Spin-orbit duality

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Summary

I. Summary of the spin-orbit duality

II. Duality as a Hopf fibration and the conformal group

III. Oscillator vs Ising model

IV. QED realization

Part I
The duality

Part I – The duality

Noether's theorem on Lorentz invariance, $\dot{J}^{\mu\nu} = 0$, decomposes

$$J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu} \tag{1}$$

This **kinematic/dynamic** complementarity is made geometric by $(1+3)$-decomposing wrt $\eta^{\mu\nu} = \frac{p^\mu p^\nu}{p^2} + h^{\mu\nu}$,

$$J^{\mu\nu} = E^{\mu\nu} + H^{\mu\nu} : \qquad \{p_\nu \star E^{\mu\nu} = 0\,,\; H^{\mu\nu} p_\nu = 0\} := \mathcal{H} \tag{2}$$

This is a **Hodge decomposition**, generalization of the $\mathbb{R}^3$ Helmholtz decomposition (into curl-free and divergence-free parts). For $J^{\mu\nu}$,

$$p_\nu \star L^{\mu\nu} = 0 \quad \text{and} \quad S^{\mu\nu} p_\nu = 0 \quad \text{(SSC)} \tag{3}$$

Algebraically, SSC $=$ $S^{\mu\nu}$ set as generators of the little group.

Part I – The duality

Hodge decomposition separates between **electric** and **magnetic** parts,

$$J \; = \; E \wedge p \; + \; \star(p \wedge H) \tag{4}$$

where

$$\begin{aligned} E^\mu &= \frac{L^{\mu\nu} p_\nu}{p^2} = n^\mu && \text{spacelike four-position} \\ H^\mu &= \frac{p^\nu \star S^{\mu\nu}}{p^2} = W^\mu && \text{Pauli-Lubanski (position) vector} \end{aligned} \tag{5}$$

Then, if $p^\mu \mapsto p^\mu$, spin-orbit duality is an **electric-magnetic duality**,

$$\boxed{\begin{aligned} n^\mu &\mapsto W^\mu \\ W^\mu &\mapsto -n^\mu \end{aligned} \qquad \Leftrightarrow \qquad J \mapsto \star J}$$

Part I – The duality

`Why is this transformation a` **meaningful** `duality?`

▷ It is an automorphism of $\mathcal{H}$ (original motivation).

▷ It preserves the Poincarè conservation laws $\dot{J} = \dot{p} = 0$.

▷ For $F^{\mu\nu}$, in the rest frame, it is the usual U(1) electromagnetic duality.

Part I – The duality

• `Algebraically`, the Lorentz algebra $\mathfrak{so}(1,3)$ is preserved.
(Hints: $\dot{J} = 0$ and $\star$ is a linear map that shifts orthonormal basis).

• `Geometrically`, $J \mapsto \star J$ is a swap between rotations and boosts, i.e. the topological invariance

$$\mathbf{RP}^3 \times \mathbf{R}^3 \;\; \mapsto \;\; \mathbf{R}^3 \times \mathbf{RP}^3$$

• `Translation generators` are preserved (hint: $\dot{p} = 0$). But spacetime transforms and those are not translations anymore. The Poincarè group transforms.

Part I – The duality

For **Poincarè generators**, their possible compositions are

$$\mathbf{W} := \frac{\star(\mathbf{J} \wedge \mathbf{P})}{\mathbf{P}^2} \qquad \text{and} \qquad \mathbf{N} := \frac{\mathbf{J} \cdot \mathbf{P}}{\mathbf{P}^2} \tag{6}$$

whereas $\mathbf{L} = \mathbf{N} \wedge \mathbf{P}$. Then, $\mathbf{W}$ generates SO(3) and $\mathbf{N}$ boosts,

$$[\mathbf{W}, \mathbf{W}] = \frac{\mathbf{J}}{\mathbf{P}^2}\,, \qquad [\mathbf{W}, \mathbf{N}] = \frac{\star\mathbf{J}}{\mathbf{P}^2}\,, \qquad [\mathbf{N}, \mathbf{N}] = -\frac{\mathbf{J}}{\mathbf{P}^2} \tag{7}$$

The duality is

$$\begin{matrix} \mathbf{N} \mapsto \mathbf{W} \\ \mathbf{W} \mapsto -\mathbf{N} \end{matrix} \qquad \Leftrightarrow \qquad \mathbf{J} \mapsto \star\mathbf{J} \tag{8}$$

It leaves the $\mathbf{W}, \mathbf{N}$ algebra invariant $\leftrightarrow$ $\mathfrak{so}(1,3)$ and $\mathcal{H}$ are preserved.
It *does not* say anything (yet) for the Poincarè algebra.

