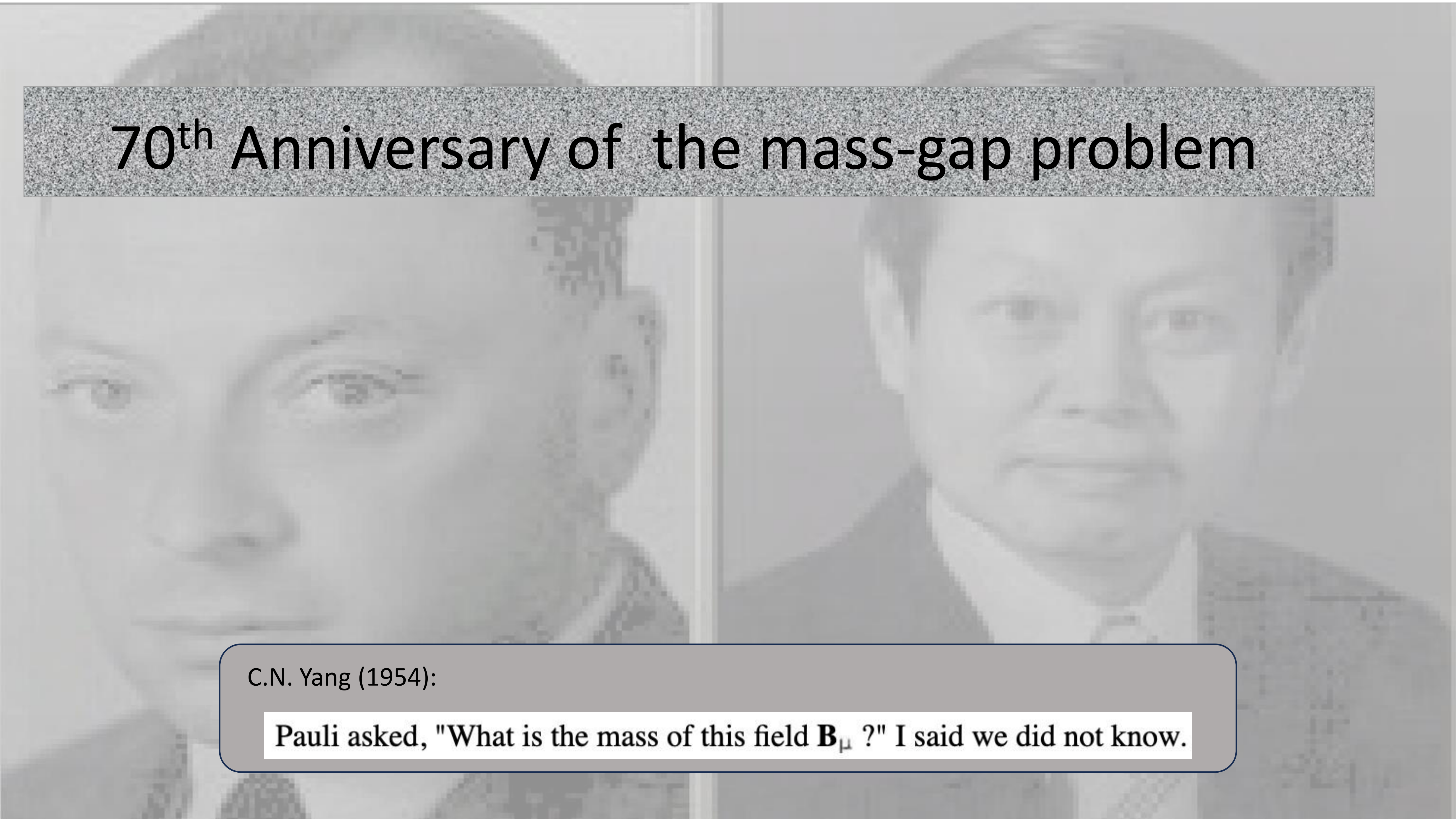
TIMECRYSTALLINE VORTICES, STAGNATION POINTS AND THE POINCARÉ INDEX FORMULA

Antti Niemi --
Nordita

70^{th} Anniversary of the mass-gap problem



C.N. Yang (1954):

Pauli asked, "What is the mass of this field $\mathbf{B}_\mu$?" I said we did not know.

SU(2) YM motivation:

$$\mathbf{A} = A\mathbf{n} + \rho \mathbf{dn} + \sigma \mathbf{n} \wedge \mathbf{dn}$$

"on-shell" decomposition

$$\psi = \rho + i\sigma$$

$$|\psi| = \text{density}$$

$$\mathbf{v} = \nabla \tan^{-1}\left(\frac{\sigma}{\rho}\right)$$

$$\begin{matrix} \text{Madelung} \\ \text{Hasimoto} \end{matrix} \rightarrow \begin{matrix} \text{fluids} \\ \text{strings} \end{matrix}$$

Gross-Pitaevskii equation with harmonic trap (NLSE):

[redacted]

Gross-Pitaevskii free energy:

[redacted]

Rigorous results (Lieb *et.al.)*

- Relevant point:

For $g \in [5,500]$ 2D Gross-Pitaevskii equation describes $N \sim 10^4$ - 10^6 ultracold Bose-Einstein condensed alkali atoms in a typical experiment with axially symmetric oblate harmonic trap

Plan:

- ✓ Vortices and Gross-Pitaevskii – textbook results
- ✓ Vortices and time crystals
- ✓ Vortices are anyons
- ✓ Topological (phase) transition
- ✓ Order parameter and the Poincaré index formula

Properties of Gross-Pitaevskii:

[redacted]

[redacted]

Poisson brackets

[redacted]

[redacted]

[redacted]

Conserved quantities:

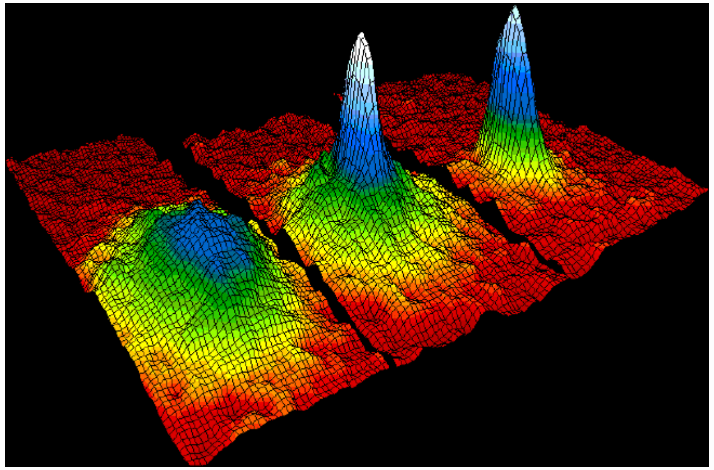
[redacted] = 1 (normalization) *number of atoms*

[redacted] *z-component of macroscopic angular momentum*

[redacted]

Topological quantity: [redacted] *vorticity*

Known results:



