Cosmological Constant and Renormalization

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27/08/2024

QCD Vacuum Structure and Confinement

26 - 30 August 2024 Naxos Island of Cyclades, Greece

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Introduction

Renormalization: evergreen subject ... Source of frustration and surprises

Surprise: Anomalies ... measure ...

This talk: possibly another surprise ... related to measure

CC problem: most severe naturalness problem in physics

Several attempts towards its solution ...

Polyakov ... and later Jackiw ... Moscow zero ...

Coleman ... Wormholes

Taylor e Veneziano ... $V \log V$

... Many other attempts ...

"Conservative" attitude: implement SUSY in the theory (SUGRA)

One-loop VDW Effective Action

* Consider Euclidean action - Einstein-Hilbert truncation

$$S_{\rm grav} = \frac{1}{16\pi G} \int \mathrm{d}^4x \sqrt{g} \, (-R + 2\Lambda)$$

* Cosmological framework: manifolds with typical length scale $l \gg M_P^{-1}$

* Gauge-invariant one-loop effective action: $\Gamma^{1l}_{\rm grav} = S_{\rm grav} + \delta S^{1l}_{\rm grav}$ geometrical approach pioneered by Vilkovisky and DeWitt

* Strategy put forward by Fradkin and Tseytlin / Taylor and Veneziano

* Particular attention to the role played by the measure

* Background field method: $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ ($\bar{g}_{\mu\nu}$ is the background)

* When $\bar{g}_{\mu\nu}$ has spherical symmetry, the one-loop VDW effective action coincides with the standard one calculated with the gauge-fixing term

$$S_{\rm gf} = \frac{1}{32\pi G\xi} \int \mathrm{d}^4x \sqrt{\bar{g}} \left[\nabla_\mu \left(h^\mu_\nu - \frac{1}{2} \delta^\mu_\nu \, h^\sigma_\sigma \right) \right]$$

after taking the limit $\xi \to 0$ at the end of the calculation

One-loop VDW Effective Action continued

Calculation of the 1-loop correction $\delta S^{1l}_{\rm grav}$ to $S_{\rm grav}$

Take the spherical background $\bar{g}_{\mu\nu} = g^{(a)}_{\mu\nu}$ (a radius of the sphere)

Note: the coordinates x are the angles parametrizing the sphere

$g^{(a)}_{\mu\nu}$ dimension $(\mathrm{length})^2$ and goes like a^2

Classical Action ($\int \mathrm{d}^4x \sqrt{g^{(a)}} = \frac{8\pi^2}{3} a^4$, $R(g^{(a)}) = \frac{12}{a^2}$)

$$S^{(a)}_{\rm grav} = \frac{\pi\Lambda}{3G} a^4 - \frac{2\pi}{G} a^2$$

Add to $S_{\rm grav} + S_{\rm gf}$ the corresponding ghost action (v_μ vector ghost fields)

$$S_{\rm ghost} = \frac{1}{32\pi G} \int \mathrm{d}^4x \sqrt{g^{(a)}}\, g^{(a)\,\mu\nu}\, v^*_\mu \left(-\nabla_\rho \nabla^\rho - \frac{3}{a^2}\right) v_\nu$$

Finally identify the 1-loop corrections to $\frac{\Lambda}{G}$ and $\frac{1}{G}$ with the coefficients of a^4 and a^2 in $\delta S^{1l}_{\rm grav}$

One-loop VDW Effective Action continued

The VDW one-loop correction $\delta S^{1l}_{\mathrm{grav}}$ to $S^{(a)}_{\mathrm{grav}}$ given by

$$e^{-\delta S^{1l}_{\mathrm{grav}}} = \lim_{\xi\to 0} \int \left[\mathcal{D}u(h)\mathcal{D}v^*_\rho\, \mathcal{D}v_\sigma\right] e^{-\delta S^{(2)}}$$

where

$$\delta S^{(2)} \equiv S_2 + S_{\mathrm{gf}} + S_{\mathrm{ghost}}$$

S_2 quadratic term in the expansion of $S_{\mathrm{grav}}[g^{(a)}_{\mu\nu} + h_{\mu\nu}]$

$$S_2 \equiv \frac{1}{32\pi G} \int \mathrm{d}^4x\, \sqrt{g^{(a)}} \left[\frac{1}{2}\tilde{h}^{\mu\nu}\left(-\nabla_\rho\nabla^\rho - 2\Lambda + \frac{8}{a^2}\right)h_{\mu\nu} + \frac{h^2}{a^2} - \nabla^\rho \tilde{h}_{\rho\mu}\nabla^\sigma \tilde{h}^\mu_\sigma\right]$$

$h \equiv g^{(a)}_{\mu\nu} h^{\mu\nu}$, $\tilde{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} g^{(a)}_{\mu\nu} h$

indexes raised with $g^{(a)\,\mu\nu}$; covariant derivatives in terms of $g^{(a)}_{\mu\nu}$

Measure $\left[\mathcal{D}u(h)\mathcal{D}v^*_\rho\,\mathcal{D}v_\sigma\right]$

$$\left[\mathcal{D}u(h)\mathcal{D}v^*_\rho\,\mathcal{D}v_\sigma\right] \equiv \prod_x \left[g^{(a)\,00}(x)\ \left(g^{(a)}(x)\right)^{-1} \left(\prod_{\alpha\leq\beta} \mathrm{d}h_{\alpha\beta}(x)\right)\left(\prod_\rho \mathrm{d}v^*_\rho(x)\right)\left(\prod_\sigma \mathrm{d}v_\sigma(x)\right)\right]$$

$g^{(a)\,00}(x)\ \left(g^{(a)}(x)\right)^{-1}$ from integration over conjugate momenta[1] (FV)

Observe: $g^{(a)}_{\mu\nu}$ can be written as $g^{(a)}_{\mu\nu} = a^2 g^{(1)}_{\mu\nu}$

$g^{(1)}_{\mu\nu}$ metric of a sphere of unitary radius, $a = 1$

$$\Longrightarrow \qquad g^{(a)\,00}(x)\ \left(g^{(a)}(x)\right)^{-1} = a^{-10}\, g^{(1)\,00}(x)\ \left(g^{(1)}(x)\right)^{-1}$$

with $g^{(1)\,00}(x)\ \left(g^{(1)}(x)\right)^{-1}$ a-independent

[1] original expression in FV is $g^{(a)\,00}(x)\ \left(g^{(a)}(x)\right)^{-\frac{3}{2}}$. Difference due to the fact that here both v and v^* are world vectors, in FV different choice. $\sqrt{g^{(a)}}$ Jacobian due to the change between these two equivalent functional integration variables (Unz).