Part I – The duality

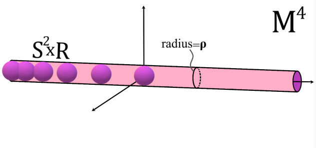
The duality **maps the coordinates** $n^\mu \mapsto \tilde{n}^\mu := W^\mu$, becoming trivial at

$$\rho = \sqrt{W^2} = \frac{S}{m} \qquad \text{or} \qquad \hat{\rho} = \frac{\hbar\sqrt{s(s+1)}}{m} \qquad \textbf{(Møller radius)} \tag{9}$$

This is a **conformal immersion** $\mathbb{R}^3 \setminus \{0\} \to \mathbb{S}^2$. The **holographic** map

$$\boxed{\mathbb{R}^{1,3} \quad \mapsto \quad \mathbb{S}^2 \times \mathbb{R}}$$

▷ ρ is a natural localization boundary: Classically, envelopes region of non-covariance. Quantum-mechanically, $\hat{\rho} \sim \lambda_C$, signifies pair production.



- In fact: defining the timelike position as $\mathbf{A} = \frac{\mathbf{DP}}{\mathbf{P}^2}$: $\mathbf{X} = \mathbf{A} + \mathbf{N}$,

$$[\mathbf{X}^\mu, \mathbf{X}^\nu] \; = \; -\frac{\mathbf{S}^{\mu\nu}}{\mathbf{P}^2} \tag{10}$$

Formally, this means a massive theory with spin is **noncommutative**. This was first seen in `relativistic mechanics` by [Pryce1948] and on the `superparticle` by [Casalbuoni1976] and [Brink&Schwarz1981].

▷ This sets the **fundamental scale** at $\hat{\rho} \sim \lambda_C$, exactly on $\mathbb{S}^2 \times \mathbb{R}$.

▷ It reaffirms $\hat{\rho}$, where duality becomes trivial, as natural QM boundary.

Part I – The duality

The dual theory on $\mathbb{S}^2 \times \mathbb{R}$ is **noncommutative**,

$$\begin{aligned} [\hat{X}^\mu, \hat{X}^\nu] &= \frac{i}{p^2}\left(\hat{X}^\mu p^\nu - \hat{X}^\nu p^\mu + \epsilon^{\mu\nu\rho\sigma}\hat{X}_\rho p_\sigma\right) \\ [\hat{X}^\mu, \hat{p}^\nu] &= i\frac{\hat{p}^\mu \hat{p}^\nu}{p^2} \end{aligned} \tag{11}$$

Minus the 3rd term, it is a κ-**deformation** of the Poincarè-Hopf algebra, with $\kappa = m$. Also,

$$[\hat{X}^i, \hat{p}^0] = -i\frac{\hat{p}^i}{\hat{p}^0} \qquad \rightarrow \qquad \textbf{Newton-Wigner localization} \tag{12}$$

In the rest frame, or in the **low-energy** regime,

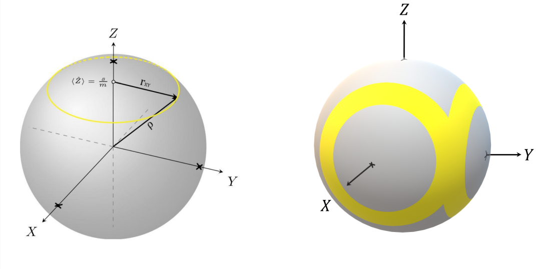
$$\begin{aligned} [\hat{X}^0, \hat{X}^i] &= -i\frac{\hat{X}^i}{m} && \rightarrow && \kappa\textbf{-Minkowski} \\ [\hat{X}^i, \hat{X}^j] &= -i\lambda_C \epsilon^{ijk}\hat{X}_k && \rightarrow && \textbf{fuzzy sphere} \end{aligned} \tag{13}$$