- Fundamental excitations are vortices

- There is no stable vortex solution of the static GP equation

[redacted]

- There are stable vortex solutions that are stationary in a uniformly rotating frame

[redacted]

chemical potential for N

- These vortices solve the time independent GP (nonlinear Schrödinger) equation

[redacted] [redacted]

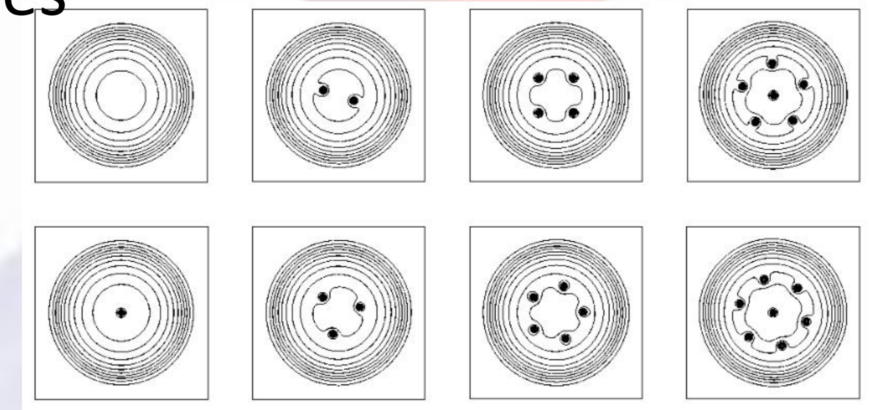
Vortices in rotating condensates:

- Numerical simulations:

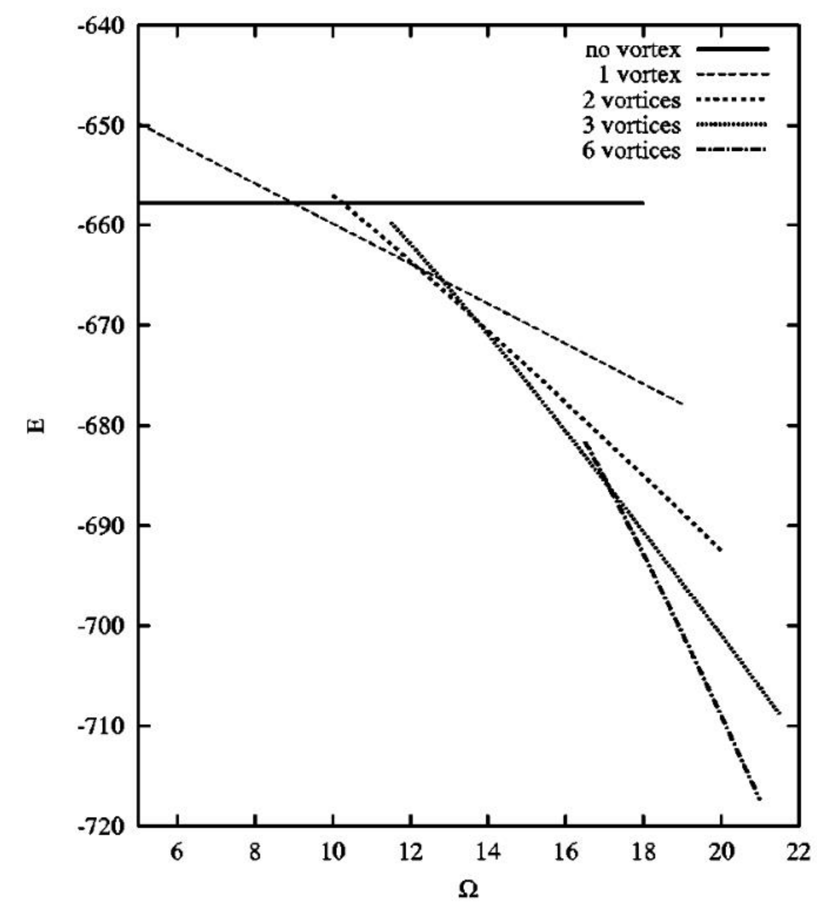
Vortices in a rotating condensate form highly symmetric lattices.

Vortex (Abrikosov) lattices are modeled as minimum energy critical points of F_{Ω}

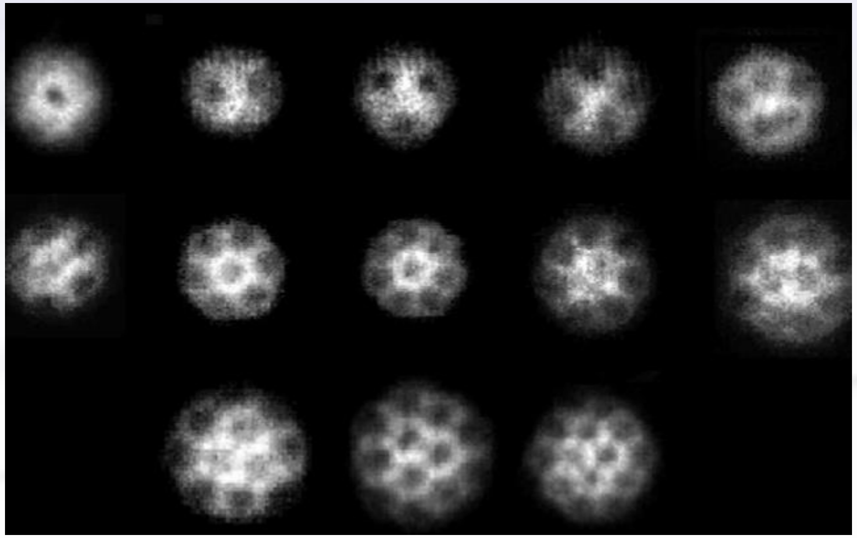
Ω increases



NOTE: Unlike Euler vortices the vortex energy is finite! No delta-functions!



- Experimental observations:



“Highly symmetric”

Vortices and time crystals

Wikipedia
The Free Encyclopedia

PARADIGM:

"A time crystal never reaches thermal equilibrium, as it is a type of nonequilibrium matter, a form of matter proposed in 2012, and first observed in 2017. This state of matter cannot be isolated from its environment—it is an open system in nonequilibrium."

$$\dot{p}_i = \frac{\partial H}{\partial q_i} = 0$$

$$\dot{q}_i = -\frac{\partial H}{\partial p_i} = 0$$

Energy minimum is critical point
⇒ There are no hamiltonian time crystals

Counterexample:
Hamiltonian with condition

[redacted]

[redacted]

Hamilton's equation has no time independent solution

Lagrange multiplier theorem:

Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be the objective function, $g\colon \mathbb{R}^n \to \mathbb{R}^c$ be the constraints function, both belonging to C^1 (that is, having continuous first derivatives). Let x^* be an optimal solution to the following optimization problem such that $\operatorname{rank}(Dg(x^*)) = c < n$ (here $Dg(x^*)$ denotes the matrix of partial derivatives, $[\partial g_j / \partial x_k]$):

$$\text{maximize } f(x)$$
$$\text{subject to: } g(x) = 0$$

Then there exists a unique Lagrange multiplier $\lambda^* \in \mathbb{R}^c$ such that $Df(x^*) = \lambda^{*T} Dg(x^*)$.

