A gauge-invariant measure for gauge fields on $\mathbb{CP}^2$

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- *Gauge and Scalar Fields on* $\mathbb{CP}^2$: *A Gauge-Invariant Analysis I. The effective action from chiral scalars*

  D. Karabali, A. Maj and V.P. Nair

  Phys. Rev. D **106**, 085012 (2022) [arXiv:2110.11926]

- *Gauge and Scalar Fields on* $\mathbb{CP}^2$: *A Gauge-Invariant Analysis II. The measure for gauge fields and a 4d WZW theory*

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Motivation

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- On $\mathbb{CP}^2$ gauge-invariant parametrization

$$A = A(H, U)$$

$$Z[A] = \int [dA]\, e^{-S[A]} = \int \cancel{[dU]}\, [dH]\, e^{-\tilde{S}[H]}$$

$[dU]$ → gauge modes

Outline

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1. Parametrization of the gauge fields

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2. Measure of integration for the gauge orbit space

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3. Results and future directions

2d toy model

2d Abelian fields: $A_\mu(x^\mu), \qquad \mu = 1, 2$

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1) Complexify: $A = A_1 + iA_2, \quad \bar{A} = A_1 - iA_2$

& parametrize: $A = \partial f, \quad \bar{A} = \bar{\partial}\bar{f}$

2) Gauge dof: $\mathrm{Re}(f)$, gauge-invariant dof: $\mathrm{Im}(f)$

4d case

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1. 4d Euclidean space has no complex structure

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2. Choose $\mathbb{CP}^2$ as the 2d complex manifold

$\mathbb{CP}^2$ basics

Homogenous coordinates: $Z = (Z_1, Z_2, Z_3),\qquad Z \neq 0$

$$(Z_1, Z_2, Z_3) \equiv (\lambda Z_1, \lambda Z_2, \lambda Z_3), \quad \lambda \in \mathbb{C} - \{0\}$$

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$$Z = Z_3 \left(\frac{Z_1}{Z_3}, \frac{Z_2}{Z_3}, 1 \right) \equiv Z_3(z^1, z^2, 1)$$

$\mathbb{CP}^2$ basics

Metric: $$ds^2 = \frac{dz \cdot d\bar{z}}{(1 + z \cdot \bar{z})} - \frac{dz \cdot \bar{z} d\bar{z} \cdot z}{(1 + z \cdot \bar{z})^2} = g_{a\bar{a}} dz^a d\bar{z}^{\bar{a}}$$

Measure: $$d\mu = \frac{2}{\pi^2} \frac{d^4 x}{(1 + z \cdot \bar{z})^3}$$

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$z \to z/r$ $$ds^2 = \frac{dz\cdot d\bar{z}}{(1+z\cdot\bar{z}/r^2)} - \frac{dz\cdot\bar{z}d\bar{z}\cdot z}{r^2(1+z\cdot\bar{z}/r^2)^2}$$

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   - group theoretic structure as $SU(3)/U(2)$ facilitates a parametrization of the gauge fields

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- functions on $SU(3)$ in terms of Wigner D-matrices:

$$F(g) = \sum_{J,A} C_A^{(J)} \, \mathcal{D}_{A,B}^{(J)}(g) = \sum_{J,A} C_A^{(J)} \, \langle J, A|\hat{g}|J, B\rangle$$

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- $U(2)$ subgroup: local isotropy group for $\mathbb{CP}^2$
  - scalars, vectors, tensors are functions on SU(3) that behave in the right way under the U(2) subgroup