Reabsorb $G^{-1/2}a^{-1}$ in $h_{\mu\nu}$ $\Longrightarrow$ $\widehat{h}_{\mu\nu} = (32\pi G)^{-1/2}\, a^{-1} h_{\mu\nu}$

$S_2 + S_{\rm gf}$ rewritten as

$$S_2+S_{\rm gf} = \int \mathrm{d}^4x\,\sqrt{g^{(1)}}\left[\frac{1}{2}\overline{h}^{\mu\nu}\left(-\nabla_\rho\nabla^\rho - 2a^2\Lambda + 8\right)\widehat{h}_{\mu\nu} + \widehat{h}^2 - \left(1-\frac{1}{\xi}\right)\nabla^\rho\overline{h}_{\rho\mu}\nabla^\sigma\overline{h}^\mu_\sigma\right]$$

with $\widehat{h} \equiv g^{(1)}_{\mu\nu}\,\widehat{h}^{\mu\nu}$, $\overline{h}_{\mu\nu} \equiv \widehat{h}_{\mu\nu} - \frac{1}{2}g^{(1)}_{\mu\nu}\,\widehat{h}$, indexes raised with $g^{(1)\,\mu\nu}$, covariant derivatives in terms of $g^{(1)}_{\mu\nu}$

Clearly $\widehat{h}_{\mu\nu}$ defined on a sphere of unitary radius

Redefinition $v_\mu \to (32\pi G)^{\frac{1}{2}}\, v_\mu$ (covariant derivatives in terms of $g^{(1)}_{\mu\nu}$)

$$S_{\rm ghost} = \int \mathrm{d}^4x\sqrt{g^{(1)}}\,g^{(1)\,\mu\nu}\,v^*_\mu\left(-\nabla_\rho\nabla^\rho - 3\right)v_\nu$$

As $\widehat{h}_{\mu\nu}$: v_μ defined on a sphere of unitary radius

Note: when written in terms of $\widehat{h}_{\mu\nu}$ and v_μ, $\delta S^{(2)} = S_2 + S_{\rm gf} + S_{\rm ghost}$ contains only dimensionless quantum fluctuation operators

Back to the measure

$$\left[\mathcal{D}u(h)\mathcal{D}v^*_\rho\,\mathcal{D}v_\sigma\right] \equiv \prod_x \left[a^{-10} g^{(1)\,00}(x)\ \left(g^{(1)}(x)\right)^{-1} \left(\prod_{\alpha\leq\beta} \mathrm{d}h_{\alpha\beta}(x)\right)\left(\prod_\rho \mathrm{d}v^*_\rho(x)\right)\left(\prod_\sigma \mathrm{d}v_\sigma(x)\right)\right]$$

From $\widehat{h}_{\mu\nu} = (32\pi G)^{-1/2}\, a^{-1} h_{\mu\nu}$

$$\prod_{\alpha\leq\beta} \mathrm{d}h_{\alpha\beta}(x) = (32\pi G)^5\, a^{10} \prod_{\alpha\leq\beta} \mathrm{d}\widehat{h}_{\alpha\beta}(x)$$

Therefore

$$\left[\mathcal{D}u(h)\mathcal{D}v^*_\rho\,\mathcal{D}v_\sigma\right] = \mathcal{N}\prod_x \left[\left(\prod_{\alpha\leq\beta} \mathrm{d}\widehat{h}_{\alpha\beta}(x)\right)\left(\prod_\rho \mathrm{d}v^*_\rho(x)\right)\left(\prod_\sigma \mathrm{d}v_\sigma(x)\right)\right]$$

with a-independent terms as $\prod_x g^{(1)\,00}(x)\ \left(g^{(1)}(x)\right)^{-1}$ included in the harmless constant $\mathcal{N}$

Disappearance of a in the measure: crucial point

Since $\widehat{h}_{\mu\nu}$ and v_μ fields on a sphere of radius $a = 1 \Longrightarrow$ bases for symmetric tensors and vectors with eigenfunctions of the

Dimensionless Laplace-Beltrami operator $\quad -\Box^{(s)}_{a=1} \equiv -a^2\,\Box^{(s)}_a$

$-\Box^{(s)}_a$ Laplace-Beltrami for sphere of radius a; s spins: $s = 0, 1, 2$

Dimensionless eigenvalues $\lambda^{(s)}_n$ and corresponding degeneracies $D^{(s)}_n$

$$\lambda^{(s)}_n = n^2 + 3n - s \qquad ; \qquad D^{(s)}_n = \frac{2s+1}{3}\left(n + \frac{3}{2}\right)^3 - \frac{(2s+1)^3}{12}\left(n + \frac{3}{2}\right)$$

where $n = s, s+1, \ldots$

Expanding $\widehat{h}_{\mu\nu}$, v^*_ρ and v_σ for $\delta S^{1l}_{\rm grav}$ we have (backup slides)

$$\delta S^{1l}_{\rm grav} = -\frac{1}{2}\log\frac{\det_1[-\Box^{(1)}_{a=1} - 3]\,\det_2[-\Box^{(0)}_{a=1} - 6]}{\det_0[-\Box^{(2)}_{a=1} - 2a^2\Lambda + 8]\,\det_2[-\Box^{(0)}_{a=1} - 2a^2\Lambda]} + \frac{1}{2}\log(2a^2\Lambda) + \mathcal{B}$$