Critical point:

- Energy: H
- Minimize energy

⇨ critical point of H

Hamilton's equation:

[redacted]

[redacted]

No "time crystal"

Constrained optimization:

- Energy H and set of conditions $G^a = 0$
- Minimize energy H subject to conditions

[redacted]

[redacted]

[redacted]

[redacted] time crystal

[redacted]

Question: *Are there non-symmetric minimal energy vortex solutions of the Gross-Pitaevskii equation that are NOT critical points of the free energy?*

- Minimize [redacted]

- Subject to conditions [redacted] ≡ 1 (normalization) [redacted]

NOTE: Macroscopic angular momentum Does not need to be integer

The minimum of *F* subject to conditions can be found as a critical point of [redacted] *"Lagrange multiplier theorem"*

[redacted] ← *"Time independent GP in rotating frame"*

[redacted]

Solve for [redacted]

Note:

[redacted]

[redacted]

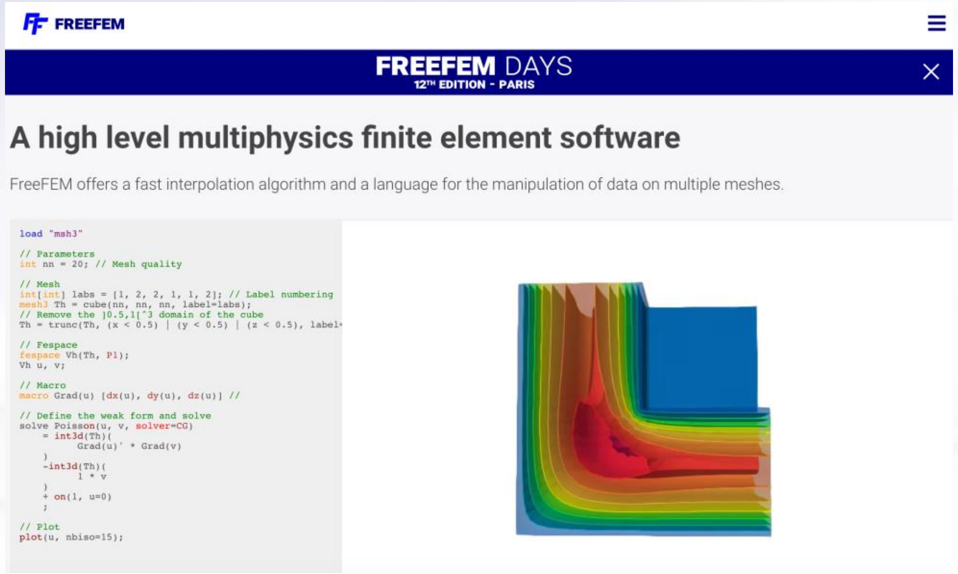
[redacted]

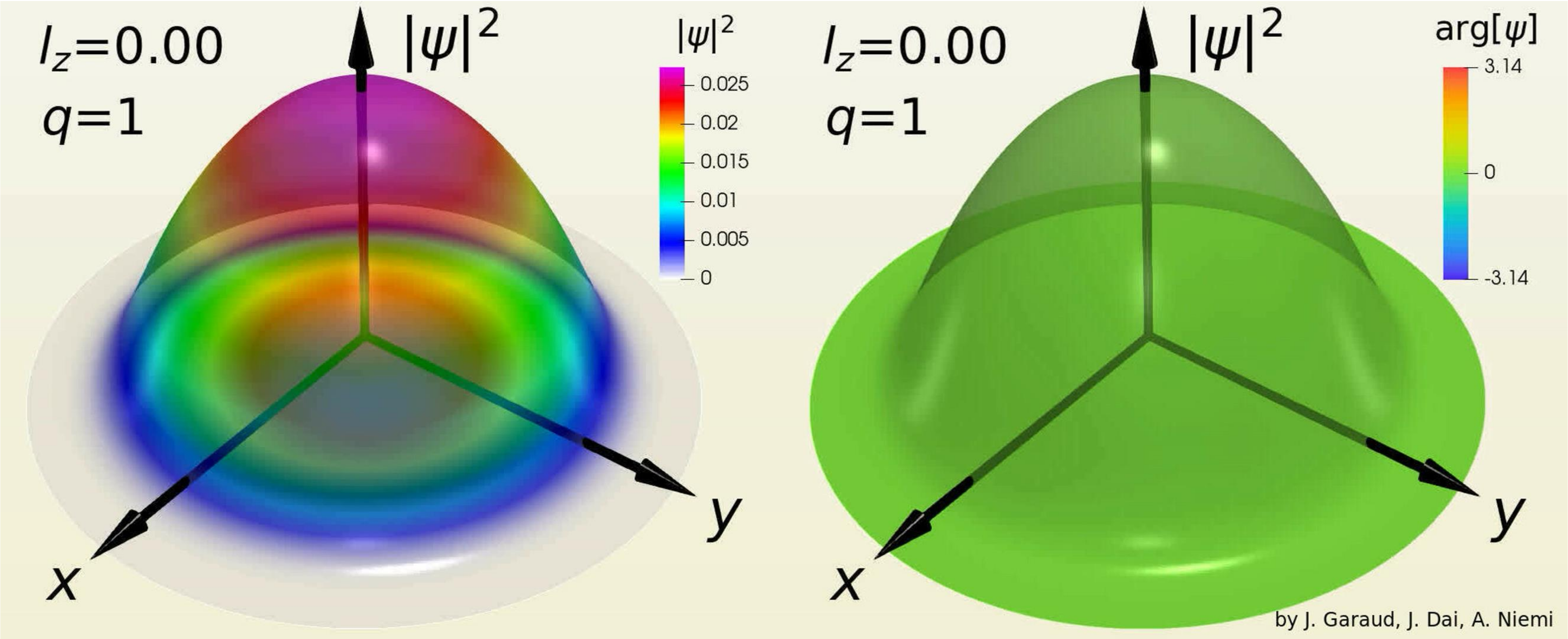
with [redacted]

[redacted]