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$\longrightarrow$ SU(2) doublets with $Y = \pm 1$

gauge fields on $\mathbb{CP}^2$

Abelian gauge fields:

$$
\begin{aligned}
A_a &= \nabla_a f(g) + g_{a\bar{a}} \epsilon^{\bar{a}\bar{b}} \bar{\nabla}_{\bar{b}} \chi(g) \\
\bar{A}_{\bar{a}} &= -\bar{\nabla}_{\bar{a}} \bar{f}(g) - g_{\bar{a}a} \epsilon^{ab} \nabla_b \bar{\chi}(g)
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Non-Abelian gauge fields:

$$\begin{aligned} A_a &= -\nabla_a M M^{-1} + M^{\dagger-1} \left(g_{a\bar{a}} \epsilon^{\bar{a}\bar{b}} \bar{\nabla}_{\bar{b}} \chi \right) M^\dagger \\ \bar{A}_{\bar{a}} &= M^{\dagger-1} \bar{\nabla}_{\bar{a}} M^\dagger + M \left(-g_{\bar{a}a} \epsilon^{ab} \nabla_b \bar{\chi} \right) M^{-1} \end{aligned}$$

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gauge group $SU(N)$: $M, M^{\dagger} \in SL(N, \mathbb{C})$

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- Under gauge transformations $U \in SU(N)$:

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pure gauge parameters: $U \in SU(N)$

gauge-invariant parameters: $\rho, \quad \chi, \quad \bar{\chi}$

$$
H = M^{\dagger} M = \rho^2
$$

Outline

1. Parametrization of the gauge fields
2. Measure of integration for the gauge orbit space
3. Results and future directions

Measure for space of gauge potentials

Measure for gauge potentials: $$dV = \prod_x [\det(g_{a\bar{a}})\, dA_a^\alpha d\bar{A}_{\bar{a}}^\alpha]$$

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$$\xi = (\theta, \theta^\dagger, \delta\tilde{\chi}, \delta\tilde{\chi}^\dagger) \equiv (\delta M M^{-1},\, M^{\dagger -1}\delta M^\dagger,\, M^{\dagger -1}\delta\chi M^\dagger,\, M\delta\bar{\chi}M^{-1})$$

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$$dV \;=\; \sqrt{\det\mathcal{M}}\,[\, d\mu(M,\, M^\dagger)\, d\chi d\bar{\chi}]$$

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metric: $$ds^2 = \frac{1}{2}\int \xi^{\dagger\alpha}\mathcal{M}_{\alpha\beta}\xi^{\beta}$$

Jacobian:

$$\mathcal{M} = \begin{pmatrix} -\bar{D}\cdot D & -2\epsilon^{ab}\bar{D}_{\bar{a}}(M^{\dagger-1}\chi M^{\dagger})\bar{D}_{\bar{b}} & 0 & -\bar{D}(M^{\dagger-1}\chi M^{\dagger})\cdot D \\ 2\epsilon^{ab}D_a(M\bar{\chi}M^{-1})D_b & -D\cdot\bar{D} & D(M\bar{\chi}M^{-1})\cdot\bar{D} & 0 \\ 0 & -D\cdot\bar{D}(M^{\dagger-1}\chi M^{\dagger}) & -D\cdot\bar{D} & 0 \\ \bar{D}\cdot D(M\bar{\chi}M^{-1}) & 0 & 0 & -\bar{D}\cdot D \end{pmatrix}$$

where $D_a\Phi = \nabla_a\Phi + [-\nabla_a M M^{-1}, \Phi]$, $\bar{D}_{\bar{a}}\Phi = \bar{\nabla}_{\bar{a}}\Phi + [M^{\dagger-1}\bar{\nabla}_{\bar{a}}M^{\dagger}, \Phi]$