$\mathcal{B}$ inessential a-independent term. The index i in $\det_i$ signals that the product of eigenvalues starts from $\lambda^{(s)}_{s+i}$.

$$\delta S^{1l}_{\rm grav} = -\frac{1}{2}\log\frac{\det_1[-\Box^{(1)}_{a=1}-3]\,\det_2[-\Box^{(0)}_{a=1}-6]}{\det_0[-\Box^{(2)}_{a=1}-2a^2\Lambda+8]\,\det_2[-\Box^{(0)}_{a=1}-2a^2\Lambda]} + \frac{1}{2}\log(2a^2\Lambda) + \mathcal{B}$$

$\frac{1}{2}\log(2a^2\Lambda)$ (from the integration over one of the modes in which $\widehat{h}_{\mu\nu}$ is decomposed (backup slides)) and $\mathcal{B}$: irrelevant for our scopes

Truly important term: the first one in the right hand side

Peculiarity: written in terms of dimensionless determinants $\Longrightarrow$

No need to introduce any arbitrary mass scale μ:

the determinants are *automatically* dimensionless

In typical calculations of $\delta S^{1l}_{\rm grav}$, the arguments in $\det_i$ are dimensionful. To take care of that an arbitrary mass scale μ is introduced

Note: although the calculation is performed for a sphere of generic radius a, the Laplace-Beltrami operators are those for a sphere of unitary radius

Note: a only comes in the combination $a^2\Lambda$.

Calculation of the fluctuation determinants

Two different strategies

First: direct calculation in terms of eigenvalues of Laplace-Beltrami ops.

Second: proper-time, as usually done

Anticipating: both calculations show that quartically and quadratically divergent contributions to the vacuum energy usually present in the literature are actually absent

$\Longrightarrow$ No need for supersymmetric embedding of the theory (SUGRA)

Calculation in terms of the eigenvalues $\lambda_n^{(s)}$

$$\delta S^{1l}_{\text{grav}} = \frac{1}{2}\sum_{n=2}^{N-2}\Big[D_n^{(2)}\log\left(\lambda_n^{(2)} - 2a^2\Lambda + 8\right) + D_n^{(0)}\log\left(\lambda_n^{(0)} - 2a^2\Lambda\right) - D_n^{(1)}\log\left(\lambda_n^{(1)} - 3\right) - D_n^{(0)}\log\left(\lambda_n^{(0)} - 6\right)\Big] + \frac{1}{2}\log(2a^2\Lambda) + \mathcal{B}$$

UV cutoff introduced as gauge invariant numerical cut N on the number of eigenvalues ($N-2$ rather than N simplifies the expression)

Note: De Sitter solution for the classical action

$$a_{\text{dS}} = \sqrt{\frac{3}{\Lambda_{\text{cc}}}}$$

a_{dS} size of the universe $\Longrightarrow$ connection between N and physical cutoff scale $\Lambda_{\text{cut}} \sim M_P$ given by

$$\Lambda_{\text{cut}} \sim M_P = \frac{N}{a_{\text{dS}}} = N\sqrt{\frac{\Lambda_{\text{cc}}}{3}}$$

Calculation in terms of the $\lambda_n^{(s)}$ continued

$N-2$: number of modes retained in the calculation of the determinants

Since the eigenvalues $\widetilde{\lambda}_n^{(s)}$ of $-\Box_a^{(s)}$ go like $\widetilde{\lambda}_n^{(s)} \equiv \frac{\lambda_n^{(s)}}{a^2} \sim \frac{n^2}{a^2}$, the requirement $n \leq N-2$ is not equivalent to require $\widetilde{\lambda}_n^{(s)} \leq \Lambda_{\rm cut}^2$

This latter choice might seem natural, since it would amount to require that the maximal eigenvalue $\widetilde{\lambda}_{\rm max}^{(s)}$ is $\widetilde{\lambda}_{\rm max}^{(s)} \sim \Lambda_{\rm cut}^2$

But this reasoning is misleading. Since the $\widetilde{\lambda}_n^{(s)}$ go like a^{-2}, such a choice would introduce an unphysical a-dependence in the implementation of the cutoff, i.e. on the background metric $g_{\mu\nu}^{(a)}$

This simple observation is fundamental to obtain the correct result for $\delta S_{\rm grav}^{1l}$, in particular to see that there are

No quartic and quadratic divergences in the vacuum energy

Calculation in terms of the $\lambda_n^{(s)}$ continued

Remarkably, sum in $S^{1l}_{\rm grav}$ obtained in closed form (backup slides)

Expanding for $N \gg 1$

$$\begin{aligned}\delta S^{1l}_{\rm grav} =& -\left(\Lambda_{\rm cc}^2 \log N^2\right) a^4 + \Lambda_{\rm cc}\left(-N^2 + 8\log N^2\right) a^2 \\ &+ \frac{N^4}{24}\left(-1 + 2\log N^2\right) + \frac{N^2}{36}\left(203 - 75\log N^2\right) - \frac{779}{90}\log N^2 + \mathcal{B} \\ &+ \frac{1}{2}\log(2a^2\Lambda_{\rm cc}) + \mathcal{F}(a^2\Lambda_{\rm cc}) + \mathcal{O}\left(N^{-2}\right)\end{aligned}$$

where $\mathcal{F}(a^2\Lambda)$ contains only UV-finite terms (no dependence on N)

