Gravitational Waves and their Detection

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Brief outline

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  - Gauge transformations and Killing equation
  - Gauge fixing condition: Harmonic gauge
  - Solution of the gravitational wave equation and their polarizations
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Brief outline

- Radiation of Gravitational waves from binary systems, their orbital decay and the merger time scale
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1.Gravitational Waves

1.1 Weak Field Approximation

- We consider a weak gravitational field, whose metric can be written as sum of Minkowski metric for flat space with a small perturbation $h_{\mu\nu}$:

$$\boxed{g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu} \quad , \quad h_{\mu\nu} << 1} \quad , \quad g^{(0)}_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1}$$

- Replacing the metric $g_{\mu\nu}$ above into the formula $g_{\mu\nu}g^{\nu\lambda} = \delta^{\lambda}_{\mu}$ we find the inverse metric $g^{\mu\nu}$ to be:

$$\boxed{g^{\mu\nu} = g^{(0)\mu\nu} - h^{\mu\nu}} \tag{2}$$

- Then we calculate for this metric the Christoffel symbols, the Riemann tensor and the Ricci tensor, neglecting powers of $h_{\mu\nu}$ higher than the first as significantly small terms, since $|h_{\mu\nu}| << 1$.

1.1 Weak Field Approximation

- We begin with the connection coefficients:

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\alpha} (\partial_\nu g_{\alpha\mu} + \partial_\mu g_{\alpha\nu} - \partial_\alpha g_{\mu\nu}),$$

which through the weak field approximation become:

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{(0)\sigma\alpha} (\partial_\nu h_{\alpha\mu} + \partial_\mu h_{\alpha\nu} - \partial_\alpha h_{\mu\nu}) \tag{3}$$

- Then we calculate the Riemann tensor:

$$R^{\sigma}_{\nu\lambda\rho} = \partial_\lambda(\Gamma^{\sigma}_{\rho\nu}) - \partial_\rho(\Gamma^{\sigma}_{\lambda\nu}) + \Gamma^{\gamma}_{\rho\nu}\Gamma^{\sigma}_{\lambda\gamma} - \Gamma^{\gamma}_{\lambda\nu}\Gamma^{\sigma}_{\rho\gamma} :$$

$$R_{\mu\nu\lambda\rho} = \frac{1}{2} (\partial_\lambda \partial_\nu h_{\mu\rho} + \partial_\rho \partial_\mu h_{\lambda\nu} - \partial_\lambda \partial_\mu h_{\rho\nu} - \partial_\rho \partial_\nu h_{\mu\lambda}) \tag{4}$$

1.1 Weak Field Approximation

- Finally, we calculate the Ricci tensor:

$$g^{(0)\mu\lambda}R_{\mu\nu\lambda\rho} = \frac{1}{2}g^{(0)\mu\lambda}(\partial_\lambda\partial_\nu h_{\mu\rho} + \partial_\rho\partial_\mu h_{\lambda\nu} - \partial_\lambda\partial_\mu h_{\rho\nu} - \partial_\rho\partial_\nu h_{\mu\lambda})$$

$$\boxed{R_{\nu\rho} = \frac{1}{2}(\partial_\lambda\partial_\nu h^\lambda_\rho + \partial_\rho\partial_\mu h^\mu_\nu - \partial_\rho\partial_\nu h + \Box h_{\rho\nu})} \tag{5}$$

1.2 Gauge transformations and Killing equation

- There are several perturbations in agreement with the weak field approximation, so we take the gauge transformations by taking an infinitesimal change in the coordinates:

$$\boxed{x^{\mu'} = x^{\mu} + \xi^{\mu}(x), \quad where \quad \xi^{\mu}(x) << 1}.$$

- In these coordinates, after some calculations, the new metric is:

$$g'^{\mu\nu}(x) = g^{\mu\nu}(x) + \xi^{\mu;\nu} + \xi^{\nu;\mu} \tag{6}$$

- So in order for the metric to be invariant under this transformation we take the killing equation:

$$\xi^{\mu;\nu} + \xi^{\nu;\mu} = 0 \tag{7}$$

1.2 Gauge transformations and Killing equation

- The perturbation of the metric under the infinitesimal coordinate transformation becomes:

$$h'^{\mu\nu} = h^{\mu\nu} - \frac{\partial \xi^{\nu}}{\partial x_{\mu}} - \frac{\partial \xi^{\mu}}{\partial x_{\nu}} \tag{8}$$