Measure

Measure for the gauge orbit space: $dV[\mathcal{C}] = e^{\Gamma(H,\chi,\bar{\chi})}[d\mu(H)d\chi d\bar{\chi}]$

$$
\begin{aligned}
\Gamma(H,\chi,\bar{\chi}) \;&= \frac{7}{2\pi r^2}\, S_{\rm wzw}(H) \;+ \left(-\frac{1}{4\epsilon} + \frac{1}{2r^2}\log\epsilon\right)\int \mathrm{Tr}\left(\bar{\nabla}\chi \cdot H\nabla\bar{\chi}H^{-1}\right) \\
&\quad + \frac{\log\epsilon}{12}\int \mathrm{Tr}\Big[(\bar{\nabla}(\nabla H H^{-1}))^2 + (\bar{\nabla}\chi\cdot H\nabla\bar{\chi}H^{-1})^2 \\
&\qquad + g^{a\bar{a}}g^{b\bar{b}}\bar{\nabla}_{\bar{a}}\left(\nabla_b H H^{-1}\right)[H\nabla_a\bar{\chi}H^{-1}, \bar{\nabla}_{\bar{b}}\chi] \\
&\qquad - g^{a\bar{a}}g^{b\bar{b}}\bar{\nabla}_{\bar{a}}\left(H\nabla_b\bar{\chi}H^{-1}\right)\mathcal{D}_a\bar{\nabla}_{\bar{b}}\chi\Big] \\
&\quad + \text{finite terms}
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$$+g^{a\bar{a}}g^{b\bar{b}}\bar{\nabla}_{\bar{a}}\left(\nabla_b H H^{-1}\right)\left[H\nabla_a\bar{\chi}H^{-1}, \bar{\nabla}_{\bar{b}}\chi\right]$$

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$$\Gamma(H,\chi,\bar{\chi}) = \frac{7}{2\pi r^2} S_{\rm wzw}(H) + \left(-\frac{1}{4\epsilon} + \frac{1}{2r^2}\log\epsilon\right)\int {\rm Tr}\left(\bar{\nabla}\chi \cdot H\nabla\bar{\chi}H^{-1}\right)$$

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$$+g^{a\bar{a}}g^{b\bar{b}}\bar{\nabla}_{\bar{a}}\left(\nabla_b H H^{-1}\right)[H\nabla_a\bar{\chi}H^{-1}, \bar{\nabla}_{\bar{b}}\chi]$$

$$-g^{a\bar{a}}g^{b\bar{b}}\bar{\nabla}_{\bar{a}}\left(H\nabla_b\bar{\chi}H^{-1}\right)\mathcal{D}_a\bar{\nabla}_{\bar{b}}\chi\Big]$$