Using $\Lambda_{\rm cut} \sim M_P = \frac{N}{a_{\rm dS}} = N\sqrt{\frac{\Lambda_{\rm cc}}{3}}$

$$\begin{aligned}\delta S^{1l}_{\rm grav} =& -\left(\Lambda_{\rm cc}^2 \log\frac{3\Lambda_{\rm cut}^2}{\Lambda_{\rm cc}}\right) a^4 + \left(-3\Lambda_{\rm cut}^2 + 8\Lambda_{\rm cc}\log\frac{3\Lambda_{\rm cut}^2}{\Lambda_{\rm cc}}\right) a^2 \\ &+ \frac{3\Lambda_{\rm cut}^4}{8\Lambda_{\rm cc}^2}\left(-1 + 2\log\frac{3\Lambda_{\rm cut}^2}{\Lambda_{\rm cc}}\right) + \frac{\Lambda_{\rm cut}^2}{12\Lambda_{\rm cc}}\left(203 - 75\log\frac{3\Lambda_{\rm cut}^2}{\Lambda_{\rm cc}}\right) - \frac{779}{90}\log\frac{3\Lambda_{\rm cut}^2}{\Lambda_{\rm cc}} + \mathcal{B} \\ &+ \frac{1}{2}\log(2a^2\Lambda_{\rm cc}) + \mathcal{F}(a^2\Lambda_{\rm cc}) + \mathcal{O}\left(\Lambda_{\rm cut}^{-2}\right)\end{aligned}$$

Calculation with proper-time

Being $(-\Box^{(s)}_{a=1}-\alpha)$ dimensionless $\Longrightarrow$ determinants regularized in terms of a dimensionless proper-time τ (lower cut: number $N_{\rm pt} \gg 1$)

$$\det_i(-\Box^{(s)}_{a=1}-\alpha) = e^{-\int_{1/N^2_{\rm pt}}^{+\infty} \frac{d\tau}{\tau}\, \mathrm{K}^{(s)}_i(\tau)}\,.$$

The kernel $\mathrm{K}^{(s)}_i(\tau)$ is

$$\mathrm{K}^{(s)}_i(\tau) = \sum_{n=s+i}^{+\infty} D^{(s)}_n\, e^{-\tau\left(\lambda^{(s)}_n-\alpha\right)}$$

After integration over τ, the sum over n done with the EML sum formula

$$\sum_{n=n_i}^{n_f} f(n) = \int_{n_i}^{n_f} dx\, f(x) + \frac{f(n_f)+f(n_i)}{2} + \sum_{k=1}^{p} \frac{B_{2k}}{(2k)!}\left(f^{(2k-1)}(n_f) - f^{(2k-1)}(n_i)\right) + R_{2p}$$

p is an integer, B_m are Bernoulli numbers, R_{2p} is the rest given by

$$R_{2p} = \sum_{k=p+1}^{\infty} \frac{B_{2k}}{(2k)!}\left(f^{(2k-1)}(n_f) - f^{(2k-1)}(n_i)\right) = \frac{(-1)^{2p+1}}{(2p)!}\int_{n_i}^{n_f} dx\, f^{(2p)}(x) B_{2p}(x-[x])$$

$B_n(x)$ are the Bernoulli polynomials, $[x]$ the integer part of x, and $f^{(i)}$ the i-th derivative of f with respect to its argument

Calculation with proper-time continued

Expanding for $N_{\rm pt} \gg 1$

$$\delta S^{1l}_{\rm grav} = -\left(\Lambda_{\rm cc}^2 \log N_{\rm pt}^2\right) a^4 + \Lambda_{\rm cc}\left(-N_{\rm pt}^2 + 8\log N_{\rm pt}^2\right) a^2 - \frac{N_{\rm pt}^4}{12} + \frac{17}{3}N_{\rm pt}^2 - \frac{1859}{90}\log N_{\rm pt}^2 + \mathcal{B} + \frac{1}{2}\log(2a^2\Lambda_{\rm cc}) + \mathcal{G}(a^2\Lambda_{\rm cc}) + \mathcal{O}\left(N_{\rm pt}^{-2}\right)$$

$\mathcal{G}(a^2\Lambda)$ contains UV-finite terms (no dependence on $N_{\rm pt}$). As before, the connection between $N_{\rm pt}$ and the dimensionful cutoff $\Lambda_{\rm pt}$ is given by

$$\Lambda_{\rm pt} \equiv \frac{N_{\rm pt}}{a_{\rm dS}} = \sqrt{\frac{\Lambda_{\rm cc}}{3}}\, N_{\rm pt} \qquad \Longrightarrow$$

$$\delta S^{1l}_{\rm grav} = -\left(\Lambda_{\rm cc}^2 \log\frac{3\Lambda_{\rm pt}^2}{\Lambda_{\rm cc}}\right) a^4 + \left(-3\Lambda_{\rm pt}^2 + 8\Lambda_{\rm cc}\log\frac{3\Lambda_{\rm pt}^2}{\Lambda_{\rm cc}}\right) a^2 - \frac{3\Lambda_{\rm pt}^4}{4\Lambda_{\rm cc}^2} + \frac{17\Lambda_{\rm pt}^2}{\Lambda_{\rm cc}} - \frac{1859}{90}\log\frac{3\Lambda_{\rm pt}^2}{\Lambda_{\rm cc}} + \mathcal{B} + \frac{1}{2}\log(2a^2\Lambda_{\rm cc}) + \mathcal{G}(a^2\Lambda_{\rm cc}) + \mathcal{O}\left(\Lambda_{\rm pt}^{-2}\right) .$$

Note: the two methods give the same result

Coefficients of a^4 and a^2 identify the one-loop corrections to $\frac{\Lambda_{cc}}{G}$ and $\frac{1}{G}$

$$\frac{\Lambda_{cc}^{1l}}{G^{1l}} = \frac{\Lambda_{cc}}{G}\left(1 - \frac{3G\Lambda_{cc}}{\pi}\log\frac{3L^2}{\Lambda_{cc}}\right) + \text{finite}$$

$$\frac{1}{G^{1l}} = \frac{1}{G}\left[1 + \frac{G}{2\pi}\left(3L^2 - 8\Lambda_{cc}\log\frac{3L^2}{\Lambda_{cc}}\right)\right] + \text{finite}$$

