# *Caustics in Self-gravitating N-body and Large Scale Structure of Universe*

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## *Caustics in self-gravitating N-body systems and large scale structure of universe*

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This figure shows galaxies discovered by the Sloan Digital Sky Survey (SDSS). Galaxy filaments forming the cosmic web consist of walls of gravitationally bound galactic superclusters that can be seen by eye. The figure shows galaxies up to around 2 billion light-years away (z=0.14). Figure Credit: M. Blanton and SDSS



The space-time distribution of galaxies as a function of redshift. This DESI data has the Earth on the left and looks back in time to the right. Every dot represents a galaxy (blue) or quasar (red). The upper wedge includes objects all the way back to about 12 billion years ago. The bottom wedge zooms in on the closer galaxies in more detail. The clumps, strands, and blank spots are real structures in the Universe showing how galaxies group together or leave voids on gigantic scales. Figure Credit: Eleanor Downing/ DESI collaboration



Figure 3: The matter power spectrum (at  $z = 0$ ) inferred from different cosmological probes showing how CMB, LSS, clusters, weak lensing, and  $Ly-\alpha$  forest all constrain matter power spectrum P(k). The spectrum measures the power of matter fluctuations on a given scale k. For the long wave length perturbations it has power-law behaviour  $P(k) \propto k^{ns}$  with the scalar spectral index  $n_s = 0.967 \pm 0.004$ , tilted away from the scale invariant  $n_s = 1$  Harisson-Zeldovich spectrum. The sound waves diminish the strength of small scale fluctuations, and power spectrum tends to fall as  $P(k) \propto k^{-3}$  for  $k \ge 2 \times 10^{-2}$ [h Mps<sup>-1</sup>].

Galaxies are not distributed uniformly in space and time, as it can be seen in Fig. 1 and Fig. 2 representing the data of the Sloan Digital Sky Survey and of the Dark Energy Spectroscopic Instrument collaboration. *Extended galaxy redshift surveys revealed that at a large-scale the Universe consists of matter*  concentrations in the form of galaxies and clusters of galaxies of *Mpc scale, as well as filaments of galaxies that are larger than 10 Mpc in length and vast regions devoid of galaxies.* The James Webb ST telescope and the Euclid mission will observe the first stars and galaxies that formed in the Universe from the epoch of recombination to the present day. The Large Scale Structure (LSS) of the Universe is this pattern of galaxies that provides information about the spectrum of matter density fluctuations shown in Fig. 3

The prevailing theoretical paradigm regarding the existence of LSS is that the initial density fluctuations of the early Universe seen as *temperature deviations* in the Cosmic Microwave Background (CMB) grow through *gravitational instability into the structure seen today in the galaxy density field*. The best constraints on the matter density fluctuations come from the study of the CMB *temperature fluctuations generated at the epoch of the last scattering of the radiation*. *The LSS of galaxies provides independent measurements of density fluctuations of similar physical scale, but at the late epoch.* The combination of CMB measurements with measurements of LSS provide independent probes of the matter power spectrum in complementary regions shown in Fig.3.

# *Light caustics on a seabed*



### *Caustics in Yang Mills Classical Mechanics*



Yang-Mills mechanical system  $\mathcal{L}_{YM} = \frac{1}{2} (\dot{x}^2 + \dot{y}^2) - \frac{1}{2} x^2 y^2$ 

### *Caustics in Yang Mills Classical Mechanics*



# Yang-Mills mechanical system  $\mathcal{L}_{YM} = \frac{1}{2} (\dot{x}^2 + \dot{y}^2) - \frac{1}{2} x^2 y^2$

### Geometrisation of Self-Gravitating N-body System Coomotrication of Solf Crowitating NLK

Let us consider a system of  $N$  massive particles with masses  $M_{\alpha}$  and the coordinates *d*<sup>2</sup>*q– ds*<sub>2</sub> *N* m *<u>xu***d***masse*</u>  $M_{\alpha}$  and the coordinates

$$
q^{\alpha}(s) = (M_1^{1/2} \vec{r_1}, ..., M_N^{1/2} \vec{r_N}), \quad \alpha = 1, ..., 3N,
$$
 (2.6)

that are defined on a Riemannian coordinate manifold  $q^{\alpha}(s) \in Q^{3N}$  and have the velocity vector that are defined on a Riemannian coordinate manifold  $a^{\alpha}(s) \in \Omega^{3N}$  and have the velocity It is fundamentally important that the definition of the coordinates *q–* includes the masses of linate manifold  $q^{\alpha}(s) \in Q^{3N}$  and have the velocity

$$
u^{\alpha}(s) = \frac{dq^{\alpha}}{ds},\tag{2.7}
$$

$$
ds^2 = g_{\alpha\beta} dq^{\alpha} dq^{\beta}, \qquad g_{\alpha\beta} = \delta_{\alpha\beta} (E - U(q)) = \delta_{\alpha\beta} W(q),
$$

 $t_1$  is a conformally flat Maupertuist metric on  $\ddot{C}$  is defined as  $\ddot{C}$  is defined as  $\ddot{C}$  $\frac{1}{2}$  $w_1$  interacting through the potential  $\cdots$   $w_n$  (see Appendix A). An  $\cdots$  and  $\cdots$  system interaction system interaction  $\cdots$ can be in a background field of the expanding  $\mathcal{C}$  of the potential function  $\mathcal{C}$ geodesic trajectories on the Riemannian manifold *Q*<sup>3</sup>*<sup>N</sup>* are defined by the following equation: eouesic trajectories on the ru

$$
\frac{d^2q^{\alpha}}{ds^2} + \Gamma^{\alpha}_{\beta\gamma}\frac{dq^{\beta}}{ds}\frac{dq^{\gamma}}{ds} = 0,
$$
\n(2.10)

where particles are interacting through the potential *U*(*q*) (see Appendix A). An N-body system In terms of the coordinate system (2.0) introduced above  $(r_a, a = 1, ..., N)$  this equatio reques to the Built-Bagrangian equation for massive particles meeting though the potential motion of  $(1, ..., N)$ .<br> $d^2\vec{x}$  a background field. In terms of the coordinate system  $(2.6)$  introduced above  $(\vec{r}_a, a = 1, ..., N)$  this equation reduces to the Euler-Lagrangian equation for massive particles interacting though the potential function  $U(\vec{r}_1, ..., \vec{r}_N)$ : reduces to the Euler-Lagrangian equation for massive particles interacting though the potential

$$
M_a \frac{d^2 \vec{r}_a}{dt^2} = -\frac{\partial U}{\partial \vec{r}_a}, \qquad a = 1, ..., N. \qquad (2.14)
$$

### consider the second derivatives terms as the perturbation that can be safely omitted in the first Geometrisation of Self-Gravitating N-body System The physical time variable *t* should be introduced by the relation

$$
ds^2 = g_{\alpha\beta} dq^{\alpha} dq^{\beta}, \qquad g_{\alpha\beta} = \delta_{\alpha\beta}(E - U(q)) = \delta_{\alpha\beta} W(q),
$$

$$
U = -G \sum_{a
$$

$$
\frac{d^2q^\alpha}{dt^2} = -\frac{\partial U}{\partial q^\alpha}
$$

*i* the benetit of using  $\frac{1}{2}$  the benefit of uning a eserdinated  $\frac{1}{2}$ reduces to the Euler-Lagrangian equation for massive particles interacting though the potential the benefit of using q coordinates !