$$+\text{finite terms} \ \ \mathcal{O}(r^2)$$

Low energy effective theory

1. Introduce an IR cutoff λ s.t. $1/\lambda < r^2$

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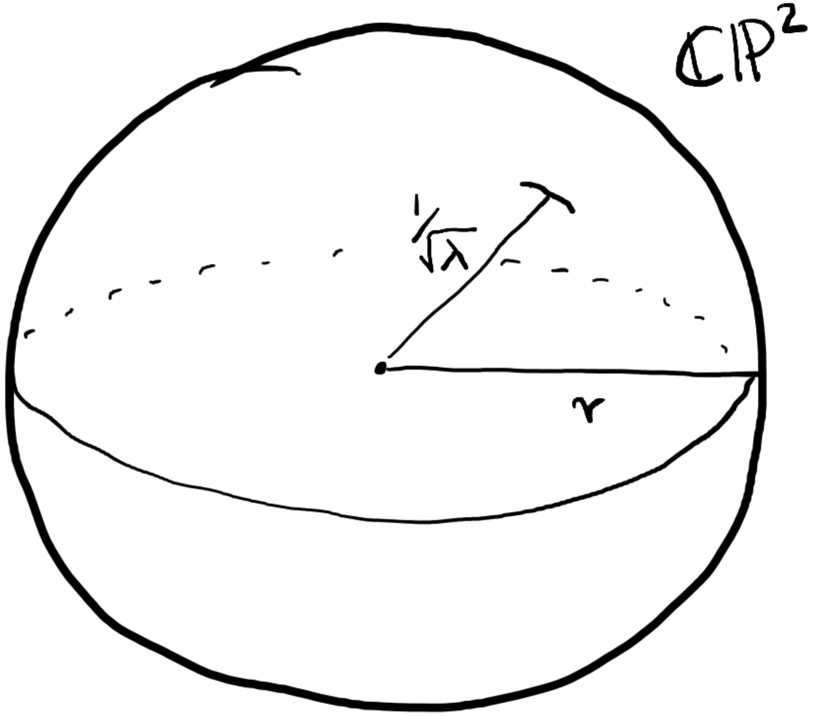
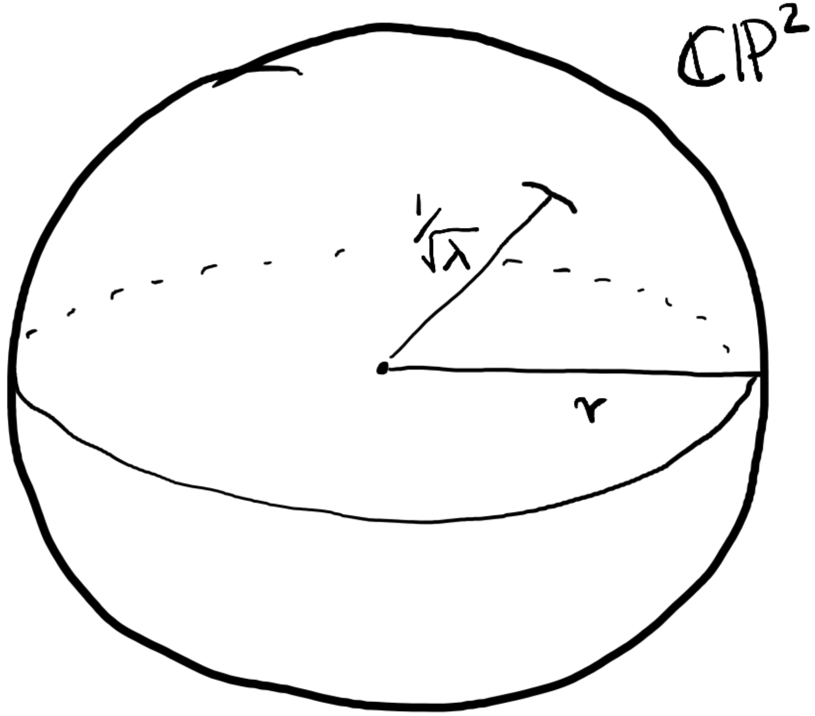


Low energy effective theory



1. Introduce an IR cutoff λ s.t. $1/\lambda < r^2$

2. Take the flat limit of $\mathbb{CP}^2$ $\quad r \to \infty$

Low energy effective theory

Measure for the gauge orbit space: $dV[\mathcal{C}] = e^{\Gamma(H,\chi,\bar{\chi})}[d\mu(H)d\chi d\bar{\chi}]$

$$
\begin{aligned}
\Gamma(H,\chi,\bar{\chi}) \;\; = \;\; & \frac{\lambda}{2\pi} S_{\mathrm{wzw}}(H) \; - \frac{1}{4\epsilon}\int \mathrm{Tr}\left(\bar{\nabla}\chi \cdot H\nabla\bar{\chi}H^{-1}\right) \\
& + \frac{\log\epsilon}{12}\int \mathrm{Tr}\Big[(\bar{\nabla}(\nabla H H^{-1}))^2 + (\bar{\nabla}\chi \cdot H\nabla\bar{\chi}H^{-1})^2 \\
& \qquad + g^{a\bar{a}} g^{b\bar{b}} \bar{\nabla}_{\bar{a}} \left(\nabla_b H H^{-1}\right)[H\nabla_a\bar{\chi}H^{-1}, \bar{\nabla}_{\bar{b}}\chi] \\
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& + \text{finite terms}
\end{aligned}
$$

Low energy effective theory

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$$+\text{finite terms} \quad \mathcal{O}\left(k^2/\lambda\right)$$

$$k^2 \ll \lambda$$

Outline

1. Parametrization of the gauge fields
2. Measure of integration for the gauge orbit space
3. Results and future directions