Part I – The duality

In QM vacuum, the duality implies

$$\langle \hat{X}^i \rangle = \frac{\langle \hat{S}^i \rangle}{m} \tag{14}$$

$2s+1$ states –on the dual fuzzy sphere– are **uncertainty rings**:



▷ Left: $\boxed{\langle \hat{Z} \rangle = \frac{\langle \hat{S}_Z \rangle}{m} = \frac{s}{m}} \xrightarrow{\langle \hat{X} \rangle = \langle \hat{Y} \rangle = 0} \{\Delta Z = 0 \ , \ \Delta X \Delta Y = r_{XY}\}.$

▷ Right: $\boxed{\langle \hat{X} \rangle = \langle \hat{Y} \rangle = \frac{s}{\sqrt{2}m}} \xrightarrow{\langle \hat{Z} \rangle = 0} \{\Delta Z, \Delta X, \Delta Y \neq 0\}.$

Part II
Duality as a Hopf fibration and the conformal group

Part II – Duality as a Hopf fibration

In Euclidean signature, since:

▷ n^μ ($n^\mu n_\mu > 0$) is an SO(4) rep, foliating $\mathbb{R}^4$ into concentric $\mathbb{S}^3$'s,

▷ $\mathbb{S}^3 \cong \mathrm{SU}(2)$ is a U(1)-bundle, since the homogeneous $\mathbb{S}^2 \cong \mathrm{SU}(2)/\mathrm{U}(1)$,

$$n^\mu \mapsto W^\mu \qquad = \qquad \textbf{1st Hopf map} \quad \mathbb{S}^3 \xrightarrow{\mathbb{S}^1} \mathbb{S}^2$$

In this view, the duality induces the conformal immersion

$$\mathbb{R}^4 \setminus \{0\} \cong \mathbb{S}^3 \times \mathbb{R} \quad \to \quad \mathbb{S}^2 \times \mathbb{R} \tag{15}$$

Part II – Duality as a Hopf fibration

Realization: SU(2) spinor $\psi: \ \psi^\dagger \psi = \mathit{const.}$, a hypersurface $\mathbb{S}^3 \subset \mathbb{C}^2$. The Hopf map is $\mathbb{S}^3 \to \mathbb{S}^2 \subset \mathbb{R}^3$,

$$\psi \quad \to \quad x^i = \psi^\dagger \sigma^i \psi \tag{16}$$

where $x^2 = (\psi^\dagger \psi)^2 = \mathit{const.} \ \Rightarrow \ x^i \in \mathbb{S}^2$.

Example: the 4D CBS superparticle with action

$$S_{CBS} = \int \mathrm{d}t \ e^{-1}(\dot{x}^\mu - i\dot{\theta}\sigma^\mu \bar{\theta} + i\theta\sigma^\mu \dot{\bar{\theta}})^2 - em^2 \tag{17}$$

feels the duality

$$x^i \quad \mapsto \quad \tilde{x}^i = W^i = \theta \sigma^i \bar{\theta} \tag{18}$$

which realizes the Hopf map.

Part II – Duality and the conformal group

$\mathbb{R}^{1,3} \mapsto \mathbb{S}^2 \times \mathbb{R}$ yields that the bulk **$\mathbf{G} =$ ISO(1,3) transforms**:

▷ SO(1,3) subgroup is preserved,

▷ translations ($\dot{p} = 0$ preserved) are realized projectively,

$$\boxed{\tilde{G} \ = \ \mathrm{SO}(2,3)}$$

- $\mathrm{SO}(2,3) \cong \mathrm{Conf}(1,2) \cong \mathrm{Conf}(\mathbb{S}^2 \times \mathbb{R})$.
- SO(1,3) is now realized as $\mathrm{Conf}(2) = \mathrm{Conf}(\mathbb{S}^2)$.
- The inverse map $\tilde{G} \mapsto G$ may be an Inonu-Wigner contraction.