The vector field  $\delta q^{\alpha}$  defined along the geodesic  $\gamma(s)$  and satisfying the above equations is called a *Jacobi field*. The equation can be written also in an alternative first-order form:

$$
\frac{D\delta q^{\alpha}}{ds} = \delta u^{\alpha}, \qquad \frac{D\delta u^{\alpha}}{ds} = -R^{\alpha}_{\beta\gamma\sigma} u^{\beta} \delta q^{\gamma} u^{\sigma}.
$$
 (3.27)

$$
\frac{d}{ds}|\delta q_{\perp}|^2 = 2\delta q_{\perp}^{\alpha} u_{\alpha;\beta} \delta q_{\perp}^{\beta}
$$

$$
\frac{d^2}{ds^2}|\delta q_{\perp}|^2 = -2K(q, u, \delta q_{\perp})|\delta q_{\perp}|^2 + 2|\delta u_{\perp}|^2.
$$

$$
K(q, u, \delta q_{\perp}) = \frac{R_{\alpha\beta\gamma\sigma}\delta q_{\perp}^{\alpha}u^{\beta}\delta q_{\perp}^{\gamma}u^{\sigma}}{|\delta q_{\perp}|^2}
$$

me sign or the sectionar curvature defines the stability or geodesic trajectones in  $\alpha$  different parts of the phase space  $(q,u)$ The sign of the sectional curvature defines the stability of geodesic trajectories in<br>different parts of the phase apace (5.51) The sign of the sectional curvature defines the stability of geodesic trajectories in different parts of the phase space (q,u)

In the regions where the sectional curvature is negative the trajectories of particles phase of deterministic chaos. In the regions where the sectional curvature is<br>positive the trajectories are stable, exhibit geodesic focusing, generating caustics. In the regions where the sectional curvature is negative the trajectories of particles<br>are unstable, are exponentially diverging, and the self-gravitating system is in a In the regions where the sectional curvature is negative the trajectories of particles are unstable, are exponentially diverging, and the self-gravitating system is in a *phase of deterministic chaos.* In the regions where the sectional curvature is

quadrupole momentum of the system. *|"q*‹*|* bution or part<br>the system.  $\frac{1}{2}$ A self-gravitating N-body system can be assigned to these distinguished regions of the phase space depending on the initial distribution of particles velocities and quadrupole momentum of the system.

Particle distribution in the phase space and the corresponding sign of the sectional curvature Particle distribution in the phase space shorter than the binary relaxation time *·collective* π *·b*. These time scales and the dynamical





$$
K(q, u, \delta q_{\perp}) < 0 \qquad K(q, u, \delta q_{\perp}) > 0
$$

$$
K(q,u,\delta q_{\perp})>0
$$

chaotic behaviour *focusing beha* approximated by the force *GM ·dyn < ·collective < ·<sup>b</sup>*

chaotic behaviour focusing behaviour  $\tau_{\text{max}} = \sqrt{\frac{W}{\sigma}} = \sqrt{v^2 + \frac{2}{\sigma^2}}$  that generating caustics *that generating caustics*  $t \hskip -1pt I$  the shortest collective relaxation time will get collective relaxation time will get  $r$ 

$$
\tau_{collective} = \sqrt{\frac{W}{(\nabla W)^2}} = \gamma \frac{\langle v^2 \rangle^{1/2}}{\pi G M n^{2/3}},
$$

$$
\tau_{galaxies} \simeq 6.14 \times 10^9 \left(\frac{\langle v^2 \rangle^{1/2}}{100 \frac{km}{s}}\right) \left(\frac{1pc^{-3}}{n}\right)^{2/3} \left(\frac{M_{\odot}}{M}\right) \, years.
$$

Astronomy and Astrophysics Astrophysics<br>160 (1986) 203--210 Astronomy and Astrophysics Astrophysics<br>160 (1986) 203--210  $t$ otronomy end Actrophysics Actrophysics <sup>È</sup>*v*<sup>2</sup>Í<sup>3</sup>*/*<sup>2</sup> 2*GMn*<sup>2</sup>*/*<sup>3</sup> <sup>13</sup>In the articles [95, 70, 71, 96, 94, 75] the formulas for the binary scattering relaxation time and the Astronomy and Astrophysics Astrophysics<br>160 (1996) 202 - 210  $100 (1300) Z00 - Z10$ 160 (1986) 203--210

*Raychaudhuri Equation and focusing*   $\overline{\mathcal{R}}$  $\ddot{V}$ *d*  $\dot{\theta}$   $\$ Raychalid Turk  $\dot{\theta}$ quation land focusing



to observe that it is equal to the expansion scalar *◊*:

of the acceleration tensor is defined as  $\mathcal{L}_{\text{max}}$ 

$$
\frac{1}{V}\frac{dV}{ds} = \frac{d\ln V}{ds} = \theta.
$$
\n(9.130)

Thus the expansion scalar  $\theta$  measures the fractional rate at which the volume of a small ball of particles forming a congruence is changing with respect to the time measured along the trajectory  $\gamma(s)$ . One can calculate the second derivative of the transversal volume:  $\alpha$  the fraction *f* and the volume of a small ball container the volume of a small ball of particles forming a congruence is changing with respect to the time measured along the rajectory  $\gamma(s)$ . One can calculate the second derivative or the transversal volume.

$$
\ddot{\mathcal{V}} = (\dot{\theta} + \theta^2) \mathcal{V}.
$$
 (9.131)