Results

Measure for the gauge orbit space: $dV[\mathcal{C}] = e^{\Gamma(H,\chi,\bar{\chi})}[d\mu(H)d\chi d\bar{\chi}]$

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$$\Gamma_{\rm log} \sim \log\epsilon \int (\partial A)^2$$

Mass terms

$$\Gamma(H,\chi,\bar{\chi}) = \frac{\lambda}{2\pi} S_{\mathrm{wzw}}(H) - \frac{1}{4\epsilon}\int \mathrm{Tr}\left(\bar{\nabla}\chi \cdot H \nabla \bar{\chi} H^{-1}\right) \quad + \ldots$$

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$$\to \frac{\lambda}{2\pi}S_{\text{wzw}}(H) + \mu_{\text{ren}}\int \mathrm{Tr}\left(\bar{\nabla}\chi \cdot H\nabla\bar{\chi}H^{-1}\right) \quad + \ldots$$

Dimensional transmutation

$$\Gamma(H,\chi,\bar{\chi}) = \frac{\lambda}{2\pi} S_{\rm wzw}(H) + \mu_{\rm ren} \int {\rm Tr}\left(\bar{\nabla}\chi \cdot H \nabla \bar{\chi} H^{-1}\right)$$

In pure YM theory need to introduce a dimensional parameter: $\Lambda_{\rm QCD}$

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- Can we express $\Lambda_{\rm QCD}$ in terms of λ and $\mu_{\rm ren}$?

$$Z = \int [d\mu(H)][d\chi][d\bar{\chi}]\, e^{\Gamma(H,\chi,\bar{\chi}) - \frac{1}{2g^2}\int {\rm Tr} F^2} \quad \rightarrow \quad \Lambda_{\rm QCD} = \Lambda_{\rm QCD}(\lambda, \mu_{\rm Ren})$$

Gluon self-energy

$$\Gamma(H,\chi,\bar{\chi}) = \frac{\lambda}{2\pi} S_{\mathrm{wzw}}(H) + \mu_{\mathrm{ren}} \int \mathrm{Tr}\left(\bar{\nabla}\chi \cdot H \nabla \bar{\chi} H^{-1}\right)$$

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Mass term in 4d Euclidean space:

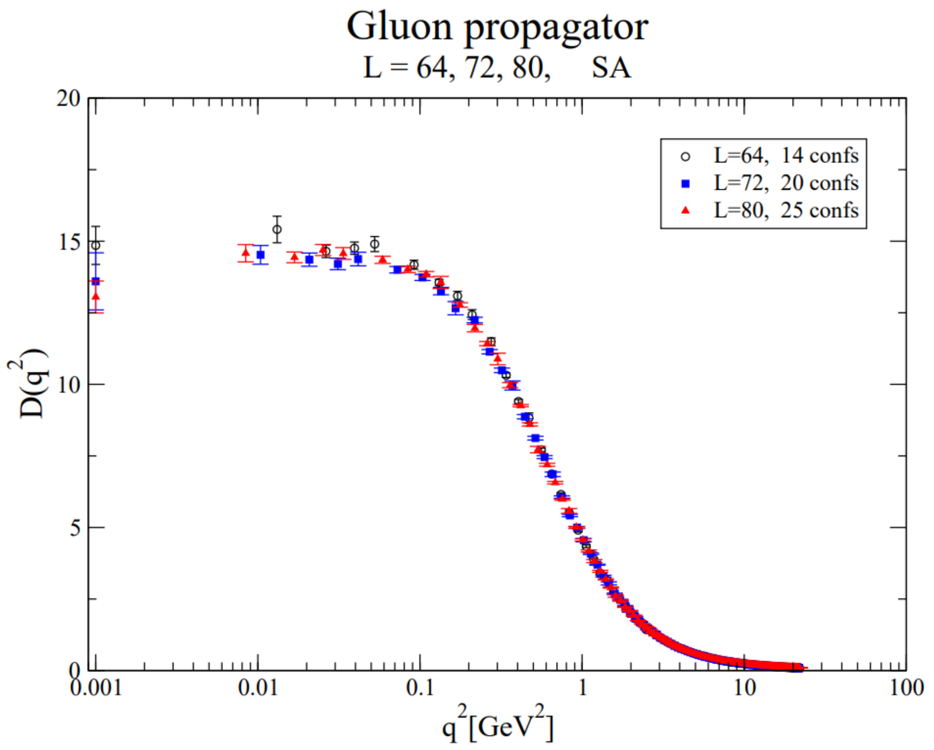
$$\int \mathrm{Tr} A_i \left(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2} \right) A_j + \cdots \sim \int \mathrm{Tr} F \left(\frac{1}{-D \cdot D} \right) F\,?$$