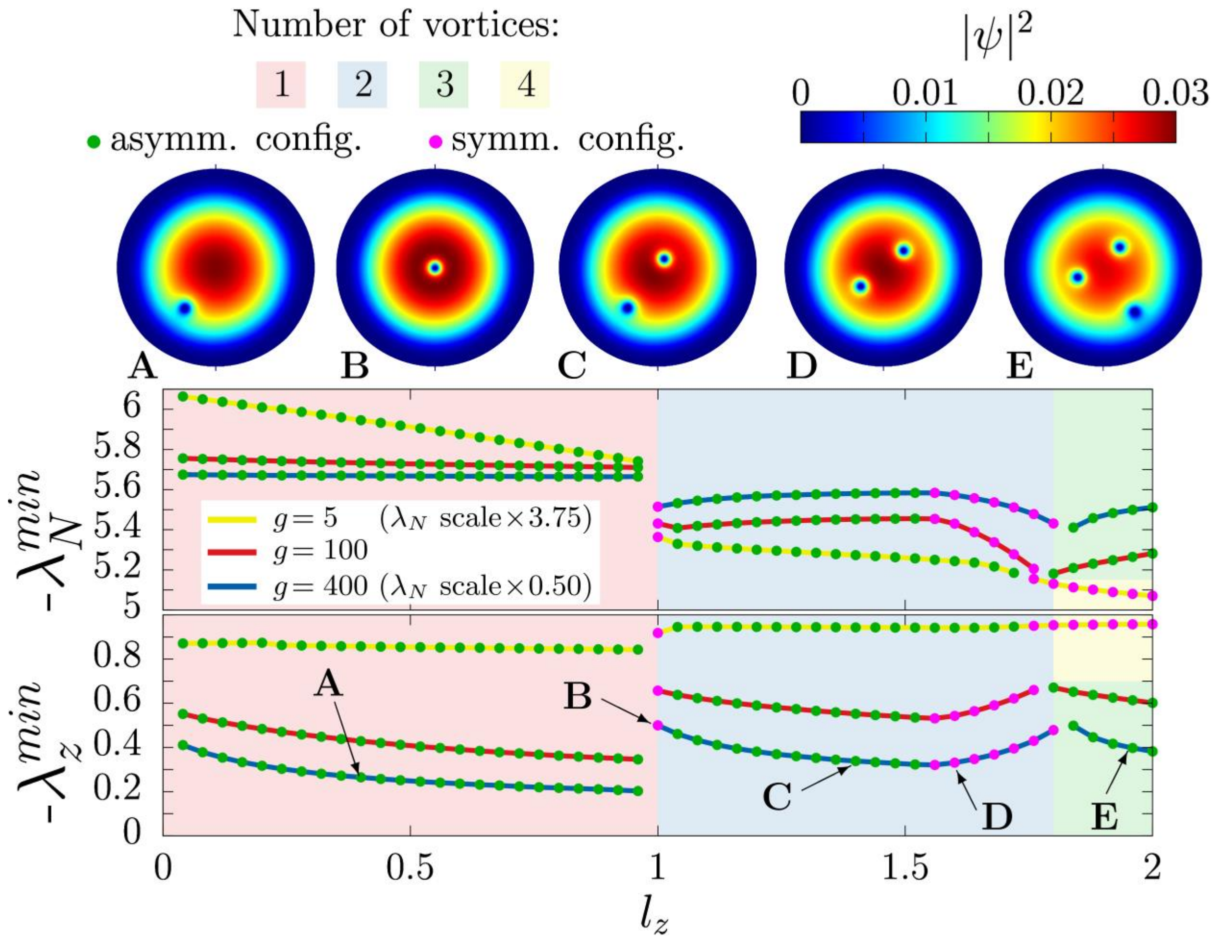
- Spontaneous symmetry breakdown: U(1)xU(1) -> U(1)
- Timecrystalline evolution is a symmetry transformation

Search for a solution numerically:





l_z=0.00
q=1
$|\psi|^2$
$|\psi|^2$
0.025
0.02
0.015
0.01
0.005
0
x
y
l_z=0.00
q=1
$|\psi|^2$
arg[ψ]
3.14
0
-3.14
x
y
by J. Garaud, J. Dai, A. Niemi