Part III
Oscillator vs Ising model

Part III – Dual Landau levels

The simplest arena is a spin-s charge in a uniform magnetic field, $B^i = \epsilon^{ijk}\partial_j A_k$, producing the **Landau levels** ($\omega_c = \frac{B}{m}$),

$$\mathcal{H} = \frac{1}{2m}\left(p^i + A^i(x^j)\right)^2 \ , \qquad E_n = \omega_c\left(n + \frac{1}{2}\right) \tag{19}$$

The duality $\mathbb{R}^3 \mapsto \mathbb{S}^2$ takes $x^i \mapsto \tilde{x}^i$, with $\tilde{x}^i \in \mathbb{S}^2$ (i.e. $\tilde{x}^2 = \rho^2$) and

$$\tilde{\mathcal{H}} = \frac{1}{2m}\left(p^i + A^i(\tilde{x}^j)\right)^2 \ , \qquad \tilde{E}_n = \frac{1}{2m\rho^2}\left(n^2 + n(2s+1) + s\right) \tag{20}$$

where Hopf map $\mathbb{S}^3 \xrightarrow{\mathbb{S}^1} \mathbb{S}^2$ takes the U(1) connection $A^i(x^j) \mapsto A^i(\tilde{x}^j)$, the potential of a **Dirac monopole** of minimum charge.

Part III – Dual Landau levels

The dual monopole problem on $\mathbb{S}^2$ has **Lowest Landau Level**:

- $\tilde{E}_0 (= E_0) = \frac{\omega}{2}$,
- $(2s+1)$-fold degenerate,
- $2s+1$ Landau orbitals, a spin-s SO(3) rep: **fuzzy sphere**.

$$\updownarrow$$

Original postulate of the duality: the vacuum on the dual $\mathbb{S}^2$ is a fuzzy sphere of $2s+1$ eigenstates. ✓

Part III – Dual Landau levels

Taking $\rho, s \to \infty$, holding $B = \frac{s}{\rho^2}$ fixed, is the **thermodynamic limit**,

$$\tilde{E}_n \quad \xrightarrow{TL} \quad E_n = \omega \left(n + \frac{1}{2} \right) \tag{21}$$

- But, what is the interpretation of TL on the dual spectrum?

▷ $\mathbb{S}^3 \to \mathbb{S}^2$ is a conformal immersion, hence $\rho \to \infty$ is the inverse map ($\mathbb{S}^2 \to \mathbb{S}^3$) wrt spectral parameters:

`TL is the duality on the spectrum.`

▷ The dual theory is conformal, hence TL is actually mandatory for the correct spectrum:

`The dual spectra match,` $\tilde{E}_n = E_n$.

Part III – Oscillator vs Ising model

For uniform $B^i = \epsilon^{ijk}\partial_j A_k$, the generic form of the Hamiltonian is

$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2}m\omega_c^2 x^2 + \omega_L B \cdot L \tag{22}$$

The duality takes $x^i \mapsto \frac{S^i}{m}$ –and also $L^i \mapsto S^i$– hence

$$\tilde{\mathcal{H}} = \frac{p^2}{2m} + \frac{\omega_c^2}{2m} S^2 + \omega_L B \cdot S \tag{23}$$

- This is an **Ising model** for just one electron:

▷ The 1st term, with p^i conjugate to $\tilde{x}^i = \frac{S^i}{m}$, only makes sense on $\mathbb{S}^2$.

▷ The 2nd term is self-interaction, a QM memory term (new $\propto$ old state).

▷ The 3rd term is the usual coupling between S^i and external B^i.

Part III – Oscillator vs Ising model

Disregarding electric repulsion (wrt the external B^i), consider N electrons, i.e. the center-of-mass position $x^i = (x_1^i + \ldots + x_N^i)/N$,

$$\mathcal{H} = \sum^{N} \left(\frac{p^2}{2m} + \frac{1}{2} m\omega_c^2 x^2 + \omega_L B \cdot L \right) + m\omega_c^2 \sum_{a \neq b}^{N} x_a \cdot x_b \qquad (24)$$

and the duality implies

$$\tilde{\mathcal{H}} = \sum^{N} \left(\frac{p^2}{2m} + \frac{\omega_c^2}{2m} S^2 + \omega_L B \cdot S \right) + \frac{\omega_c^2}{m} \sum_{a \neq b}^{N} S_a \cdot S_b \qquad (25)$$

- This is an **Ising model** for N electrons:

▷ The new term is the known inter-site interaction. It is between all possible spin-lattice sites: i.e. not only for next-neighbor (short-range) interactions but for long-range ones too.