L is equivalently either Λ_{cut} or Λ_{pt} $(\sim M_P)$

Unexpected result: *only* logarithmic corrections to $\rho = \frac{\Lambda_{cc}}{8\pi G}$

Moreover: Taking for G the natural value $G \sim M_P^{-2}$ we see that quantum corrections do not spoil the naturalness of this relation

No naturalness problem with the renormaliz. of the Newton constant

$$G \sim G^{1l} \sim \frac{1}{M_P^2}$$

Vacuum energy

$$\frac{\Lambda_{cc}^{1l}}{G^{1l}} = \frac{\Lambda_{cc}}{G}\left(1 - \frac{3G\Lambda_{cc}}{\pi}\log\frac{3L^2}{\Lambda_{cc}}\right)$$

Quantum correction to the vacuum energy $\rho = \frac{\Lambda_{cc}}{8\pi G}$ goes like

$\log M_P$ rather than M_P^4

Usual result: $\rho \sim M_P^4 \implies$ bare value of $\rho \sim M_P^4$ with a coefficient that must be enormously fine-tuned for it to cancel (quite exactly) the one-loop generated M_P^4 correction

Our result: loop corrections $\rightarrow$ only mild (log) correction to $\rho \implies$

In pure gravity no naturalness problem arises: the bare cosmological constant Λ_{cc} does not need to be $\sim M_P^2$. We may naturally have $\Lambda_{cc} \ll M_P^2$, and so

$$\Lambda_{cc}^{1l} \sim \Lambda_{cc}$$

Incorrect identification of the cutoff

What is at the origin of our unexpected result?

Why usually quartic and quadratic divergences found?

Connect for a moment $N_{\rm pt}$ and $\Lambda_{\rm pt}$ through

$$\Lambda_{\rm pt} = \frac{N_{\rm pt}}{a} \qquad \left(\text{rather than through } \Lambda_{\rm pt} = \frac{N_{\rm pt}}{a_{\rm dS}}\right)$$

which corresponds to the (incorrect) identification of $\Lambda_{\rm pt}$ with the maximal eigenvalue $\widetilde{\lambda}^{(s)}_{\rm max}$ ... then for $\delta S^{1l}_{\rm grav}$ we obtain

$$\delta S^{1l}_{\rm grav} = -\left[\Lambda^2_{\rm cc} \log\left(\Lambda^2_{\rm pt}\, a^2\right)\right] a^4 + \Lambda_{\rm cc}\left[-\Lambda^2_{\rm pt}\, a^2 + 8 \log\left(\Lambda^2_{\rm pt}\, a^2\right)\right] a^2 - \frac{\Lambda^4_{\rm pt}}{12}\, a^4 + \frac{17}{3}\Lambda^2_{\rm pt}\, a^2 - \frac{1859}{90} \log\left(\Lambda^2_{\rm pt}\, a^2\right)$$

Trivially rewritten as

$$\delta S^{1l}_{\rm grav} = -\left[\frac{\Lambda^4_{\rm pt}}{12} + \Lambda_{\rm cc}\Lambda^2_{\rm pt} + \Lambda^2_{\rm cc}\log\left(\Lambda^2_{\rm pt}\, a^2\right)\right] a^4 + \left[\frac{17}{3}\Lambda^2_{\rm pt} + 8\Lambda_{\rm cc}\log\left(\Lambda^2_{\rm pt}\, a^2\right)\right] a^2 - \frac{1859}{90}\log\left(\Lambda^2_{\rm pt}\, a^2\right) .$$

known result found with heat-kernel (Taylor, Veneziano ; Fradkin, Tseytlin)

What we have just seen is that implementing the cut in the fluctuation determinants taking as physical cutoff the maximal eigenvalues $\widetilde{\lambda}^{(s)}_{\rm max}$ introduces in $\delta S^{1l}_{\rm grav}$ spurious, unphysical dependence on the metric $g^{(a)}_{\mu\nu}$

The connection between $N_{\rm pt}$ and $\Lambda_{\rm pt}$ must be realised through $a_{\rm dS}$

$a_{\rm dS}$ is the size of the universe

Additional comments

$\frac{1}{2}\log(2a^2\Lambda_{\rm cc})$ and $\mathcal{G}(a^2\Lambda_{\rm cc})$ are negligible $\mathcal{O}(1)$ contributions to $\delta S^{1l}_{\rm grav}$

The constant terms (proportional to a^0) in principle could be interpreted as corrections to $\int \mathrm{d}^4x\sqrt{g}\, R^2$ rather then as constants to be discarded

... Due to the high symmetry of the background considered (sphere), it is impossible to distinguish between constant terms and corrections to R^2

... since our universe seems to be well described by the Einstein-Hilbert action (with cosmological constant) even at large energy scales, we rather expect these terms to be interpreted as inessential constants ...

This question should be further investigated ...

Matter contribution

Consider the free theory of a real scalar field ϕ of mass m defined on the classical gravitational background $g^{(a)}_{\mu\nu}$ (sphere of radius a)

$$S = \frac{\pi\Lambda}{3G}a^4 - \frac{2\pi}{G}a^2 + \int \mathrm{d}^4x \sqrt{g^{(a)}}\left[\frac{1}{2}g^{(a)\,\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}m^2\phi^2\right]$$

Write $\phi(x) = \Phi + \eta(x)$, Φ constant background. Effective gravitational action $S^{\mathrm{eff}}_{\mathrm{grav}}$ with quantum fluctuations of ϕ included

$$S^{\mathrm{eff}}_{\mathrm{grav}}(a) = \frac{\pi\Lambda}{3G}a^4 - \frac{2\pi}{G}a^2 + \delta S_{\mathrm{grav}}$$

with δS_{grav} given by

$$e^{-\delta S_{\mathrm{grav}}} = \int\prod_x\left[\left(g^{(a)\,00}(x)\right)^{\frac{1}{2}}\left(g^{(a)}(x)\right)^{\frac{1}{4}}\mathrm{d}\eta(x)\right]e^{-\int \mathrm{d}^4x\sqrt{g^{(a)}}\left[-\frac{1}{2}\eta\Box\eta+\frac{1}{2}m^2\eta^2\right]}$$