*The vanishing of the volume element at q characterises q as a conjugate point*. It follows en by a logarithmic derivative of the volume element (<mark>9.130</mark>)<br>*d* lp )<sup>;</sup> that the expansion scalar  $\theta$  given by a logarithmic derivative of the volume element (9.130)

$$
\theta = \frac{d \ln \mathcal{V}}{ds} \tag{10.146}
$$

 $\tan$  **l**  $\arctan$  $f \gamma(s)$  at v and positive just to t  $\overline{\text{h}}$ *◊*2  $\overline{a}$ *V* : 1 is a continuous function at all points of  $\gamma(s)$  at which  $V \neq 0$ , while  $\theta$  becomes unbounded near and the Raychaudhuri equation (9.125) can be written in the following form:  $\frac{1}{\sqrt{2}}$ point *q* at which  $V = 0$  with large and positive just to the future of *q* and large and negative just to the past of *q* on  $\gamma(s)$ 



 $\frac{1}{2}$ *<sup>W</sup>–W—* <sup>≠</sup> <sup>3</sup>*<sup>N</sup>* <sup>≠</sup> <sup>4</sup> and the scalar curvature *R* by the second contraction: the particle second as  $\Gamma$  and order as four MI is a clear contained  $\frac{d}{dt}$  and it is not and it is not accelerate that the section of a section  $\frac{d}{dt}$ *Raychaudhuri Equation for N-body system* 

$$
ds^2 = g_{\alpha\beta} dq^{\alpha} dq^{\beta}, \qquad g_{\alpha\beta} = \delta_{\alpha\beta} (E - U(q)) = \delta_{\alpha\beta} W(q),
$$

The self-gravitating system of *I*v particles interacts through the gravitational potential and contraction of the vertices vectors: where particles are interacting through the potential *U*(*q*) (see Appendix A). An N-body system *R*<sub></sub> The self-gravitating system of *N* particles interacts through the gravitational potential function of the form  $\int N$  parti *|* <sup>2</sup> *<sup>R</sup>–—u–u—|min* <sup>=</sup> <sup>≠</sup>(3*<sup>N</sup>* <sup>≠</sup> 4)  $e$  gravitation *|* nal potential

$$
U = -G \sum_{a
$$

The Raychaudhuri equation  $(10.157)$  will take the following form:  $\mathbf{t}$  the following form:  $\mathbf{t}$  form:  $\mathbf{t}$  for  $\mathbf{t}$ 

$$
\frac{d\theta}{ds} = -\frac{3(3N-2)}{4W^2} \left( (uW')^2 - \frac{3N-4}{3(3N-2)}|W'|^2 \right) - \frac{1}{3N-1} \theta^2 - \theta_{\alpha\beta} \theta^{\alpha\beta} + \n+ \frac{3N-2}{2W} \left( (uW''u) + \frac{1}{3N-2}|W''| \right). \tag{11.166}
$$

In the case of spherically symmetric evolution  $\theta^{\alpha \beta} = 0$  [106] and the equation will take the following form:  $\frac{10}{10}$ *ˆr<sup>i</sup> <sup>a</sup>ˆr b r*5 *ab ˆ*<sup>2</sup>*U MaM<sup>c</sup>* 4*fiGMaM<sup>c</sup>* In the case of spherically symmetric evolution  $\theta^{\alpha\beta} = 0$  [106] and the equation will take the

$$
\frac{d\theta}{ds} = -(3N-1)\frac{(\nabla W)^2}{2W^3} - \frac{1}{3N-1}\theta^2,\tag{11.171}
$$

aU <sup>=</sup> <sup>≠</sup>(3*<sup>N</sup>* <sup>≠</sup> 1)(Ò*W*)<sup>2</sup> <sup>2</sup>*W*<sup>3</sup> <sup>≠</sup> <sup>1</sup> y systel *◊*2 *Raychaudhuri Equation for N-body system* <sup>2</sup>*W*<sup>3</sup> <sup>≠</sup> <sup>1</sup>

When the number of particles is large  $N\gg 1$  we will have When the number of particles is large  $N \gg 1$  we will have

$$
\frac{d\theta}{ds} = -3N \frac{(\nabla W)^2}{2W^3} - \frac{1}{3N} \theta^2.
$$

<u>unction</u> It is convenient to introduce the function  $B^2$ 

$$
B^{2}(s) = (3N)^{2} \frac{(\nabla W)^{2}}{2W^{3}}
$$

**d**<br>  $\frac{11.1.17}{2}$ so that the equation  $(11.172)$  will take the following form:

$$
\frac{d\theta}{ds} = -\frac{1}{3N}(\theta^2 + B^2(s)).
$$

Solution of Raychaudhuri Equation for N-body system

$$
\theta(s) = B \tan \left( \arctan \frac{\theta(0)}{B} - \frac{B}{3N} s \right),\,
$$

The expansion scalar  $\theta(s)$  becomes singular at the proper times  $s_n$ : The expansion scalar  $\theta(s)$  becomes singular at the proper times  $s_n$ :

$$
s_{caustics} = \frac{3N}{B} \left( \arctan \frac{\theta(0)}{B} + \frac{\pi}{2} + \pi n \right), \quad n = 0, \pm 1, \pm 2, ... \tag{11.176}
$$

As far as the expansion scalar  $\theta(s)$  tends to infinity at a certain epoch  $s_{\text{scust}}$  $\frac{1}{\sqrt{11}}$ As far as the expansion scalar  $\theta(s)$  tends to infinity at a certain epoch  $s_{caustics}$ , it follows that the volume element that is occupied by the galaxies decreases and tends to zero creating the *regions in space of large galactic densities. a* densities.<br> *B* densities. <sup>3</sup>*<sup>N</sup> <sup>s</sup>*

$$
\mathcal{V}(s) = \mathcal{V}(0) \left[ \frac{\cos \left( \arctan \frac{\theta(0)}{B} - \frac{B}{3N} s \right)}{\cos \left( \arctan \frac{\theta(0)}{B} \right)} \right]^{3N}.
$$

#### the volume element characterises the appearance of conjugate points and caustics. The density of galaxies defined as *flg*(*s*) = *NMg/V*(*s*) allows to calculate the density contrast *"* = *"fl/fl* as *Solution of Raychaudhuri Equation for N-body system s* æ *scaustics, ◊*(*s*) ¥ *guation1* <sup>9</sup>*<sup>N</sup>* (*<sup>s</sup>* <sup>≠</sup> *<sup>s</sup>caustics*) + *...* (11.183)

The ratio of densities during the evolution from the initial volume  $V(0)$  to the volume  $V(s)$ at the epoch *s* will give us the density contrast: and becomes unbounded near *scaustics* at which *V* = 0 (see Figs. 9, 14). The conjugate degree *r* defined in (10.149) for the caustic (11.183) has the maximal value

$$
\delta_{caustics}(s) + 1 = \frac{\rho(s)}{\rho(0)} = \frac{\mathcal{V}(0)}{\mathcal{V}(s)} = \left[\frac{\cos\left(\arctan\frac{\theta(0)}{B}\right)}{\cos\left(\arctan\frac{\theta(0)}{B} - \frac{B}{3N}s\right)}\right]^{3N}.\tag{11.179}
$$

As one can see, at the epoch  $(11.176)$  where the expansion scalar  $\theta(s)$  becomes singular, the trigonometric function in the denominator tends to zero and the density contrast is increasing and tends to infinity, the phenomenon similar to the spherical top-hat model. *scaustics* we have the following expression: 3*N* the spherical top-hat model.