▷ How to interpret its independence of inter-site distance?

Part IV
QED realization

Part IV – QED realization

In field theory, the simplest example is **massive spinors** in external A^μ,

$$S = \int \mathrm{d}^4 x \; i\bar{\psi}\not{D}\psi - m\bar{\psi}\psi \tag{26}$$

▶ In analogy, we understand the duality to:

▷ leave the kinetic term invariant,
▷ shift A^μ into a monopole,
▷ transform the mass term.

▶ The **mass term** should somehow transform, since:

▷ the dual theory on $\mathbb{S}^2 \times \mathbb{R}$ is conformal, $\tilde{G} = \mathrm{SO}(2{,}3)$,
▷ $\bar{\psi}\psi$ is the **probability density**, a field analog of position.

Part IV – QED realization

There is an elegant way to realize the duality. The generalized momenta $\Pi_\mu = i\partial_\mu\psi, \bar{\Pi}_\mu = i\partial_\mu\bar{\psi}$ define a kind of **generalized field coordinates**,

$$\Psi^\mu := \frac{\gamma^\mu\psi}{2\sqrt{-p^2}} \quad \text{and} \quad \overline{\Psi}^\mu := -\frac{\bar{\psi}\gamma^\mu}{2\sqrt{-p^2}} \,, \tag{27}$$

- Those make sense, because:

▷ $\overline{\Psi}\cdot\Psi = \frac{\bar{\psi}\psi}{m^2}$ is the probability density, analog of position,

▷ $[\overline{\Psi}^\mu, \Psi^\nu] = -\frac{i\bar{\psi}\,\mathsf{S}^{\mu\nu}\psi}{p^2}$, same as the underlying noncommutative algebra.

Part IV – QED realization

We may even extract a **spacelike coordinate** N^μ, analog of n^μ, by considering the projector $A_{\mu\nu} = i^2 \overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu / p^2$,

$$\overline{\Psi} \cdot \mathrm{N} \;=\; \overline{\Psi} \cdot \Psi - \overline{\Psi} \cdot (A \cdot \Psi) \;=\; \frac{\bar{\psi}\psi}{m^2} - \frac{1}{4}\frac{\bar{\psi}\psi}{m^2} \;=\; \frac{3}{4}\frac{\bar{\psi}\psi}{m^2}\,, \tag{28}$$

The numerical factors naturally decompose into timelike/spacelike dof. Manipulating the Dirac equation, we obtain an explicit expression,

$$\mathrm{N}^\mu := \frac{\mathbf{S}^{\mu\nu}\partial_\nu\psi}{p^2} \quad \text{and} \quad \overline{\mathrm{N}}^\mu := \frac{\partial_\nu\bar{\psi}\,\mathbf{S}^{\nu\mu}}{p^2}\,. \tag{29}$$

Moreover, it turns out we may isolate the spatial dof into $\bar{\psi}\psi$,

$$\bar{\psi}\psi \;\to\; \overline{\Psi} \cdot \mathrm{N} \tag{30}$$

Part IV – QED realization

We may even define an analog of orbital angular momentum acting on Dirac spinors,

$$\mathfrak{L}^{\mu\nu} := \frac{\mathbf{S}^{\mu\rho}\partial_\rho}{p^2}\partial_\nu - \frac{\mathbf{S}^{\nu\rho}\partial_\rho}{p^2}\partial_\mu \tag{31}$$

Then, the total angular momentum generator,

$$\mathfrak{J}^{\mu\nu} \ = \ \mathfrak{L}^{\mu\nu} \ + \ \mathfrak{S}^{\mu\nu} \ , \tag{32}$$

where $\mathfrak{S}^{\mu\nu} = \mathbf{S}^{\mu\nu}/2$, satisfies the Lorentz algebra. Hence, the duality $J^{\mu\nu} \mapsto \star J^{\mu\nu}$ is (in this representation) $\mathfrak{J}^{\mu\nu} \mapsto \star\mathfrak{J}^{\mu\nu}$. Equally,

$$\boxed{\mathrm{N}^\mu \quad \mapsto \quad \mathrm{W}^\mu}$$

where $\mathrm{W}^\mu = (i\,\overrightarrow{\partial}_\nu \star \mathbf{S}^{\mu\nu}\psi)/p^2$.