As before $\left(g^{(a)\,00}(x)\right)^{\frac{1}{2}}\left(g^{(a)}(x)\right)^{\frac{1}{4}}$ from integration over conjugate momenta (Fradkin, Vilkovisky). Now, since $g^{(a)}_{\mu\nu} = a^2 g^{(1)}_{\mu\nu} \implies$

$\left(g^{(a)\,00}(x)\right)^{\frac{1}{2}}\left(g^{(a)}(x)\right)^{\frac{1}{4}} = a\left(g^{(1)\,00}(x)\right)^{\frac{1}{2}}\left(g^{(1)}(x)\right)^{\frac{1}{4}}$, no dimensionful parameter in

$$e^{-\delta S_{\mathrm{grav}}} = \mathcal{N}\int\prod_x\left[\mathrm{d}\,\widehat{\eta}(x)\right]e^{-\int \mathrm{d}^4x\sqrt{g^{(1)}}\left[-\frac{1}{2}\widehat{\eta}\left(\Box^{(0)}_{a=1}\right)\widehat{\eta}+\frac{1}{2}a^2m^2\widehat{\eta}^2\right]}$$

$-\Box^{(0)}_{a=1}$: Laplace-Beltrami operator for sphere of unitary radius

$\widehat{\eta} \equiv a\eta$: dimensionless fluctuation field

$\mathcal{N}$: inessential a-independent constant

Expanding $\widehat{\eta}(x)$ in terms of the eigenfunctions[2] $\phi^{(i)}_n(x)$ (i degeneracy index and $n = 0, 1, \dots$) of $-\Box^{(0)}_{a=1}$: $\widehat{\eta} = \sum_{n,i} a^{(i)}_n \phi^{(i)}_n$

$$e^{-\delta S_{\rm grav}} = \mathcal{N} \int \prod_{n,i} \mathrm{d}a^{(i)}_n \, e^{-\frac{1}{2}\sum_{n,i}\left[a^{(i)}_n\right]^2\left(\lambda^{(0)}_n + a^2 m^2\right)}$$

and then ($\mathcal{C}$ inessential a-independent constant)

$$S^{\rm eff}_{\rm grav} = \frac{\pi\Lambda}{3G} a^4 - \frac{2\pi}{G} a^2 + \frac{1}{2} \log\left[\det\left(-\Box^{(0)}_{a=1} + a^2 m^2\right)\right] + \mathcal{C} \,.$$

[2]The $\phi^{(i)}_n$ are normalized as $\int \mathrm{d}^4x \sqrt{g^{(1)}}\, \phi^{(i)}_n(x)\, \phi^{(j)}_m(x) = \delta^{ij}\delta_{nm}$.

Calculate determinant with direct product of $\lambda_n^{(0)}$ up to $n = N - 2$ as before, and expand for $N \gg 1$

$$S^{\rm eff}_{\rm grav} = \frac{\pi}{3}\left(\frac{\Lambda}{G} - \frac{m^4}{8\pi}\log N^2\right)a^4 - 2\pi\left[\frac{1}{G} - \frac{m^2}{24\pi}\left(N^2 + 2\log N^2\right)\right]a^2$$
$$+ \frac{N^4}{48}\left(-1 + 2\log N^2\right) - \frac{N^2}{72}\left(13 + 3\log N^2\right) - \frac{29}{180}\log N^2 + \mathcal{C}$$
$$+ \mathcal{H}(a^2 m^2) + \mathcal{O}\left(N^{-2}\right)$$

Similarity with the result obtained in the pure gravity case : evident

Consider the vacuum energy term. Once again : if N *correctly* related to $\Lambda_{\rm cut} \sim M_P$ through $\Lambda_{\rm cut} \sim M_P = \frac{N}{a_{\rm dS}} = N\sqrt{\frac{\Lambda_{\rm cc}}{3}}$, $\rho = \frac{\Lambda_{\rm cc}}{8\pi G}$ receives only mild logarithmically divergent correction

$$\delta\left(\frac{\Lambda_{\rm cc}}{G}\right) = -\frac{m^4}{8\pi}\log\frac{3\Lambda^2_{\rm cut}}{\Lambda_{\rm cc}}$$

... However ...

If again we perform the *incorrect* replacement of N as $N = a\,\Lambda_{\rm cut}$, again we generate spurious quartically and quadratically divergent terms. For instance:

$$-\frac{N^4}{48} \quad\longrightarrow\quad -\frac{\Lambda_{\rm cut}^4}{48}\,a^4$$

Quartically divergent contribution to $\frac{\Lambda_{\rm cc}}{G}$

$$\frac{N^2}{12}\,m^2\,a^2 \quad\longrightarrow\quad \frac{\Lambda_{\rm cut}^2}{12}\,m^2\,a^4$$

Quadratically divergent contribution to $\frac{\Lambda_{\rm cc}}{G}$

Conclusions

The **absence** of quartic and quadratic divergences in

our result for the vacuum energy

even when the presence of matter is taken into account

possibly a progress towards the solution of the CC problem

Naturally

the question of how to dispose of the terms $m^4 \log \Lambda_{\rm cut}$

needs to be further investigated

maybe along the lines put forward in the present work

ADDITIONAL SLIDES

Expansion of $\hat{h}_{\mu\nu}$, v^*_ρ and v_σ

We indicate with $h_n^{\mu\nu(i)}$ (transverse-traceless), $\xi_n^{\mu(i)}$ (transverse) and $\phi_n^{(i)}$ the pure spin-2, spin-1 and spin-0 eigenfunctions of the Laplace-Beltrami operator on the sphere of unitary radius that are normalized as