Gluon self-energy

Soft gluon mass

Gluon self-energy

Soft gluon mass



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  - I.L. Bogolubsky, et.al., Phys.Lett. B676, 69-73 (2009)
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  - P.O. Bowman, et.al., Phys.Rev. D76:094505 (2007)

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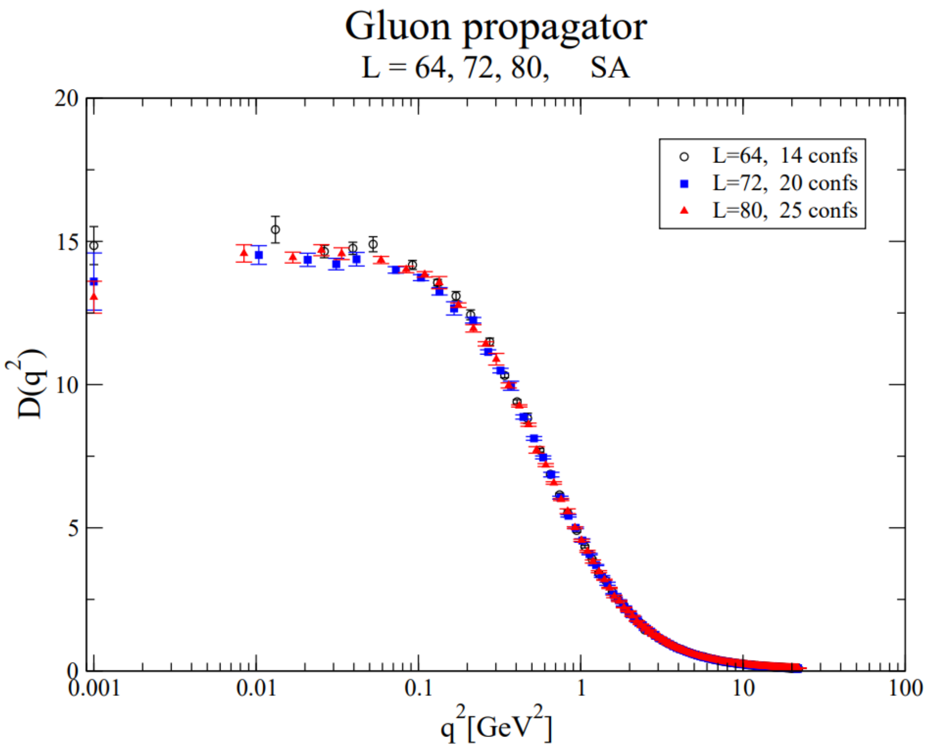

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  - A.C. Aguilar et.al., Front.Phys. 11(2), 111203 (2016)
  - A.C. Aguilar et.al., Phys.Rev. D78:025010 (2008)

WZW term

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$$\Gamma(H) \simeq \frac{\lambda}{2\pi} S_{\text{wzw}}(H) - \frac{\log \epsilon}{96} \int \text{Tr} F^2(H)$$