One can calculate the volume element per galaxy defined in (9.134), (10.158), thus arctan *◊*(0)  $\frac{1}{\mathbf{H}^{\mathcal{I}}}$ In terms of physical time  $(2.12)$  the characteristic time scale of generation of gravitational caustics is

$$
\tau_{caustics} = \frac{3\pi N}{2B\sqrt{2}W} = \frac{\pi}{2} \sqrt{\frac{W}{(\nabla W)^2}}.
$$
\n(11.186)

### *Newtonian Cosmological Mechanics are positive.* Thus we have to estimate the quantities *<sup>W</sup>* and (Ò*W*)<sup>2</sup> in the above equation.

Let us consider the evolution of a spherical shell of radius  $R_0$  that expands with the Universe, so that  $R = R_0 a(t)$  and  $a(t)$  is the scale factor in the Newtonian cosmological model of the expanding Universe. One can derive the evolution of  $a(t)$  by using mostly the Newtonian mechanics and accepting two results from the general relativity: The Birkhoff's theorem stated that for a spherically symmetric system the force due to gravity at radius *R* is determined only by the mass interior to that radius and that the energy contributes to the gravitating mass density through the matter density  $\rho_m$  at zero pressure,  $p = 0$ , and the energy density of radiation/relativistic particles,  $\rho_r = 3p/c^2$ , where  $p = \epsilon/3$  is pressure and  $\epsilon = \rho_r c^2$  is energy density. The expansion of the sphere will slow down due to the gravitational force of the matter inside:

$$
\frac{d^2R}{dt^2} = -\frac{GM}{R^2} = -\frac{G}{R^2}\frac{4\pi}{3}R^3\rho = -\frac{4\pi G}{3}R\rho,\tag{8.99}
$$

where  $\rho = \rho_m + 3P/c^2$ . Since  $R = R_0$  *a(t)* and  $R_0$  is a constant, one can get the evolution equation for the scale factor  $a(t)$  that reproduces the Friedmann equation:

$$
\ddot{a} = -\frac{4\pi G}{3} (\rho_m + \frac{3P}{c^2})a. \tag{8.100}
$$

$$
W(t) = T = \sum_{g=1}^{N} \frac{M_g v_g^2}{2} = \frac{N M_g R_0^2 \dot{a}^2(t)}{2}.
$$

of the appearance of galactic caustics, the regions of the space where the density of galaxies

The square of the force acting on a unit mass of the galaxies is

=

3*fi*

$$
(\nabla W)^2(t) = \sum_{g=1}^N \frac{1}{M_g} F_g^2 = \frac{N}{M_g} \left( \frac{GMM_g}{R_0^2 a^2(t)} \right)^2 = \frac{N}{M_g} \left( \frac{4\pi GM_g}{3} R_0 a(t) \rho(t) \right)^2,
$$

We can evaluate the quantities entering into this equation by considering a self-gravitating system of N galaxies of the mass  $M<sub>g</sub>$  each. The kinetic energy W of the galaxies was found in (8.101) and the square of the force acting on a unit mass of the galaxies  $(\nabla W)^2$  in (8.102). Thus we will obtain ve can evaluate the quantities enter *–* 4*fiGfl*(*t*)  $\frac{1}{2}$  $\int$ *–* 4*fiGfl*(*t*)  $\frac{1}{2}$  in  $\left( 8.102 \right)$ .

$$
\tau_{caustics} = \frac{\alpha'}{4\pi G\rho(t)} H(t),\tag{11.187}
$$

Ô2 <del>. It is in a good agreement with our previous result, and it is in a good agreement with our previous result, a<br>The contract of the contra</del>

where the numerical coefficient  $\alpha =$  $\overline{\phantom{a}}$ where the numerical coefficient  $\alpha = \sqrt{1/10}$ . This general result for the characteristic time scale of the appearance of galactic caustics, the regions of the space where the density of galaxies of the appearance of galactic caustics, the regions of the space where the density of galaxies is large, means that the appearance of caustics depends on the given epoch of the Universe is large, means that the appearance of caustics depends on the given epoch of the Universe expansion. The formula has a universal character and depends only on the density of matter expansion. The formula has a universal character and depends only on the density of matter and the Hubble parameter<sup><sup>18</sup>. These are time-dependent parameters that are varying during</sup> the evolution of the Universe from the recombination epoch to the present day. Let us calculate the evolution of the Universe from the recombination epoch to the present day. Let us calculate this time scale during the *matter-dominated epoch* when  $\overline{1}$ 1*/*10. This general result for the characteristic time scale

$$
\rho_m(t) = \rho_0 \frac{a_0^3}{a^3(t)}.\tag{8.104}
$$

In that case the equation  $(8.100)$  has the following form In that case the equation  $(8.100)$  has the following form:

$$
\dot{a}^2 = \frac{A^2}{a} - kc^2, \qquad A^2 = \left(\frac{8\pi G}{3}\right)\rho_0 a_0^3, \qquad k = 1, 0, -1,\tag{8.105}
$$

and for the flat Universe, *k* = 0, we will get: and for the flat Universe,  $k = 0$ , we will get:

$$
a_m(t) = \left(\frac{3A}{2}\right)^{2/3} t^{2/3}, \qquad H_m(t) = \frac{2}{3t}, \qquad \rho_m(t) = \frac{1}{6\pi G t^2}.
$$
 (8.106)

 $B = \begin{bmatrix} 1 & \cdots & \cdots & 1 \\ 1 & \cdots & 1 \\ 0 & \cdots & 1 \end{bmatrix}$  we will find that  $\begin{bmatrix} 1 & \cdots & 1 \\ 0 & \cdots & 1 \\ 0 & \cdots & 0 \end{bmatrix}$  is provided that  $\begin{bmatrix} 1 & \cdots & 1 \\ 0 & \cdots & 1 \\ 0 & \cdots & 0 \end{bmatrix}$  is provided to  $\begin{bmatrix} 1 & \cdots & 1 \\ 0 & \cdots & 1 \\ 0 & \cdots$ portional to the given epoch *t*: By substituting these values into the general formula  $(8.103)$  we will find that  $\tau_{caustics}$  is proportional to the given epoch *t*:

$$
\tau_{caustics} = \alpha \frac{2}{3H(t)} = \alpha t.
$$
\n(8.107)

This result means that the time required to generate galactic caustics is very short at the early stages of the Universe expansion, at the recombination epoch, and linearly increases with the expansion time. At the present epoch,  $a = a_0$ , this time scale is large and is proportional to the Hubble time:

$$
\tau_{0\ caustics} = \alpha \frac{2}{3H_0},\tag{8.108}
$$

where for a flat, matter-dominated Universe we substituted the expression for the matter density equal to the critical density:

$$
\rho_c = \frac{3H_0^2}{8\pi G}.\tag{8.109}
$$

Considering the *radiation-dominated epoch* one can obtain the identical functional time dependence, with  $\alpha =$  $\overline{\phantom{a}}$ 2*/*5.