Part IV – QED realization

Hence, the duality transforms the mass term,

$$\bar{\Psi} \cdot \mathrm{N} \quad \mapsto \quad \bar{\Psi} \cdot \mathrm{W} \tag{33}$$

or, wrt Dirac spinors,

$$\begin{aligned} m\,\bar{\psi}\psi \quad &\mapsto \quad i\,\frac{\bar{\psi}\gamma^{\mu}}{2}(\partial^{\nu} \star \mathbf{S}_{\mu\nu})\psi \\ &= \frac{i}{4}\bar{\lambda}\gamma^{\alpha}\left[e^{b}_{\beta}\,\nabla^{\beta} e^{a}_{\alpha}\right]\sigma_{ab}\lambda \end{aligned} \tag{34}$$

Here, $\gamma^{\mu} = \gamma^{a} e^{\alpha}_{a} e^{\mu}_{\alpha}$: e^{α}_{a} = 3D vielbein and e^{μ}_{α} = 4D/3D duality map. Also, λ are Weyl spinors. Finally,

$$\boxed{m\,\bar{\psi}\psi \quad \mapsto \quad \frac{i}{4}\bar{\lambda}\,\gamma^{\alpha}\,\omega_{\alpha}{}^{ab}\,\sigma_{ab}\,\lambda}$$

where ω is the **spin connection** on $\mathbb{S}^2 \times \mathbb{R}$.

Part IV – QED realization

Hence, the duality transforms the action,

$$S = \int_{\mathbb{R}^{1,3}} i\bar{\psi}\not{D}\psi - m\bar{\psi}\psi \quad \mapsto \quad \tilde{S} = \int_{\mathbb{S}^2\times\mathbb{R}} i\,\bar{\lambda}\gamma^\alpha \left(D_\alpha - \frac{1}{4}\omega_\alpha{}^{ab}\,\sigma_{ab}\right)\lambda$$

where N_f massive 4D Dirac spinors realize $2N_f$ massless 3D Weyl's.

▶ Hence, the 4D mass term transforms:

▷ in analogy with position, representing the probability density,

▷ into a massless structure, since the dual theory must be conformal,

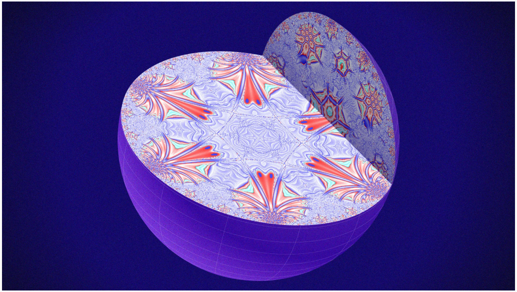
▷ into exactly the spin connection needed for the dual $\mathbb{S}^2 \times \mathbb{R}$.

Part IV – Nested holography

The dual theory has $\tilde{G} = SO(2,3) = \mathrm{Conf}(1,2) = \mathrm{Isom}(\mathrm{AdS}_4)$. Seeing it as the conformal class of $\mathbb{S}^2 \times \mathbb{R}$, it's the **conformal boundary of AdS$_4$**.

- The $\mathbb{S}^2 \times \mathbb{R}$ cylinder continues inside to AdS$_4$.
- The **AdS/CFT duality**, we realize a **nested holography**:

$$\begin{array}{c}\text{massive QFT}\\ \text{on } \mathbb{R}^{1,3}\end{array} \; \xrightleftharpoons[\text{duality}]{\text{spin-orbit}} \; \begin{array}{c}\text{(massless) CFT}\\ \text{on } \mathbb{S}^2 \times \mathbb{R}\end{array} \; \xrightleftharpoons[\text{duality}]{\text{AdS/CFT}} \; \begin{array}{c}\text{supergravity}\\ \text{on } \mathrm{AdS}_4 \times X^6\end{array}$$



The end

thanks!