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$$Z \simeq \int [d\mu(H)]e^{\frac{\lambda}{2\pi}S_{\rm wzw}(H) - \frac{1}{2g^2}\int \mathrm{Tr}F^2(H)}$$

4d WZW action

$$S_{\rm wzw}(H) \;=\; \frac{1}{2\pi}\int g^{a\bar{a}}\,{\rm Tr}(\nabla_a H\,\bar{\nabla}_{\bar{a}} H^{-1}) - \frac{i}{24\pi}\int \omega\wedge {\rm Tr}(H^{-1}dH)^3$$

Kähler 2-form on $\mathbb{CP}^2$: $\omega = i\, g_{a\bar{a}}\, dz^a\, d\bar{z}^{\bar{a}}$

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- Also appears in Kähler-Chern-Simons theory, quantum Hall systems, string theory etc.
- Low energy dynamics dominated by antiself-dual instantons.

Casimir energy

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Lattice simulations of Casimir energy for 4d Yang-Mills theory

M.N. Chernodub et. al., Phys. Rev. D **108**, 014515 (2023)

- good fit for a massive scalar field

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$$S(H) = -\frac{\lambda}{2\pi} S_{\mathrm{wzw}}(H) + \frac{1}{2g^2}\int \mathrm{Tr} F^2(H) \xrightarrow{H = e^{t\cdot\theta}} \frac{1}{2}\int (\nabla\phi)^2 + m^2\phi^2 + \mathcal{O}(\phi^3)$$

with $m^2 = \frac{\lambda g^2}{8\pi^2}$

WZW theory and confinement

WZW theory and confinement

In 2+1 Yang Mills theory can show area law of Wilson loop:

$$\begin{aligned}\langle W(C)\rangle &= \langle \mathrm{Tr}\,\mathcal{P}e^{\oint_C \nabla H H^{-1}}\rangle \approx \mathrm{Tr}\,\exp\left[\frac{1}{2}\langle \oint_C \nabla H H^{-1} \oint_C \nabla H H^{-1}\rangle\right] \\ &= \mathrm{Tr}\exp\left[\mathcal{N}\int d\mu(H)\, e^{2c_A S^{(2d)}_{\mathrm{wzw}}(H) - \frac{8\pi}{e^4 c_A}\int \mathrm{Tr}(\bar{\nabla}(\nabla H H^{-1}))^2}\left(\frac{1}{2}\oint_C \nabla H H^{-1}\oint_C \nabla H H^{-1}\right)\right] \\ &\sim e^{-\sigma_R\,\mathrm{Area}(C)}, \qquad \sigma_R = e^4\frac{c_A c_R}{4\pi}\end{aligned}$$

D. Karabali, C. Kim, V.P. Nair, *Phys. Lett.* **B434**, 103 (1998)

WZW theory and confinement

In 4d Yang Mills theory?

$$\begin{aligned}\langle W(C)\rangle &= \langle \mathrm{Tr}\,\mathcal{P}e^{\oint_C \nabla H H^{-1}}\rangle \approx \mathrm{Tr}\,\exp\left[\frac{1}{2}\langle \oint_C \nabla H H^{-1} \oint_C \nabla H H^{-1}\rangle\right] \\ &= \mathrm{Tr}\exp\left[\mathcal{N}\int d\mu(H)\, e^{\frac{\lambda}{2\pi}S_{\rm wzw}(H)-\frac{1}{2g^2}\int \mathrm{Tr}F^2(H)}\left(\frac{1}{2}\oint_C \nabla H H^{-1}\oint_C \nabla H H^{-1}\right)\right] =?\end{aligned}$$

WZW theory and confinement

In 4d Yang Mills theory?