Let us compare the above time scales with the Jeans gravitational instability of a uniformly distributed matter. Jeans developed a Newtonian theory of instability of a uniformly distributed matter in a non-expanding infinite space, and Lifshitz considered small perturbations of a homogeneously expanding Universe in the theory of the general relativity. Bonnor demonstrated that in the Newtonian cosmological model of an expanding Universe forms into a slower *power-growth rate "*(*t*) <sup>≥</sup> *At*<sup>2</sup>*/*<sup>3</sup> <sup>+</sup> *Bt*≠<sup>1</sup> *A*  $\frac{1}{2}$  and  $\frac{1}{2}$  and  $\frac{1}{2}$  are *a*  $\frac{1}{2}$  and  $\frac{1}{2}$  are *n*  $\frac{1}{2}$  and  $\frac{1}{2}$  and  $\frac{1}{2}$  are *R*  $\frac{1}{2}$  and  $\frac{1}{2}$  are *R*  $\frac{1}{2}$  and  $\frac{1}{2}$  are *R*  $\frac{1}{2}$  and  $\frac{1}{2}$  are *R* result connected words and is proportion for the Lifeshitz connected perturbation for the expanding Universe connected model of an expanding Universe pare the above time scales with the Jeans gravitational mstability of a nly distributed r 11<br>1 T

the Jeans exponential growth of density perturbation  $\delta(t) \sim A e^{t/\tau_{Jeans}} + B e^{-t/\tau_{Jeans}}$  transforms into a slower *power-growth rate*  $\delta(t) \sim At^{2/3} + Bt^{-1} = Aa(t) + Ba(t)^{-3/2}$  and that his result coincides with the Lifshitz' exact solution for the long wave length perturbations. forms into a slower power-growth rate  $o(t) \sim At$   $t^2 + Dt^2 = Au(t) + Du(t)^2$  and that ms If we depend to the distribution of the day wave tength per turbations.

$$
\tau_{Jeans} \sim \frac{1}{\sqrt{4\pi G\rho_c}}, \qquad \tau_{collapse} \sim 2t_{turn} \propto \sqrt{\frac{2}{G\rho_{lump}}}.
$$

bation of the uniformly distributed matter in the uniformly distributed matter in the ideal-gas approximation  $\mathcal{L}_{\mathcal{A}}$ 

1<br>1<br>1

and is proportional to the Hubble time, where *fl<sup>c</sup>* is time-independent matter density (8.109).

*<u>Julians</u>* where for a flat, matter-dominated Universe we substituted the expression for the matter density equal to the critical density: flat matter deminated Huiverse we substituted the expression for the matter where for a flat, matter-dominated Universe we substituted the expression for  $\frac{1}{2}$ 

2

$$
\rho_c = \frac{3H_0^2}{8\pi G}.\tag{8.110}
$$

is the speed of sound) are increasing due to the gravitational interaction that play a dominant role against the pressure. The gravitational collapse time scale in the spherical top-hat model Û 2 dence, with *–* = Considering the *radiation-dominated epoch* one can obtain the identical functional time depen-<sup>OI</sup> 2011 complete the second in the spherical top material in the details in the next two metals in the second term in the second term in the second term in The gravitational collapse time scale in the spherical top-hat model

# *Polarisation of the Vacuum*

 *Dark Energy and Cosmological Inflation* 

George Savvidy Demokritos National Research Centre, Athens, Greece Yerevan Physics Institute, Yerevan, Armenia

> QCD Vacuum Structure and Confinement axos, Greece 26-30 August, 2024

Gauge field theory vacuum and cosmological inflation without scalar field *Annals Phys.* **436** (2022) 168681

*Stability of the Yang Mills Vacuum State*  Nucl.Phys. *B* **990** (2023) 116187

What is the Influence of the

*Vacuum Energy Density* 

*on the Cosmological Evolution?* 



[24] Y. B. Zel'dovich, *The Cosmological constant and the theory of elementary particles,* Sov. Phys. Usp. **11** (1968) 381

[30] S. Weinberg, *The Cosmological constant problem,* Rev. Mod. Phys. **61** (1989) 1-23 5. welliberg, *The* 

with the physical companions of cosmology, called the call V. Mukhanov, Physical Foundations of Cosmology, Cambridge University Press, New York, 2005. [25] S. Weinberg, *The Cosmological constant problem,* Rev. Mod. Phys. **61** (1989) 1-23

 *Zero Point Energy of a Quantised Field Is there Energy Density in the Vacuum ?* 



$$
U_{vacuum} = \hbar \sum_{k} \frac{\omega_k}{2}
$$

*There is Energy Density in the Vacuum, it is Zero Point Energy*  $P_{\text{p}}$ theory Density in the vacuum, it is 2  $\blacksquare$ ) Point Energy the  $\mathcal{A}^{\mathcal{A}}$  contribution to the values is the values of  $\mathcal{A}^{\mathcal{A}}$ *'* Œ  $i$ t is  $Z$  $\overline{\mathbf{c}}$  $\frac{1}{2}$  *Point Enerav A*<sup>2</sup> = ≠*"*21*x*<sup>1</sup> ≠ *"*23*x*<sup>3</sup> ≠ *"*24*x*<sup>4</sup> = 2*bx*<sup>1</sup>

re is Energy Density in the Vacuum, it is Zero Point Energy  
\nLamb shift - 1947 Casimir effect 1948  
\n
$$
U^{\infty}_{\gamma} = \sum \frac{1}{2} \hbar \omega_k e^{-\gamma \omega_k}
$$
\n
$$
\lim_{\gamma \to 0} [U^{\infty}_{\gamma}(J) - U^{\infty}_{\gamma}(0)] = U_{phys} \qquad U_{phys} = \hbar c \pi^2 \frac{Area}{720a^3}
$$