Vortices are timecrystalline – a vortex always moves

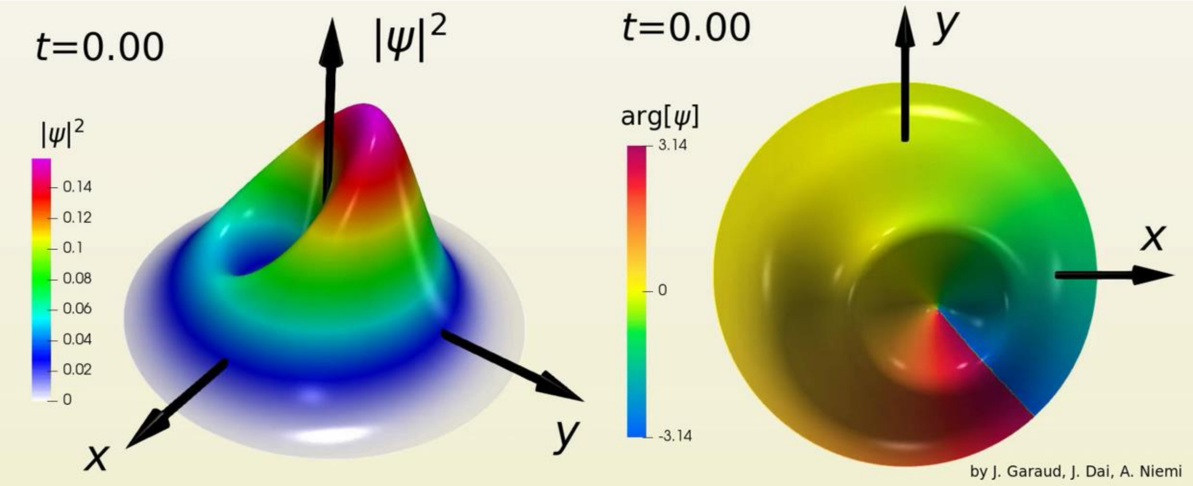
Vortices are anyons

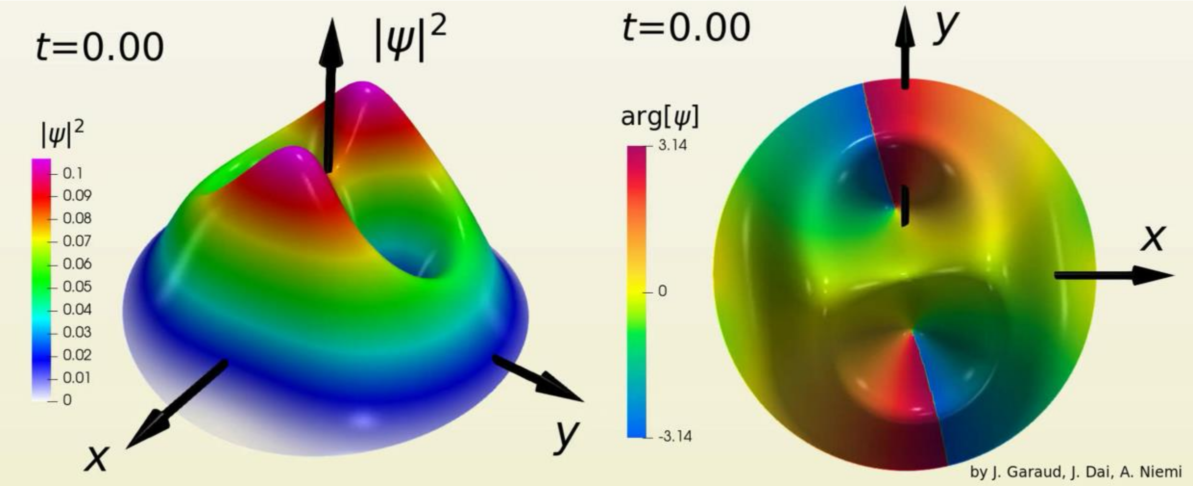
WIKIPEDIA
The Free Encyclopedia

“In two dimensions, exchanging identical particles twice is not equivalent to leaving them alone. The particles’ wavefunction after swapping places twice may differ from the original one; particles with such unusual exchange statistics are known as anyons. By contrast, in three dimensions, exchanging particles twice cannot change their wavefunction, leaving us with only two possibilities: bosons, whose wavefunction remains the same even after a single exchange, and fermions, whose exchange only changes the sign of their wavefunction.”

Time evolution:

[redacted]

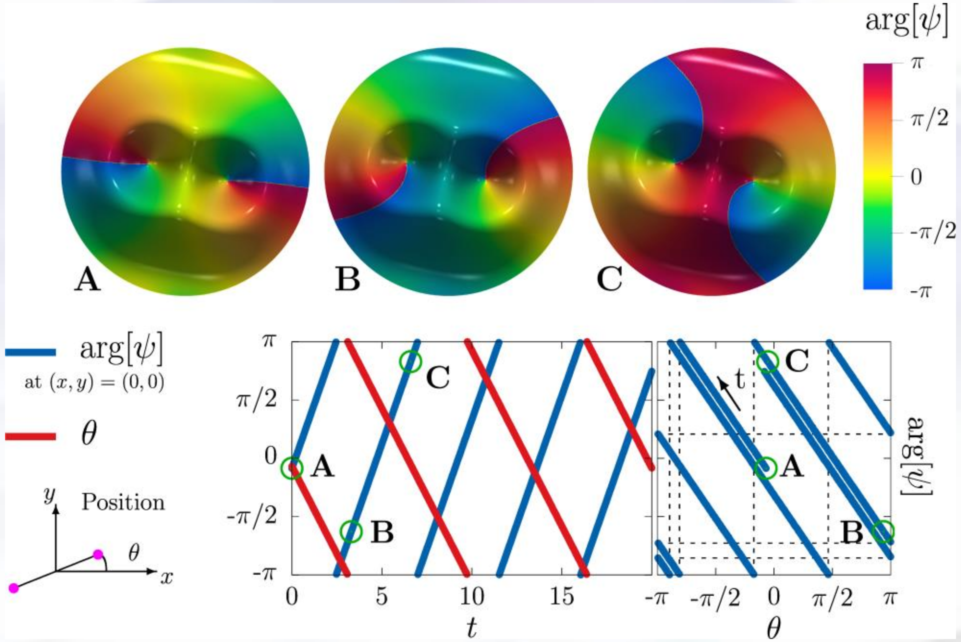




Vortices are timecrystalline anyons: [redacted]

[redacted] ”Kinetic angular momentum”

[redacted] "Statistical gauge field”



Accrued phase of wavefunction depends on [redacted] as statistical parameter

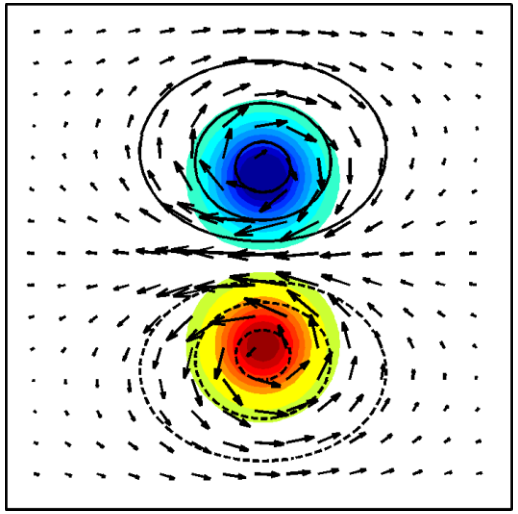
Saddle points, topological (phase) transitions And the Poincaré index formula

conventional (super)fluid vortices:

Order parameter $\psi(\mathbf{x}, t)$ solves a (variant of) NLSE

At (anti)vortex core $|\psi(\mathbf{x})| = 0$

$$\mathbf{v}(\mathbf{x}, t) = \nabla \arg[\psi](\mathbf{x}, t)$$



vortex

Opposite circulations

antivortex

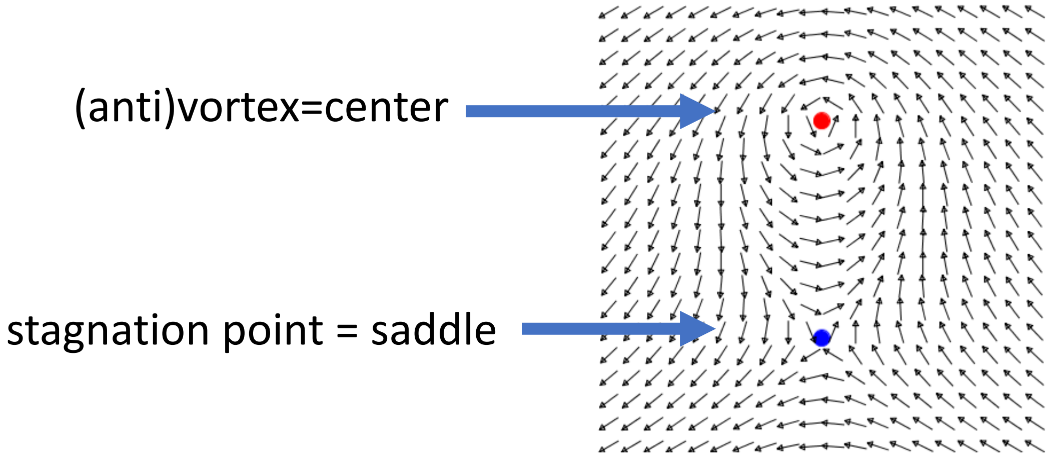
Circulation:

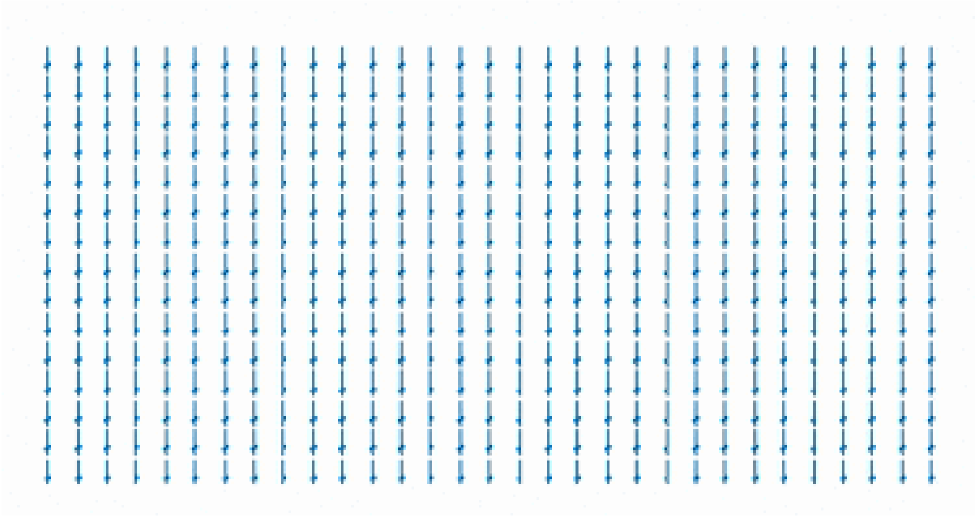
$$n_{\mathbf{v}}(p; \Gamma) = \frac{1}{2\pi} \oint_{\Gamma} d\mathbf{l} \cdot \mathbf{v} \quad \in \mathbb{Z}$$

Circulation = +1 for a single vortex, and -1 for a single antivortex

Normally, no other topological invariant -- but is this the full story??

Vortices and saddles (stagnation points) as topological structures:





center: $(v_x, v_y) = (y, -x)$ saddle: $(v_x, v_y) = (y, x)$

<u>Two different integral invariants</u>:

Circulation:

$$n_{\mathbf{v}}(p;\Gamma) = \frac{1}{2\pi} \oint_\Gamma d\mathbf{l} \cdot \mathbf{v} \quad \in \mathbb{Z}$$

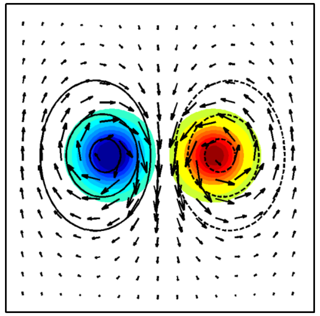
$= \pm 1$ center
$= 0$ saddle

Winding number:

$$i_{\mathbf{v}}(p;\Gamma) = \frac{1}{2\pi} \oint_\Gamma \frac{v_x dv_y - v_y dv_x}{v_x^2 + v_y^2} \in \mathbb{Z}\,.$$

$= +1$ center
$= -1$ saddle

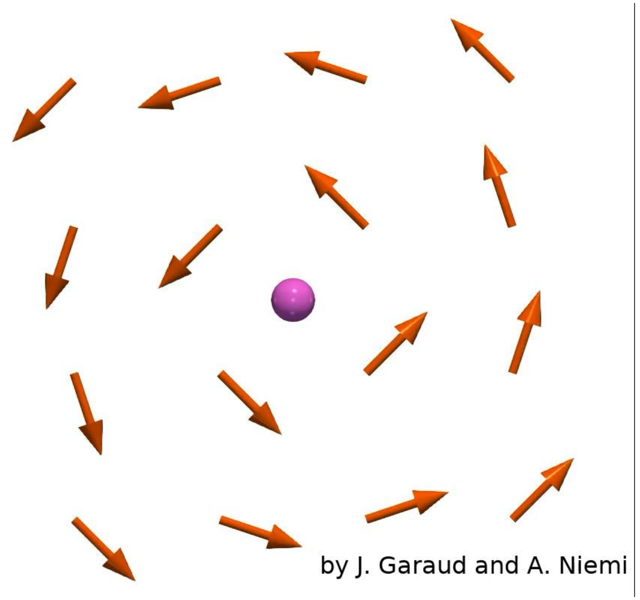
Vector fields, critical points and the winding number:



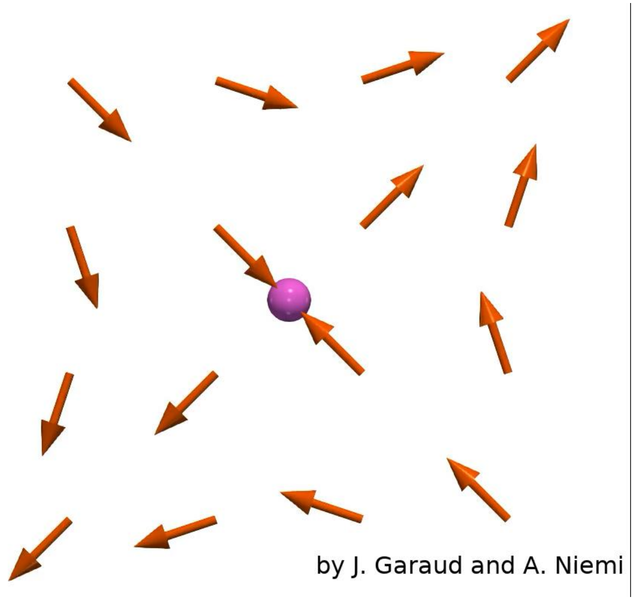
$$i_{\mathbf{v}}(p;\Gamma) = \frac{1}{2\pi}\oint_{\Gamma} \frac{v_x dv_y - v_y dv_x}{v_x^2 + v_y^2} \in \mathbb{Z}\,.$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$(v_x, v_y) \sim (-y, x)$ initially