1.3 Gauge fixing condition: Harmonic gauge

- We now fix the gauge in order to simplify the formula of the Ricci tensor (5) and relate the general relativity with the classical gravitation. We choose the harmonic gauge condition: $\Box x^{\mu} = 0$, which through the weak field approximation becomes (proof in the appentix A):

$$\frac{\partial \Psi^{\mu}_{\nu}}{\partial x^{\mu}} = 0, \quad where \quad \Psi^{\mu}_{\nu} = h^{\mu}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu} h, \quad \nu = 0,1,2,3 \tag{9}$$

- These are actually four conditions $(\nu = 0,1,2,3)$ and do not violate the initial condition on $h_{\mu\nu}$ being small, while:

$$\Psi^{\mu}_{\mu} = h^{\mu}_{\mu} - \frac{1}{2}\delta^{\mu}_{\mu} h = h - \frac{1}{2} \cdot 4h = h - 2h = -h$$

- Calculating the second derivative of $\Psi_{\mu\nu}$ and doing some more calculation we find that:

$$\frac{\partial^2 h^{\mu}_{\nu}}{\partial x^{\rho} \partial x^{\mu}} + \frac{\partial^2 h^{\mu}_{\rho}}{\partial x^{\nu} \partial x^{\mu}} - \frac{\partial^2 h}{\partial x^{\rho} \partial x^{\nu}} = 0$$

1.3 Gauge fixing condition: Harmonic gauge

- So the Ricci tensor (5) is simplified to the following formula:

$$R_{\nu\rho} = \frac{1}{2}\Box h_{\rho\nu} \tag{10}$$

- As soon as we calculate: $h'^{\mu}_{\nu} = h^{\mu}_{\nu} - \frac{\partial \xi^{\mu}}{\partial x^{\nu}} - \frac{\partial \xi_{\nu}}{\partial x_{\mu}}$ and $h' = h - 2\frac{\partial \xi^{\mu}}{\partial x^{\mu}}$, we apply the harmonic gauge condition in the transformed coordinates in order to find the new conditions that appear:

$$\frac{\partial \Psi'^{\mu}_{\nu}}{\partial x^{\mu}} = \frac{\partial h'^{\mu}_{\nu}}{\partial x^{\mu}} - \frac{1}{2}\delta^{\mu}_{\nu}\frac{\partial h'}{\partial x^{\mu}} = 0$$

- Substituting and doing analytically the calculations we end up in the formula:

$$\Box \xi_{\nu} = 0 \tag{11}$$

So, ξ_{μ} satisfies the killing equation and the above residual symmetry condition.

1.4 Solution of the gravitational wave equation and their polarizations

- From the Einstein equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi k}{c^4}T_{\mu\nu} \tag{12}$$

in the void $T_{\mu\nu} = 0 \Rightarrow R_{\mu\nu} = 0$ (proof in the appentix B of the thesis) and from the formula (10):

$$\Box h_{\mu\nu} = 0 \tag{13}$$

in one direction: $(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})h_{\mu\nu} = 0$

$$\boxed{h_{\mu\nu} = h_{\mu\nu}(t \pm \frac{x}{c})} \tag{14}$$

1.4 Solution of the gravitational wave equation and their polarizations

- We reach the classical wave equation and we prove the existence of gravitational waves.
- Thus, the disturbance on the metric of our spacetime is a wave which propagates with the speed of light c.