$$\delta^{ij}\delta_{nm} = \int \mathrm{d}^4x\sqrt{g^{(1)}}\, h_n^{\mu\nu(i)}(x) h_{\mu\nu}^{m(j)}(x) = \int \mathrm{d}^4x\sqrt{g^{(1)}}\, \xi_n^{\mu(i)}(x)\xi_\mu^{m(j)}(x) = \int \mathrm{d}^4x\sqrt{g^{(1)}}\, \phi_n^{(i)}(x)\,\phi_m^{(j)}(x)\,, \tag{1}$$

corresponding to the eigenvalues $\lambda_n^{(2)}$, $\lambda_n^{(1)}$ and $\lambda_n^{(0)}$ respectively. The modes $\{h_n^{\mu\nu}, v_n^{\mu\nu}, w_n^{\mu\nu}, z_n^{\mu\nu}\}$, with

$$\begin{aligned}
v_n^{\mu\nu} &= \left[\frac{1}{2}\left(\lambda_n^{(1)} - 3\right)\right]^{-\frac{1}{2}} \nabla^{(\mu}\xi_n^{\nu)}\,, \quad n = 2, \dots\,, \\
w_n^{\mu\nu} &= \left[\lambda_n^{(0)}\left(\frac{3}{4}\lambda_n^{(0)} - 3\right)\right]^{-\frac{1}{2}} \left(\nabla^\mu\nabla^\nu - \frac{1}{4}g^{(1)\,\mu\nu}\Box\right)\phi_n\,, \quad n = 2, \dots\,, \\
z_n^{\mu\nu} &= \frac{1}{2}g^{(1)\,\mu\nu}\phi_n\,, \quad n = 0, 1, 2, \dots\,,
\end{aligned} \tag{2}$$

of which we do not write explicitly the degeneracy indexes form the orthonormal basis for symmetric tensors.

Moreover, defining the longitudinal vector modes

$$l_n^\mu = \left(\lambda_n^{(0)}\right)^{-\frac{1}{2}} \nabla^\mu \phi_n\,, \quad n = 1, 2, \ldots\,, \tag{3}$$

the latter, together with the transverse modes ξ_n^μ, form the orthonormal basis for vectors.

Expand the graviton field $\widehat{h}^{\mu\nu}$ as [8]

$$\widehat{h}^{\mu\nu} = \sum_{n=2}^{\infty} a_n h_n^{\mu\nu} + \sum_{n=2}^{\infty} b_n v_n^{\mu\nu} + \sum_{n=2}^{\infty} c_n w_n^{\mu\nu} + \sum_{n=0}^{\infty} e_n z_n^{\mu\nu} \tag{4}$$

$$\widehat{h} \equiv g_{\mu\nu}^{(1)} \widehat{h}^{\mu\nu} = 2 \sum_{n=0}^{\infty} e_n \phi_n\,, \tag{5}$$

and the ghost field v^μ as

$$v^\mu = \sum_{n=1}^{\infty} g_n\, \xi_n^\mu + \sum_{n=1}^{\infty} f_n\, l_n^\mu \tag{6}$$

so that we have

$$
\begin{aligned}
64\pi G\,(S_2 + S_{\rm gf}) = &\sum_{n=2}^{\infty} a_n^2 \left[\lambda_n^{(2)} - 2a^2\Lambda + 8\right] \\
&+ \sum_{n=2}^{\infty} b_n^2 \left[\xi^{-1}\left(\lambda_n^{(1)} - 3\right) - 2a^2\Lambda + 6\right] \\
&+ \sum_{n=2}^{\infty} c_n^2 \left[\xi^{-1}\left(\frac{3}{4}\lambda_n^{(0)} - 6\right) - \frac{\lambda_n^{(0)}}{2} - 2a^2\Lambda + 6\right] \\
&+ \sum_{n=0}^{\infty} e_n^2 \left[\frac{-3+\xi^{-1}}{2}\lambda_n^{(0)} + 2a^2\Lambda\right] \\
&+ \sum_{n=2}^{\infty} 2e_n c_n(\xi^{-1} - 1)\left[\lambda_n^{(0)}\left(\frac{3}{4}\lambda_n^{(0)} - 3\right)\right]^{\frac{1}{2}}
\end{aligned}
\qquad (7)
$$

$$
32\pi G\, S_{\rm ghost} = \sum_{n=1}^{\infty} g_n^*\, g_n \left(\lambda_n^{(1)} - 3\right) + \sum_{n=1}^{\infty} f_n^*\, f_n \left(\lambda_n^{(0)} - 6\right)\ . \qquad (8)
$$

Therefore, the functional measure in (**??**) can be written as (defined as)

$$\mathcal{D}\widehat{h}_{\mu\nu}\,\mathcal{D}v_\rho^*\,\mathcal{D}v_\sigma \equiv \frac{1}{V_{SO(5)}}\prod_{n=2}^{\infty}\mathrm{d}a_n\prod_{n=2}^{\infty}\mathrm{d}b_n\prod_{n=2}^{\infty}\mathrm{d}c_n\prod_{n=0}^{\infty}\mathrm{d}e_n\prod_{n=2}^{\infty}\mathrm{d}g_n^*\prod_{n=2}^{\infty}\mathrm{d}g_n\prod_{n=1}^{\infty}\mathrm{d}f_n^*\prod_{n=1}^{\infty}\mathrm{d}f_n\,, \tag{9}$$

Notice that there is no integration over the zero modes g_1^* and g_1 of S_{ghost} [16]. The corresponding ghost fields are proportional to the ten Killing vectors ξ_1^μ. These zero eigenmodes correspond to residual gauge invariances which are not eliminated by gauge fixing in the presence of an $SO(5)$ spherical symmetry. Overcounting has been compensated by inserting the explicit group-volume factor $V_{SO(5)}$ in Eq. (9) (see, e.g., [17]).