$$
\begin{aligned}
\langle W(C)\rangle &= \langle \mathrm{Tr}\,\mathcal{P}e^{\oint_C \nabla H H^{-1}}\rangle \approx \mathrm{Tr}\ \exp\left[\frac{1}{2}\langle \oint_C \nabla H H^{-1} \oint_C \nabla H H^{-1}\rangle\right] \\
&= \mathrm{Tr}\,\exp\left[\mathcal{N}\int d\mu(H)\, e^{\frac{\lambda}{2\pi}S_{\mathrm{wzw}}(H)-\frac{1}{2g^2}\int \mathrm{Tr}F^2(H)}\left(\frac{1}{2}\oint_C \nabla H H^{-1}\oint_C \nabla H H^{-1}\right)\right] =?
\end{aligned}
$$

Calculations too complicated here...

WZW theory and confinement

2+1 Yang Mills theory:

WZW theory and confinement

2+1 Yang Mills theory:

- Limit of $e^2 \to \infty$:

$$\begin{aligned}
\langle W(C)\rangle &= \langle \mathrm{Tr}\,\mathcal{P} e^{\oint_C \nabla H H^{-1}}\rangle \approx \mathrm{Tr}\,\exp\left[\frac{1}{2}\langle \oint_C \nabla H H^{-1} \oint_C \nabla H H^{-1}\rangle\right] \\
&= \mathrm{Tr}\exp\left[\mathcal{N}\int d\mu(H)\, e^{2c_A S^{(2d)}_{\mathrm{wzw}}(H) - \frac{8\pi}{e^4 c_A}\int \mathrm{Tr}(\bar{\nabla}(\nabla H H^{-1}))^2} \left(\frac{1}{2}\oint_C \nabla H H^{-1} \oint_C \nabla H H^{-1}\right)\right] \\
&\sim e^{-\sigma_R\,\mathrm{Area}(C)}
\end{aligned}$$

$$\sigma_R = e^4 \frac{c_A c_R}{4\pi}$$

WZW theory and confinement

2+1 Yang Mills theory:

➢Limit of $e^2 \to \infty$:

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&= \mathrm{Tr}\exp\left[\mathcal{N}\int d\mu(H)\, e^{2c_A S^{(2d)}_{\rm wzw}(H) - \cancel{\frac{8\pi}{e^4 c_A}\int \mathrm{Tr}(\bar{\nabla}(\nabla H H^{-1}))^2}} \left(\frac{1}{2}\oint_C \nabla H H^{-1} \oint_C \nabla H H^{-1}\right)\right] \\
&\sim e^{-\sigma_R\,\mathrm{Area}(C)} \longrightarrow 0 \qquad\qquad \sigma_R = e^4\frac{c_A c_R}{4\pi} \longrightarrow \infty
\end{aligned}$$

WZW theory and confinement

2+1 Yang Mills theory:

➢Limit of $e^2 \rightarrow \infty$:

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➢ Wilson loop vanishes in absence of F^2 term

WZW theory and confinement

4d Yang Mills theory:

WZW theory and confinement

4d Yang Mills theory:

➢ In absence of F^2 term (pure 4d WZW theory) Wilson loop vanishes:

$$\begin{aligned} \langle W(C)\rangle &= \langle \mathrm{Tr}\,\mathcal{P} e^{\oint_C \nabla H H^{-1}}\rangle \approx \mathrm{Tr}\,\exp\left[\frac{1}{2}\langle \oint_C \nabla H H^{-1} \oint_C \nabla H H^{-1}\rangle\right] \\ &\approx \mathrm{Tr}\exp\left[\mathcal{N}\int d\mu(H)\, e^{CS_{\rm wzw}(H)}\left(\frac{1}{2}\oint_C \nabla H H^{-1}\oint_C \nabla H H^{-1}\right)\right] \\ &\longrightarrow 0 \end{aligned}$$

Thank you!

Gribov problem?

- Is the parametrization of $\mathcal{A}/\mathcal{G}_*$ in terms of H, χ and $\bar{\chi}$ global?

  Most likely, no.

- Expect the measure to be invariant under possible transition functions between coordinate patches.