Looking more into the nature of these waves:

- Since $\Psi^{\mu}_{\nu} = \Psi^{\mu}_{\nu}(t \pm \frac{x}{c})$, $\frac{\partial \Psi^{\mu}_{\nu}}{\partial x^{\mu}} = 0$ becomes:

$$\frac{\partial \Psi^{0}_{\nu}}{\partial x^{0}} + \frac{\partial \Psi^{1}_{\nu}}{\partial x^{1}} = 0, \quad \frac{\partial \Psi^{0}_{\nu}}{\partial (ct)} + \frac{\partial \Psi^{1}_{\nu}}{\partial x^{1}} = 0$$

$$\Psi^{0}_{\nu} - \Psi^{1}_{\nu} = 0, \quad \nu = 0, 1, 2, 3$$

1.4 Solution of the gravitational wave equation and their polarizations

- We take Ψ^{μ}_{ν} in the transformed coordinates:

$$\Psi'^{\mu}_{\nu} = \Psi^{\mu}_{\nu} - \partial_{\nu}\xi^{\mu} - \partial^{\mu}\xi_{\nu} + \delta^{\mu}_{\nu}\partial_{\rho}\xi^{\rho}$$

- Open the conditions above for the various μ, ν
- Take propagation of the wave in one direction, let it be x-axis: $\xi^{\mu} = \xi^{\mu}(t - \frac{x}{c})$ and we finally end up in the following conditions:

$$\Psi^{0}_{0} = \Psi^{0}_{1} = 0, \quad \Psi^{0}_{1} = \Psi^{1}_{1} = 0, \quad \Psi^{0}_{2} = \Psi^{1}_{2} = 0, \quad \Psi^{0}_{3} = \Psi^{1}_{3} = 0$$

$$\Psi^{2}_{2} + \Psi^{3}_{3} = 0$$

1.4 Solution of the gravitational wave equation and their polarizations

- So, Ψ^{μ}_{ν} which is a $\mathrm{symmetric}$ matrix is:

$$\Psi^{\mu}_{\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \Psi^{2}_{2} & \Psi^{2}_{3} \\ 0 & 0 & \Psi^{3}_{2} & \Psi^{3}_{3} \end{pmatrix}$$

- Ψ^{μ}_{ν} is traceless: $\Psi^{0}_{0} + \Psi^{1}_{1} + \Psi^{2}_{2} + \Psi^{3}_{3} = 0, \quad \Psi^{\mu}_{\mu} = -h = 0$
- So:

$$\boxed{\Psi^{\mu}_{\nu} = h^{\mu}_{\nu}} \tag{15}$$

$$or: \quad h^{\mu}_{\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h^{3}_{3} & h^{2}_{3} & 0 \\ 0 & h^{3}_{2} & h^{2}_{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h^{3}_{3} & h^{2}_{3} & 0 \\ 0 & h^{2}_{3} & -h^{3}_{3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

1.4 Solution of the gravitational wave equation and their polarizations

$$h^{\mu}_{\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h^3_3 & 0 & 0 \\ 0 & 0 & -h^3_3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = h^3_3 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad and \tag{16}$$

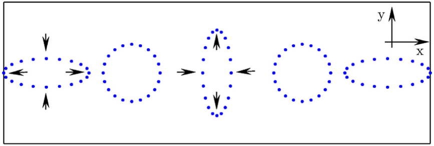
$$h^{\mu}_{\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & h^2_3 & 0 \\ 0 & h^2_3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = h^2_3 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{17}$$

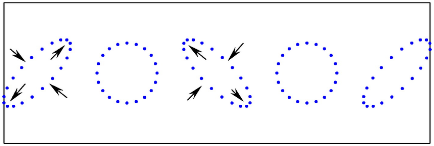
- Only 2 degrees of freedom from the 16 we initially got, taking all the symmetries of the system in account
- These 2 degrees of freedom-matrices correspond to the two ways the gravitational waves oscillate-2 polarizations:

1.4 Solution of the gravitational wave equation and their polarizations

<u>Polarization of Gravitational Waves</u>

- plus polarization
- cross polarization

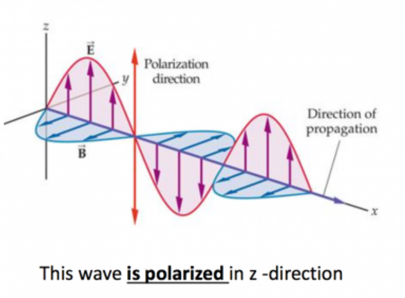




<u>Polarization of Electromagnetic Waves</u>

Considering propagation in x-axis:

- polarization in z-axis
- polarization in y-axis



1.4 Solution of the gravitational wave equation and their polarizations

- Substituting h^{μ}_{ν} in Fourier form: $h^{\mu}_{\nu}(x) = h^{\mu}_{\nu}(k)e^{ik_{\mu}x^{\mu}}$ into:
  - the harmonic gauge: $\frac{\partial \Psi^{\mu}_{\nu}}{\partial x^{\mu}} = \frac{\partial h^{\mu}_{\nu}}{\partial x^{\mu}} = 0 \rightarrow \boxed{k_{\mu}h_{\mu\nu}(k) = 0}$:

    the wave vector k_{μ}, which shows the direction of propagation, is perpendicular to the plane of the wave's oscillation, exactly as in EM.

  - the formula (13): $\Box h^{\mu}_{\nu} = 0 \rightarrow \boxed{k^{\mu}k_{\mu} = 0}$: the wavevector k_{μ} is null:

$\rightarrow$ the gravitational waves propagates with the speed of light
$\rightarrow$ the graviton is massless.
$\rightarrow$ $k^{\mu} = (\omega, 0, 0, \omega) = \omega(1, 0, 0, 1)$: taking the propagation of the wave only in x-direction

1.4 Solution of the gravitational wave equation and their polarizations

- The energy flux of a gravitational wave is given by the energy-momentum tensor:

$$T^{\mu\nu} = \frac{c^4}{32\pi k} h_{\rho}^{\lambda,\mu} h_{\lambda}^{\rho,\nu} \tag{18}$$

- When it propagates through the x-axis, $h_{\mu\nu}$ depends only on x,t, so as the energy flux: cT^{01} and since the only non zero components are the $h_3^3 = -h_2^2$ and $h_3^2 = h_2^3$:

$$cT^{01} = \frac{c^5}{32\pi k} h_{\rho}^{\lambda,0} h_{\lambda}^{\rho,1} \rightarrow$$

$$cT^{01} = \frac{c^3}{16\pi k}((\dot{h}_3^2)^2 + \frac{1}{4}(\dot{h}_2^2 - \dot{h}_3^3)^2) \tag{19}$$

2.Radiation of Gravitational Waves and their intensity

2.1 Flat Gravitational Waves and Einstein Equation

Instead of vacuum, now we consider some bodies moving and producing gravitational waves, so now:

- $T_{\mu\nu} \neq 0$
- (10):$R_{\mu\nu} = \frac{1}{2}\Box h_{\mu\nu}$
- Gauge condition (9):$\Psi_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\delta_{\mu\nu}h \Rightarrow \Box\Psi^{\mu}_{\nu} = \Box h^{\mu}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu}\Box h$
- Einstein equation: $R_{\mu\nu} = \frac{8\pi k}{c^4}(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T) \Rightarrow$

$$\left.\begin{array}{l} \frac{1}{2}\Box h^{\mu}_{\nu} = \frac{8\pi k}{c^4}(T^{\mu}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu}T) \quad and \\ \frac{1}{2}\Box h = \frac{8\pi k}{c^4}(T - \frac{1}{2}\cdot 4T) = -\frac{8\pi k}{c^4}T \end{array}\right\} \Rightarrow$$

2.1 Flat Gravitational Waves and Einstein Equation

- $\frac{1}{2}\Box\Psi^{\mu}_{\nu} = \frac{8\pi k}{c^4}T^{\mu}_{\nu} + O(h^2)$, or $\frac{1}{2}\Box\Psi^{\mu}_{\nu} = \frac{8\pi k}{c^4}\tau^{\mu}_{\nu}$,
  where $\tau_{\mu\nu}$ is the energy-momentum tensor in the weak gravitational field and contains terms of $T_{\mu\nu}$ and terms of second order of h.
- This equation is analogous to the corresponding one for the potential $A_{\mu}(\vec{r}, t)$ of electromagnetic waves:

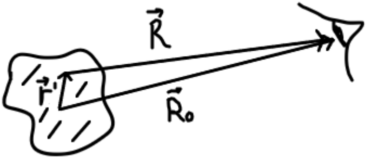
$$\Box A_{\mu}(\vec{r}, t) = \frac{4\pi}{c} j_{\mu}(\vec{r}, t)$$

the solution of which is:

$$A_{\mu}(\vec{R_0}, t) \approx \frac{1}{cR_0} \int j_{\mu}(\vec{r'}, t - \frac{R_0}{c}) \quad d\vec{r'}^3, \quad so:$$

$$\Psi^{\mu}_{\nu}(\vec{r}, t) \approx -\frac{4k}{R_0 c^4} \int \tau^{\mu}_{\nu}(\vec{r'}, t - \frac{R_0}{c}) \quad dV' \tag{20}$$

2.1 Flat Gravitational Waves and Einstein Equation



- From the harmonic gauge condition: $\frac{\partial \Psi^{\mu}_{\nu}}{\partial x^{\mu}} = 0$:

$$\frac{\partial \tau^{\mu}_{\nu}}{\partial x^{\mu}} = 0$$

- In order to evaluate the integral above we separate the space and time components of this condition:

$$\frac{\partial \tau_{ij}}{\partial x^{j}} - \frac{\partial \tau_{i0}}{\partial x^{0}} = 0, \quad \frac{\partial \tau_{0j}}{\partial x^{j}} - \frac{\partial \tau_{00}}{\partial x^{0}} = 0$$

2.1 Flat Gravitational Waves and Einstein Equation

- We integrate after multiplying by x^k the first condition and $x^i x^j$ the second one, so combining them we get that:

$$\boxed{\int \tau_{ik} dV = \frac{1}{2}\frac{\partial^2}{\partial x_0^2}\int \tau_{00} x^i x^j dV}$$

- Now, τ_{00} is the energy density of our system, thus:

$$\tau_{00} = \mu c^2 = \mu(x', y', z'; t - \frac{R_0}{c}) \cdot c^2$$

- Therefore, the weak gravitational field is:

$$(18) : \Psi_\nu^\mu = -\frac{4k}{R_0 c^4}\int \tau_\nu^\mu (t - \frac{R_0}{c}) \quad dV' \Rightarrow \Psi_{ij} = -\frac{2k}{R_0 c^4} \cdot \frac{\partial^2}{\partial t^2}\int \mu x^i x^j dV$$

$$h_{ij} = -\frac{2k}{R_0 c^4} \cdot \frac{\partial^2}{\partial t^2}\int \mu x^i x^j dV \qquad (21)$$

since: $\Psi_{ij} = h_{ij}(15)$

2.2 Energy Density Flow and the Quadrupole Momentum

- We insert the quadrupole momentum D_{ij}:

$$D_{ij} = \int \mu(3x^i x^j - \delta^{ij} x_\kappa^2) dV \quad , \quad D_{ii} = 0 \tag{22}$$

- So now we can calculate the energy flow through x^1 direction of a weak gravitational wave, produced by some moving bodies, as it propagates through this direction, depending on the quadrupole momentum of these bodies:

$$cT^{01} = \frac{c^3}{16\pi k}(\dot{h}_{23}^2 + \frac{1}{4}(\dot{h}_{22} - \dot{h}_{33})^2)$$

- $h_{23} = -\frac{2k}{R_0 c^4} \cdot \frac{\partial^2}{\partial t^2} \int \mu x^2 x^3 dV = -\frac{2k}{3R_0 c^4} \ddot{D}_{23}$

  $h_{22} - h_{33} = -\frac{2k}{R_0 c^4} \cdot \frac{\partial^2}{\partial t^2} \int \mu(x_2^2 - x_3^2) dV = -\frac{2k}{3R_0 c^4}(\ddot{D}_{22} - \ddot{D}_{33})$

2.2 Energy Density Flow and the Quadrupole Momentum

- $$cT^{01} = \frac{k}{36\pi R_0^2 c^5}[\dddot{D}_{23}^2 + \frac{1}{4}(\dddot{D}_{22} - \dddot{D}_{33})^2] \quad (23)$$

- Thus, the flow of radiation through a spherical angle $R_0^2 d\Omega$ is:

$$dI = cT^{01} \cdot R_0^2 d\Omega$$

$$dI = \frac{k}{36\pi c^5}[\dddot{D}_{23}^2 + \frac{1}{4}(\dddot{D}_{22} - \dddot{D}_{33})^2]d\Omega \quad (24)$$

- We can notice that when a mass is spherically symmetric, so $D_{ij} = 0$, it doesn't produce gravitational radiation.