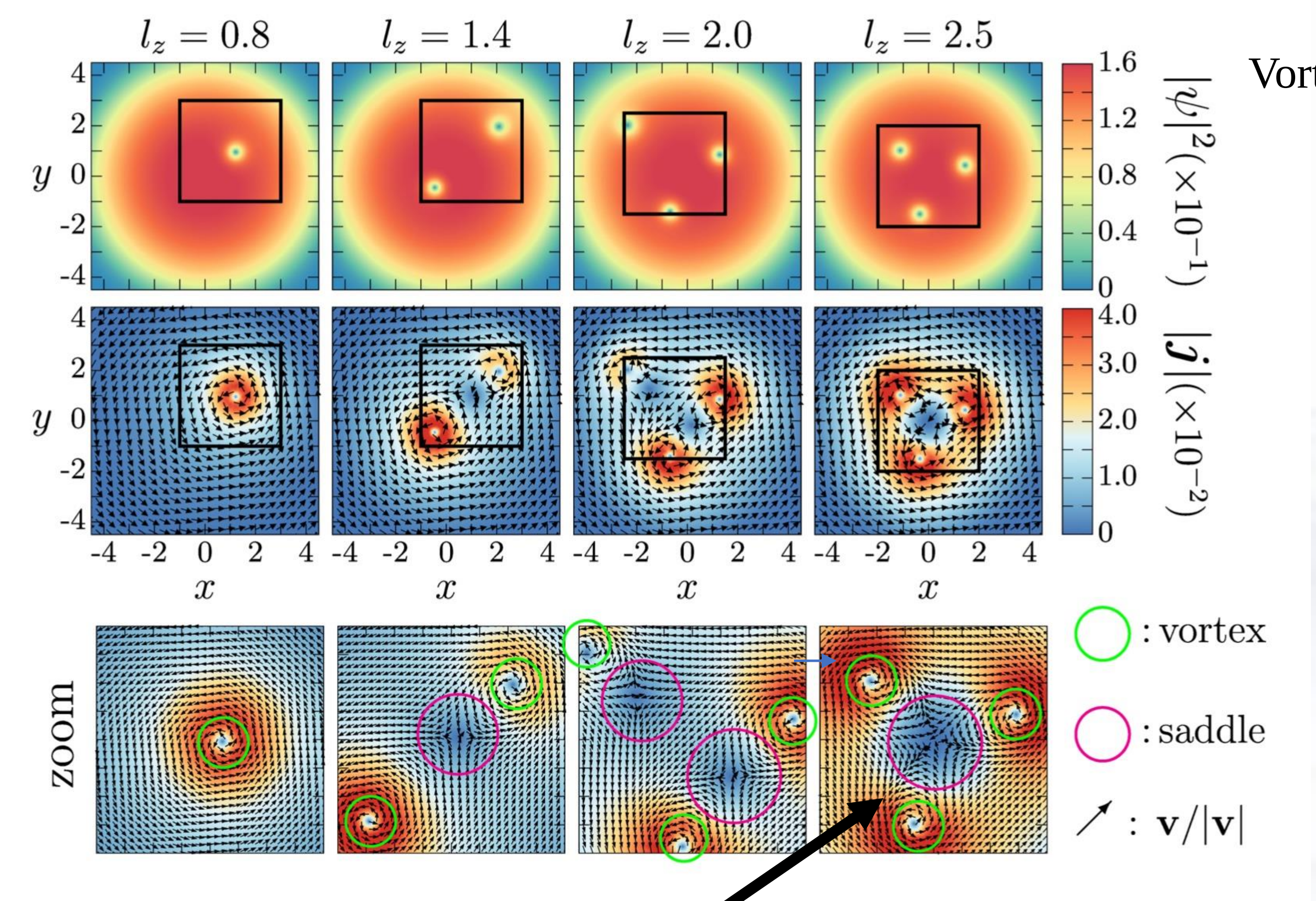
$(v_x, v_y) \sim (y, x)$ initially



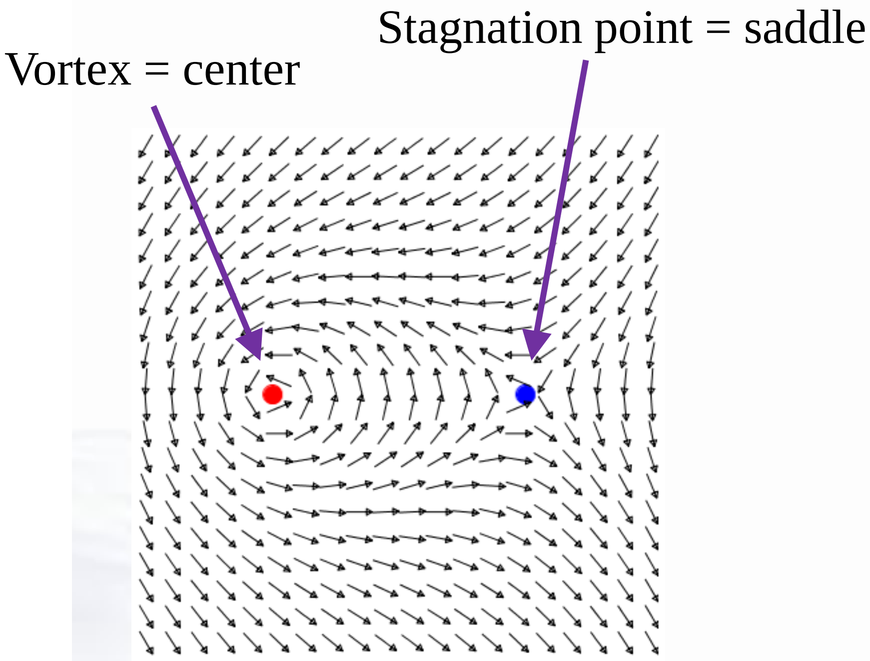
counterclockwise center, source, sink, clockwise center
All have winding number +1

Only saddle has winding number -1

Vortices and stagnation points (saddles) for small angular momentum:



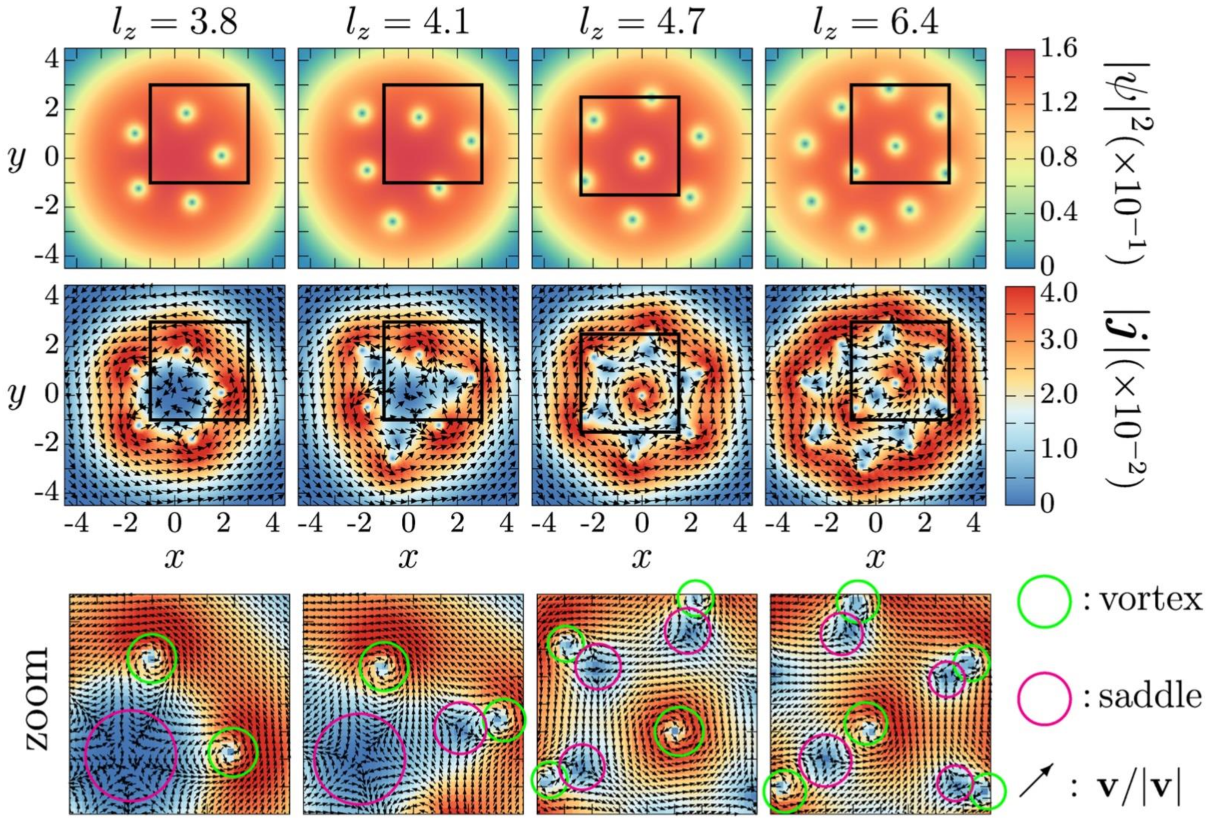
$$\mathbf{v} = \nabla \arg \psi(x, y)$$



Vortices form lattices but Saddles aggregate

Vortices and saddles at large angular momentum:

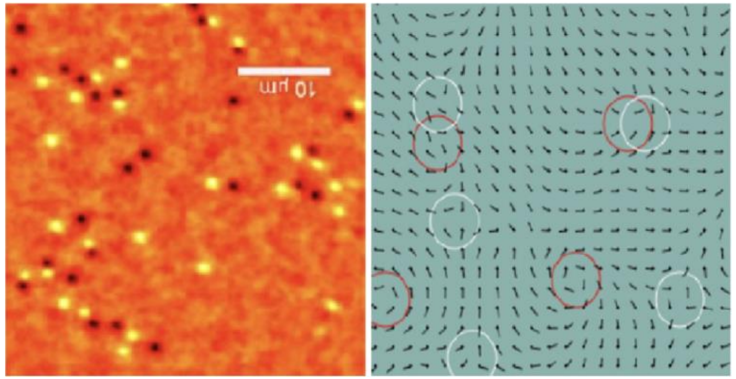
Change in local topology:



Kosterlitz-Thouless XY Model

$$\mathbf{u}_{XY} = (\cos[\arg\psi], \sin[\arg\psi])$$

Energy of vortices and saddles (logarithmically) divergent



small l_z $\sim$ small β large l_z $\sim$ large β

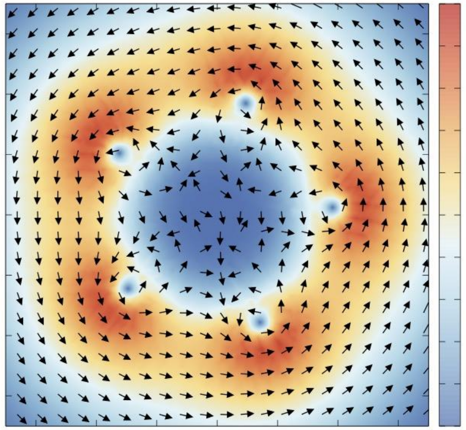
$$\mathbf{v} = \nabla \arg\psi(x, y)$$

Topological phase transition

Saddle aggregation → l_z increases → Vortex-saddle pairing

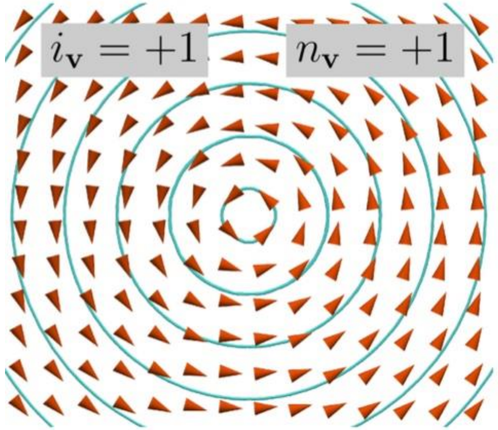
<u>IMPORTANT DIFFERENCE</u>: Energy of vortices diverges but energy of saddles finite in London limit of GP!

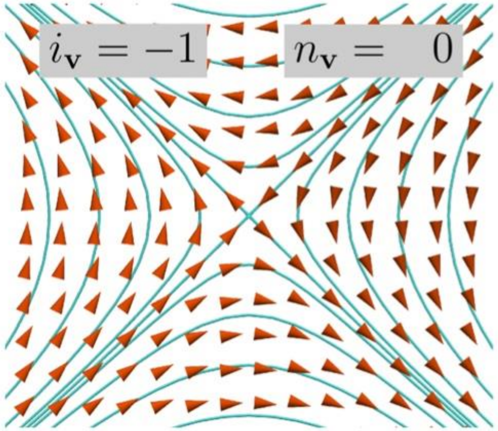
How to resolve centers and saddles in a given region:

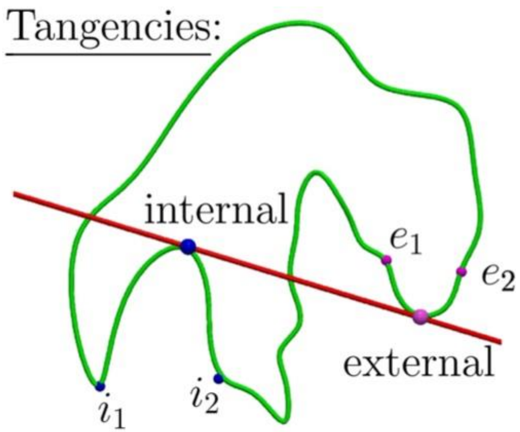


$$i_{\mathbf{v}}(p;\Gamma) = \frac{1}{2\pi}\oint_{\Gamma} \frac{v_x dv_y - v_y dv_x}{v_x^2 + v_y^2} \in \mathbb{Z}\,.$$ (Winding number)

$$n_{\mathbf{v}}(p;\Gamma) = \frac{1}{2\pi}\oint_{\Gamma} d\mathbf{l}\cdot\mathbf{v} \quad \in \mathbb{Z}$$ (Circulation)







Order parameter: Poincare index formula (for winding number)

$$\sum_{j=1}^{k} i_{\mathbf{v}}(p_j;\Gamma) = \mathrm{Index}(\Gamma) \;\equiv\; \mathcal{X}_{\Gamma} + \frac{1}{2}\left(I_{\Gamma} - E_{\Gamma}\right)$$

(for disk $\mathcal{X}_{\Gamma} = 1$)

Lessons:

- (Free) energy minima do not need to be critical points

  This can take place when there are symmetries with conditions

- Energy minima that are not critical points are often time dependent

  Timecrystalline dynamics, in Hamiltonian or Schrödinger context

- Time dependent minimum energy trajectories are symmetries

  Spontaneous symmetry breaking seems to be prerequisite for time crystals

- New physical phenomena

  Anyonic and timecrystalline vortices, topological phase transition in cold atoms, …

- Examples of Poincaré Index formula and other index theorems

  Centers, saddles and topology of 2D vector fields

What about D=3 (and higher) dimensions?