Thanks for you explanation of Donoghue's article. Apart from my misguided remark about normally mentioned is the zero-point energy. When caliological vacuum energy density from eray density from a quantum field tion calculation of the zero-point energies and momenta is 8*fiG*  $\overline{S}$  *it*  $|O|$  $\overline{a}$ 3*c*<sup>4</sup>  $10h + 100$ The Cosmological vacuum energy density from a quantum field  $\overline{a}$ 

$$
E_0 = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2} \omega_p \sim \frac{1}{16\pi^2} \Lambda^4 \approx 1.44 \times 10^{110} \frac{g}{s^2 cm}
$$

as Donoghue also write in (6) and (7), I still the article is the article is write in fact (6) and (7) and (7) are just (6) and (7) are just (6) and (7) and (7) are just (6) and (7) are just (6) and (7) are just (6) and (7 momentum integral at a scale  $\sim$  (United States) shergy Density in Universe a well-defined physically motivated prescription allowing to obtain a finite, gauge and renormalisation lim  $\mathcal{G}_{\mathcal{F}}$  invariant results when investigating the vacuum fluctuations of  $\mathcal{F}_{\mathcal{F}}$ *in l*  $\int$ *Critical Energy Density in Universe* 

$$
\epsilon_{crit} = 3 \frac{c^4}{8\pi G} \left(\frac{H_0}{c}\right)^2 \approx 7.67 \times 10^{-9} \frac{g}{s^2 cm}
$$

*Vacuum energy contribution to the energy density of the universe '* Œ *"*<br>
= *i* <sup>1</sup>  $\overline{\mathcal{U}}$ ~*Êke*≠*"Ê<sup>k</sup>* (0.8)

$$
\epsilon_{crit} = 3 \frac{c^4}{8\pi G} \left(\frac{H_0}{c}\right)^2 \approx 7.67 \times 10^{-9} \frac{g}{s^2 cm}
$$
 100%

$$
\epsilon_{\Lambda} = 3 \frac{c^4}{8\pi G} \left(\frac{H_0}{c}\right)^2 \Omega_{\Lambda} \approx 5.28 \times 10^{-9} \frac{g}{s^2 cm}
$$
 68%

#### *The Yang-Mills Theory Vacuum Energy Density*  G.S. 1977, 2020 where *<sup>H</sup>˛* and *<sup>E</sup>˛* are magnetic and electric fields. This expression should be compared with the one-loop entry in pure Subdivident in pure Super Subdividently with its nonzero trace  $\tau$ 45*m*4*–*<sup>2</sup> + *...* (1.10) *<sup>T</sup>*<sup>00</sup> © *'*(*F*) = *<sup>F</sup>* <sup>+</sup> *b g*<sup>2</sup>  $\gamma$ <sup>*V*</sup>  $\Gamma$ ln <sup>2</sup>*g*2*<sup>F</sup> um Energy Density Fr*  $2020$

$$
\mathcal{L}_g = -\mathcal{F} - \frac{11N}{96\pi^2} g^2 \mathcal{F} \Big( \ln \frac{2g^2 \mathcal{F}}{\mu^4} - 1 \Big) , \qquad \mathcal{F} = \frac{\vec{\mathcal{H}}_a^2 - \vec{\mathcal{E}}_a^2}{2} > 0, \quad \mathcal{G} = \vec{\mathcal{E}}_a \vec{\mathcal{H}}_a = 0 .
$$
  

$$
\mathcal{L}_q = -\mathcal{F} + \frac{N_f}{48\pi^2} g^2 \mathcal{F} \Big[ \ln(\frac{2g^2 \mathcal{F}}{\mu^4}) - 1 \Big]
$$



<sup>96</sup>*fi*<sup>2</sup> *<sup>g</sup>*2*<sup>F</sup>*

$$
2g^{2} \mathcal{F}_{vac} = \mu^{4} \exp \left(-\frac{96\pi^{2}}{b \ g^{2}(\mu)}\right) = \Lambda_{YM}^{4},
$$

*Tµ‹* = *T Y M*

*<sup>T</sup>ij* <sup>=</sup> *"ij* <sup>Ë</sup><sup>1</sup>

where  $b = 11N - 2N_f$ .

$$
T_{\mu\nu} = T_{\mu\nu}^{YM} \left[ 1 + \frac{b g^2}{96\pi^2} \ln \frac{2g^2 \mathcal{F}}{\mu^4} \right] - g_{\mu\nu} \frac{b g^2}{96\pi^2} \mathcal{F}, \qquad \mathcal{G} = 0.
$$

*YM Quantum Energy Momentum Tensor T* <sup>96</sup>*fi*<sup>2</sup> ln <sup>2</sup>*g*2*<sup>F</sup> µ*4 where *b* = 11*N* ≠ 2*N<sup>f</sup>* . The vacuum energy density has therefore the following form: YM Quantum Energy Momentum Tensor the contributions to the energy density from radiation, elementary particles of the Standard Model or

$$
T_{\mu\nu} = T_{\mu\nu}^{YM} \Big[ 1 + \frac{b g^2}{96\pi^2} \ln \frac{2g^2 \mathcal{F}}{\mu^4} \Big] - g_{\mu\nu} \frac{b g^2}{96\pi^2} \mathcal{F}, \qquad \mathcal{G} = 0,
$$

ln <sup>2</sup>*g*2*<sup>F</sup>*

$$
\epsilon(\mathcal{F}) = \ \mathcal{F} + \frac{b \ g^2}{96\pi^2} \mathcal{F} \Big( \ln \frac{2g^2 \mathcal{F}}{\mu^4} - 1 \Big), \qquad \qquad p(\mathcal{F}) = \frac{1}{3} \mathcal{F} + \frac{1}{3} \frac{b \ g^2}{96\pi^2} \mathcal{F} \Big( \ln \frac{2g^2 \mathcal{F}}{\mu^4} + 3 \Big).
$$

$$
\mathcal{F} = \frac{1}{4} g^{\alpha\beta} g^{\gamma\delta} G^a_{\alpha\gamma} G_{\beta\delta} \geq 0 \qquad \qquad \mathcal{G} = G^*_{\mu\nu} G^{\mu\nu} = 0
$$

#### *Yang-Mills Quantum Equation of State*  ln <sup>2</sup>*g*2*<sup>F</sup>* ang-Mills Quantu *Fouation of State*



$$
\epsilon(\mathcal{F}) = \ \mathcal{F} + \frac{b \ g^2}{96\pi^2} \mathcal{F} \Big( \ln \frac{2g^2 \mathcal{F}}{\mu^4} - 1 \Big), \qquad \qquad p(\mathcal{F}) = \frac{1}{3} \mathcal{F} + \frac{1}{3} \frac{b \ g^2}{96\pi^2} \mathcal{F} \Big( \ln \frac{2g^2 \mathcal{F}}{\mu^4} + 3 \Big).
$$