2.3 Radiation Flow and the polarizations of Flat Gravitational Waves

- We consider the tensor polarization e_{ij} for the gravitational wave, as we can see it from the formulas (15), (16):

$$e_{ij}^{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_{ij}^{(2)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{25}$$

with: $e_{ii} = 0, \quad e_{ij}^{(\lambda)} \cdot e_{ij}^{(\lambda)} = 1$ and $e_{ij} \cdot n_j = 0$, where $n_j = \dfrac{x_j}{|x|}$.

So now:

$$dI = \frac{k}{72\pi c^5} \sum_{\lambda_1, \lambda_2} (\dddot{D}_{ij} \cdot e_{ij}^{(\lambda)})^2 d\Omega \tag{26}$$

2.4 Average of Radiation over polarizations

- The total radiation in all directions per unit time is $\frac{dI}{dt} \cdot 4\pi$ and we should average over all:
  - directions and
  - polarizations.
- First, we average over polarizations:

$$\overline{dI} = \frac{k}{72\pi c^5} \sum_{\lambda_1,\lambda_2} \dddot{D}_{ij} \dddot{D}_{kl} \cdot \overline{e_{ij}^{(\lambda)} e_{kl}^{(\lambda)}} d\Omega = 2 \cdot \frac{k}{72^5} \dddot{D}_{ij} \dddot{D}_{kl} \cdot \overline{e_{ij} e_{kl}} d\Omega$$

- $\overline{e_{ij} e_{kl}} = \frac{1}{4}[n_i n_j n_k n_l + (n_i n_j \delta_{kl} + n_k n_l \delta_{ij}) - (\delta_{jl} n_i n_k + \delta_{ik} n_j n_l + \delta_{jk} n_i n_l +$

$$+\delta_{il} n_j n_k) - \delta_{ij}\delta_{kl} + (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})]$$

- Substituting and doing the calculations:
  $\overline{dI} = \frac{k}{36\pi c^5}[\frac{1}{4}(\dddot{D}_{ij} n_i n_j)^2 + \frac{1}{2} \cdot \dddot{D}_{kj} \dddot{D}_{kj} - \dddot{D}_{il} \dddot{D}_{kl} n_i n_k]$

2.5 Average of Radiation over Directions

- We then average over all directions n_i:

$$\overline{dI} = \frac{k}{36\pi c^5}[\frac{1}{4}\dddot{D}_{ij}\dddot{D}_{kl}\overline{n_i n_j n_k n_l} + \frac{1}{2}\cdot\dddot{D}_{kj}\dddot{D}_{kj} - \dddot{D}_{il}\dddot{D}_{kl}\overline{n_i n_k}]$$

- $\overline{n_i n_j} = \frac{1}{3}\delta_{ij}, \quad \overline{n_i n_j n_k n_l} = \frac{1}{15}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$
- So, after the calculations:

$$\overline{dI} = \frac{k}{36\pi c^5}\cdot\frac{1}{5}\dddot{D}_{ij}^2$$

- Finally, the total radiation in all directions per unit time is:

$$4\pi\cdot\frac{dI}{dt} = \frac{k}{45c^5}\cdot\dddot{D}_{ij}^2, \quad D_{ii} = 0 \tag{27}$$

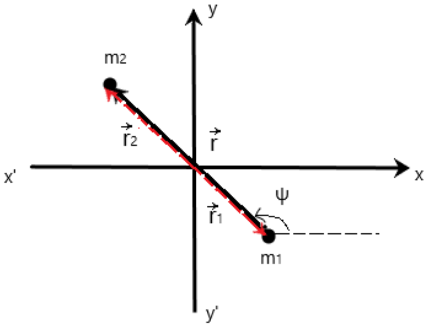
2.5 Average of Radiation over Directions