Sum over the eigenvalues in closed form

$$
\begin{aligned}
F(a^2\Lambda) = {} & 9\Lambda a^2 - \frac{1}{6}\Lambda\sqrt{8\Lambda a^2+9}\log\Gamma\left(\frac{7}{2}-\frac{1}{2}\sqrt{8\Lambda a^2+9}\right)a^2 - 5\Lambda\psi^{(-2)}\left(\frac{1}{2}\left(\sqrt{8a^2\Lambda-15}+7\right)\right)a^2 \\
& - 5\Lambda\psi^{(-2)}\left(\frac{7}{2}-\frac{1}{2}\sqrt{8a^2\Lambda-15}\right)a^2 - \Lambda\psi^{(-2)}\left(\frac{1}{2}\left(\sqrt{8\Lambda a^2+9}+7\right)\right)a^2 \\
& - \Lambda\psi^{(-2)}\left(\frac{7}{2}-\frac{1}{2}\sqrt{8\Lambda a^2+9}\right)a^2 + \frac{1}{6}\Lambda\log\Gamma\left(\frac{1}{2}\left(\sqrt{8\Lambda a^2+9}+7\right)\right)\sqrt{8\Lambda a^2+9}a^2 \\
& - 5\log(120) + \frac{49\log(A)}{3} - 2\sqrt{\frac{11}{3}}\log\Gamma\left(\frac{1}{2}\left(\sqrt{33}+7\right)\right) \\
& - \frac{5}{6}\left(a^2\Lambda-5\right)\sqrt{8a^2\Lambda-15}\log\Gamma\left(\frac{7}{2}-\frac{1}{2}\sqrt{8a^2\Lambda-15}\right) \\
& - \frac{1}{6}\sqrt{8\Lambda a^2+9}\log\Gamma\left(\frac{7}{2}-\frac{1}{2}\sqrt{8\Lambda a^2+9}\right) + 3\psi^{(-4)}(1) + 3\psi^{(-4)}(6) + \psi^{(-4)}\left(\frac{7}{2}-\frac{\sqrt{33}}{2}\right) \\
& + \psi^{(-4)}\left(\frac{1}{2}\left(\sqrt{33}+7\right)\right) - 5\psi^{(-4)}\left(\frac{1}{2}\left(\sqrt{8a^2\Lambda-15}+7\right)\right) - 5\psi^{(-4)}\left(\frac{7}{2}-\frac{1}{2}\sqrt{8a^2\Lambda-15}\right) \\
& - \psi^{(-4)}\left(\frac{1}{2}\left(\sqrt{8\Lambda a^2+9}+7\right)\right) - \psi^{(-4)}\left(\frac{7}{2}-\frac{1}{2}\sqrt{8\Lambda a^2+9}\right) + \frac{15\psi^{(-3)}(1)}{2} - \frac{15\psi^{(-3)}(6)}{2} \\
& - \frac{1}{2}\sqrt{33}\psi^{(-3)}\left(\frac{1}{2}\left(\sqrt{33}+7\right)\right) - \frac{5}{2}\sqrt{8a^2\Lambda-15}\psi^{(-3)}\left(\frac{7}{2}-\frac{1}{2}\sqrt{8a^2\Lambda-15}\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2}\sqrt{8\Lambda a^2+9}\,\psi^{(-3)}\left(\frac{7}{2}-\frac{1}{2}\sqrt{8\Lambda a^2+9}\right)+\frac{33\psi^{(-2)}(1)}{4}+\frac{33\psi^{(-2)}(6)}{4}\\
&+\frac{49}{12}\psi^{(-2)}\left(\frac{7}{2}-\frac{\sqrt{33}}{2}\right)+\frac{49}{12}\psi^{(-2)}\left(\frac{1}{2}\left(\sqrt{33}+7\right)\right)+\frac{175}{12}\psi^{(-2)}\left(\frac{1}{2}\left(\sqrt{8a^2\Lambda-15}+7\right)\right)\\
&+\frac{175}{12}\psi^{(-2)}\left(\frac{7}{2}-\frac{1}{2}\sqrt{8a^2\Lambda-15}\right)-\frac{13}{12}\psi^{(-2)}\left(\frac{1}{2}\left(\sqrt{8\Lambda a^2+9}+7\right)\right)\\
&-\frac{13}{12}\psi^{(-2)}\left(\frac{7}{2}-\frac{1}{2}\sqrt{8\Lambda a^2+9}\right)+\frac{1}{2}\psi^{(-3)}\left(\frac{7}{2}-\frac{\sqrt{33}}{2}\right)\sqrt{33}+2\log\Gamma\left(\frac{7}{2}-\frac{\sqrt{33}}{2}\right)\sqrt{\frac{11}{3}}\\
&+\frac{5}{6}\left(a^2\Lambda-5\right)\log\Gamma\left(\frac{1}{2}\left(\sqrt{8a^2\Lambda-15}+7\right)\right)\sqrt{8a^2\Lambda-15}\\
&+\frac{5}{2}\psi^{(-3)}\left(\frac{1}{2}\left(\sqrt{8a^2\Lambda-15}+7\right)\right)\sqrt{8a^2\Lambda-15}\\
&+\frac{1}{6}\log\Gamma\left(\frac{1}{2}\left(\sqrt{8\Lambda a^2+9}+7\right)\right)\sqrt{8\Lambda a^2+9}\\
&+\frac{1}{2}\psi^{(-3)}\left(\frac{1}{2}\left(\sqrt{8\Lambda a^2+9}+7\right)\right)\sqrt{8\Lambda a^2+9}+\frac{7\zeta(3)}{4\pi^2}-\frac{2}{3}\zeta'(-3)-\frac{20801}{1080}
\end{aligned}
$$

$$g_{\mu\nu}^{(a)} = \begin{pmatrix} a^2 & 0 & 0 & 0 \\ 0 & a^2 \sin^2\theta_1 & 0 & 0 \\ 0 & 0 & a^2 \sin^2\theta_1 \sin^2\theta_2 & 0 \\ 0 & 0 & 0 & a^2 \sin^2\theta_1 \sin^2\theta_2 \sin^2\theta_3 \end{pmatrix}$$

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