#### Yang-Mills Quantum Equation of State *a*˙ *a* Yang-Mills Quantum Equation of State

$$
p = \frac{1}{3}\epsilon + \frac{4}{3}\frac{b \, g^2 \mathcal{F}}{96\pi^2} \Lambda_{YM}^4 \qquad \text{and} \qquad w = \frac{p}{\epsilon} = \frac{\ln \frac{2g^2 \mathcal{F}}{\Lambda_{YM}^4} + 3}{3\left(\ln \frac{2g^2 \mathcal{F}}{\Lambda_{YM}^4} - 1\right)}
$$

general parametrisation of the equation of state  $p = w\epsilon$ These are comoving coordinates; the universe expands or contracts as *a*(*t*) increases or decreases, and

*T <sup>µ</sup>‹ vac* <sup>=</sup> <sup>≠</sup>*gµ‹ <sup>b</sup>* <sup>192</sup>*fi*<sup>2</sup> <sup>2</sup>*g*2*Fvac,* (2.18) the american equations (1.1) into the following form  $E$ *Friedman Equations*

$$
\dot{\epsilon} + 3\frac{\dot{a}}{a}(\epsilon + p) = 0,
$$

$$
\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^4}(\epsilon + 3p).
$$

*General Relativity and Yang-Mills Vacuum Energy Density*  momentum tensor (2.11) has the following form:

$$
S = -\frac{c^3}{16\pi G} \int R\sqrt{-g}d^4x + \int (\mathcal{L}_q + \mathcal{L}_g) \sqrt{-g}d^4x.
$$

$$
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4} \Big[ T_{\mu\nu}^{YM} \Big( 1 + \frac{b g^2}{96\pi^2} \ln \frac{2g^2 \mathcal{F}}{\mu^4} \Big) - g_{\mu\nu} \frac{b g^2}{96\pi^2} \mathcal{F} \Big].
$$

*c* ontril *fi* and the YM vaccion of the YM vaccion *a* cuum field to the ene gy balance of the universe The contribution of the YM vacuum field to the energy balance of the universe

192*fi*2<sup>4</sup>

*Y M .* (2.21)

*Friedmann Evolution Equations in YM, QCD Vacuum* log *<sup>a</sup>*<sup>4</sup> log *<sup>a</sup>*<sup>4</sup> *Y M, p* <sup>=</sup> *<sup>A</sup> <sup>a</sup>*<sup>4</sup> be defined through the covariant conservation of the energy-momentum tensor: *Tµ‹*;*‹* = 0*.* It is the

$$
a(\tau)=a_0\,\, \tilde{a}(\tau), \quad \ ct=L\,\, \tau,
$$

$$
\frac{d\tilde{a}}{d\tau} = \pm \sqrt{\frac{1}{\tilde{a}^2} \left( \log \frac{1}{\tilde{a}^4} - 1 \right) - k\gamma^2}, \qquad k = 0, \pm 1, \qquad \gamma^2 = \left( \frac{L}{a_0} \right)^2.
$$

$$
\frac{1}{L^2} = \frac{8\pi G}{3c^4} \mathcal{A} \Lambda_{YM}^4 \equiv \Lambda_{eff} \ ,
$$

$$
{\cal A} = \frac{b}{192\pi^2} = \frac{11N - 2N_f}{192\pi^2}.
$$



$$
0 \leq \gamma^2 < \gamma_c^2
$$

$$
\gamma^2 = \gamma_c^2 = \frac{2}{\sqrt{e}}
$$

$$
\gamma_c^2 < \gamma^2
$$

*s*<br>2*cm* (0.10) *cm* (1.10) *cm* (1.11) **cm** *f the*  $\overline{a}$ 8*fiG*  $\sqrt{2}$ *c* **200 and the Eff** Polarisation of the YM vacuum and the Effective Lagrangians  $\frac{1}{2}$   $\frac{1}{2}$ 

$$
\epsilon_{YM} = 3 \frac{c^4}{8 \pi G} \frac{1}{L^2}, \qquad \frac{1}{L^2} = \frac{8 \pi G}{3c^4} \; \frac{11N - 2N_f}{196 \pi^2} \; \Lambda_{YM}^4
$$

 $\Lambda_{YM}^4$  is the dimensional transmutation scale of YM theory *'Y M* = 3 8*fiG c* ¥ <sup>5</sup>*.*<sup>28</sup> ◊ <sup>10</sup>≠<sup>9</sup> *<sup>g</sup>* rale of YM theory

$$
\epsilon_{YM} = 3 \frac{c^4}{8\pi G} \frac{1}{L^2} = \begin{cases} 9.31 \times 10^{-3} & eV \\ 9.31 \times 10^{29} & QCD \\ 9.31 \times 10^{97} & GUT \\ 9.31 \times 10^{110} & Planck \end{cases} \frac{g}{s^2 cm}
$$

*< Tµ‹ >*= 4*÷µ‹* (0.14) Tha VM yor wim anaray dansity is wall dafinad and is finita  $T$  the calculation of the eective Lagrangian in  $\mathcal{L}$  by Heisenberg and Euler was the first example of the first and the telesion oneigy sichlery to moleculation and renormalization and renormali the YM vacuum energy density is well defined and is finite

> *< Tµ‹ <sup>&</sup>gt;*<sup>=</sup> 4(≠1*,* <sup>1</sup>*/*3*,* <sup>1</sup>*/*3*,* <sup>1</sup>*/*3)*,* (0.13) group invariant results when investigating the vacuum fluctuations of quantised fields. It appears that only the dierence between vacuum energy in the presence and in the abse[nce of the e](https://arxiv.org/abs/2109.02162)xternal sources e-Print: 2109.02162G.S. Eur.Phys.J.C. 80 (2020) 165

**Type II Solution — Initial Acceleration of Finite Duration** The interval in which ˜*a* takes its values is now infinite:  $\eta$  is the evolution equation equation equation  $\eta$ 

$$
\frac{d\tilde{a}}{d\tau} = \pm \sqrt{\frac{1}{\tilde{a}^2} \left( \log \frac{1}{\tilde{a}^4} - 1 \right) - k\gamma^2}, \qquad k = 0, \pm 1, \qquad \gamma^2 = \left( \frac{L}{a_0} \right)^2.
$$

$$
\tilde{a}^4 = \mu_2^4 e^{b^2}, \qquad b \in [0, \infty],
$$

$$
\frac{db}{d\tau} = \frac{2}{\mu_2^2} e^{-\frac{b^2}{2}} \left(\frac{\gamma^2 \mu_2^2}{b^2} (e^{\frac{b^2}{2}} - 1) - 1\right)^{1/2}.
$$

With the boundary conditions at *·* = 0 where *b*(0) = 0 (˜*a*(0) = *µ*2) we will get the integral represen-

With the boundary conditions at *·* = 0 where *b*(0) = 0 (˜*a*(0) = *µ*2) we will get the integral represen-

$$
\mu_2^2 = -\frac{2}{\gamma^2} W_- \Big( -\frac{\gamma^2}{2\sqrt{e}} \Big), \qquad 0 \le \gamma^2 < \frac{2}{\sqrt{e}}
$$

$$
0 \le \gamma^2 < \frac{2}{\sqrt{e}} \text{ and } \tilde{a} \ge \mu_2.
$$

14

τ

2

**Type II Solution and Initial Acceleration of Finite Duration** 





The time interval is *·* œ [0*,*Œ], and as *·* æ Œ, we have

*·, a*˜ ƒ *"·* = *ct.* (5.77)

$$
a(t) \simeq ct, \qquad a(\eta) \simeq a_0 e^{\eta}.
$$
 (5.87)

*<sup>b</sup>*2(*·* ) <sup>ƒ</sup> 4 ln *"*

τ

*·, a*˜ ƒ *"·* = *ct.* (5.77)

**Type II Solution — Initial Acceleration of Finite Duration** 

$$
\epsilon + 3p = -\frac{2\mathcal{A}}{\mu_2^4} e^{-b^2(\tau)} (b^2(\tau) + \gamma^2 \mu_2^2 - 2) \Lambda_{YM}^4, \qquad b \in [0, +\infty],
$$



The r.h.s  $\epsilon + 3p$  of the Friedmann acceleration equation (1.4) always negative  $\ddot{1}$ 

# *Evolution of Energy Density and Pressure*

$$
\epsilon = \frac{\mathcal{A}}{\tilde{a}^4(\tau)} \Big( \log \frac{1}{\tilde{a}^4(\tau)} - 1 \Big) \Lambda_{YM}^4, \qquad p = \frac{\mathcal{A}}{3\tilde{a}^4(\tau)} \Big( \log \frac{1}{\tilde{a}^4(\tau)} + 3 \Big) \Lambda_{YM}^4.
$$



### *Type II Solution — Effective Parameter w*



For the equation of state  $p = w\epsilon$  one can find the behaviour of the effective parameter  $w$ 

$$
w_{II} = \frac{b^2(\tau) + \gamma^2 \mu_2^2 - 4}{3(b^2(\tau) + \gamma^2 \mu_2^2)}, \qquad -\frac{1}{3} < w_{II},
$$

**Type II Solution** Initial Acceleration of Finite Duration <sup>2</sup> ≠ 2*W*≠<sup>1</sup>  $\overline{a}$ <sup>≠</sup> *"*2*µ*<sup>2</sup>  $\overline{\phantom{a}}$ *A* eration of Fire *exation* of 2 <sup>2</sup> )  $\iota$  $\overline{\phantom{a}}$ <sup>≠</sup> *"*2*µ*<sup>2</sup> *A*<br>*Pieration of Finit*<br>*Pieration of Finit*  $\overline{C}$ <sup>2</sup> ) (5.89) <sup>≠</sup> *"*2*µ*<sup>2</sup> <sup>4</sup> exp (<sup>1</sup> <sup>≠</sup> *"*2*µ*<sup>2</sup> <sup>2</sup> ) (5.89)



 $\int_{a}$  a  $\int_{a}$  The number of e-foldings

 $\frac{1}{1}$   $\frac{2}{1}$  (4.611)  $\frac{2}{1}$   $\frac{1}{1}$   $\frac{2}{1}$   $\frac{1}{1}$   $\frac{2}{1}$   $\frac{1}{1}$   $\frac{2}{1}$   $\frac{1}{2}$   $\frac{1}{1}$   $\frac{2}{1}$ *"*<sup>2</sup> is in the interval 0 <sup>Æ</sup> *"*<sup>2</sup> *<sup>&</sup>lt;* <sup>Ô</sup> typical parameters around  $\gamma^2 = 1.211$ ,  $\mu_2^2 \simeq 1.75$  we get  $\tau_s = 10^{23}$  and  $\mathcal{N} \simeq 53$ .  $\mathcal{N} = \ln \frac{a(\tau_s)}{a(0)}$ . <sup>2</sup> <sup>ƒ</sup> <sup>1</sup>*.*75 we get *·<sup>s</sup>* = 10<sup>23</sup> and *<sup>N</sup>* <sup>ƒ</sup> 53. The duration of the intypical parameters around  $\gamma^2 = 1.211$ ,  $\mu_2^2 \simeq 1.75$  we get  $\tau_s = 10^{20}$  and  $\mathcal{N} \simeq 53$ .  $\mathcal{N} = 10$  $=$   $\ln \frac{a}{a}$ *L*<sup>2</sup> typical parameters around  $\gamma^2 = 1.211$ ,  $\mu_2^2 \simeq 1.75$  we get  $\tau_s = 10^{23}$  and  $\mathcal{N} \simeq 53$ .  $\mathcal{N} = \ln \frac{a(\tau_s)}{a(0)}$ . inflation in the case of the GUM scale *Y M* = *GUM* = 1016*GeV* is of order

$$
t_s^{GUM} = \frac{L_{GUM}}{c} \tau_s \simeq 4.2 \times 10^{-13} \text{ sec}, \qquad \text{where } L_{GUM} \simeq 1.25 \times 10^{-25} \text{ cm}
$$

$$
a(0) = L_{GUM} \frac{\mu_2}{\gamma} \simeq 1.5 \times 10^{-25} \text{ cm}, \qquad a(t_s) = L_{GUM} \frac{\mu_2}{\gamma} e^{\mathcal{N}} \simeq 1.25 \times 10^{-2} \text{ cm},
$$

*ria* the regime of the exponential growth will continuously transformed in ne regime of the exponential growth will continuously transformed in the scale factor<sup>‡</sup> The regime of the exponential growth will continuously transformed into the linear in time growth of  $\mu$ *µ*2 *µ µ* is a matrix of the exponential growth will continuously transionitied into the following form  $\frac{1}{2}$ xponential growth will continuously transformed into the linear in time growth of

$$
a(t) \simeq ct, \qquad a(\eta) \simeq a_0 e^{\eta}.
$$
 (5.87)

*Thank You !*