- For the electromagnetic radiation: $I = \frac{2}{3c^3}\ddot{d}^2$, where d is the dipole momentum, there must be just acceleration and is proportional to $1/c^3$.
- In the formula above there is third time derivative of the quadrupole momentum: there must be acceleration of acceleration.
- The gravitational radiation is proportional to $1/c^5$.
- $[\frac{dE}{dt}] = \frac{cm^3}{g \cdot sec^2} \cdot \frac{sec^5}{cm^5} \cdot (\frac{\frac{g}{cm^3} \cdot cm^2 \cdot cm^3}{sec^3})^2 = \frac{g \cdot cm^2}{sec^2} \cdot \frac{1}{sec}$

3.Radiation of Gravitational Waves from Binary Systems, their Orbital Decay and the Merger Time Scale

3.1 Gravitational Wave Radiation of a Binary System

- We consider two bodies rotating around each other in a (x,y) plane:



- $\vec{r} = \vec{r}_2 - \vec{r}_1$ and the reduced mass is : $\mu = \dfrac{m_1 \cdot m_2}{m_1 + m_2}$
- $\vec{r}_1 = -\dfrac{m_2}{m_1 + m_2}\vec{r}, \quad \vec{r}_2 = \dfrac{m_1}{m_1 + m_2}\vec{r}$

3.1 Gravitational Wave Radiation of a Binary System

- $x = r \cdot cos\psi, \quad y = r \cdot sin\psi, \quad z = 0$
- $F_{1\to 2} = F_{2\to 1} = k\dfrac{m_1 m_2}{r^2} = F_\kappa = \dfrac{\mu v^2}{r} = \dfrac{m_1 m_2}{m_1 + m_2} \cdot \dfrac{\omega^2 \cdot r^2}{r}$

$$\boxed{\omega = \sqrt{\frac{k(m_1 + m_2)}{r^3}}}$$

- The total radiation that these two rotating bodies emit is:

$$(27) : \overline{\frac{dE}{dt}} = \frac{k}{45c^5} \cdot \overline{\dddot{D}_{ij}^2}$$

where we take the time average over the period: $T = \dfrac{2\pi}{\omega}$.

- So we need to calculate the average of the squares of derivatives of quadrupole momentum of this system:

■ $D_{xx} = m_1(3x_1x_1 - x_1^2 - y_1^2 - z_1^2) + m_2(3x_2x_2 - x_2^2 - y_2^2 - z_2^2)$

3.1 Gravitational Wave Radiation of a Binary System

$$D_{xx} = \frac{m_1 \cdot m_2^2}{(m_1+m_2)^2}(2x^2 - y^2) + \frac{m_2 \cdot m_1^2}{(m_1+m_2)^2}(2x^2 - y^2)$$

$$D_{xx} = \mu \cdot r^2(3cos^2\psi - 1), \quad \dddot{D}_{xx} = 24\mu r^2\omega^3 \cdot sin\psi \cdot cos\psi, \ \overline{\dddot{D}^2_{xx}} = 72\mu^2 r^4 \omega^6$$

Correspondingly:

- $\overline{\dddot{D}^2_{yy}} = 72\mu^2 r^4 \omega^6$
- $\overline{\dddot{D}^2_{xy}} = 72\mu^2 r^4 \omega^6$
- $D_{xz} = 3m_1x_1z_1 + 3m_2x_2z_2 = 0 = D_{yz}, \quad \dot{D}_{zz} = 0$

- So, eventually, the total radiation the system loses while the bodies are rotating is:

$$\overline{\frac{dE}{dt}} = \frac{32k^4 m_1^2 m_2^2 (m_1+m_2)}{5c^5 r^5} \qquad (28)$$

3.2 Orbital Decay and Merger Time

- For simplicity, we consider that the two masses are equal, so the reduced mass then is: $\mu = \frac{m \cdot m}{m + m} = \frac{m}{2}$ and the total radiation:

$$\boxed{\frac{dE}{dt} = \frac{64k^4m^5}{5c^5r^5}} \tag{29}$$

- From the classical Newton's law of gravity:

$$E = -k\frac{m_1m_2}{2r}, \quad \frac{dE}{dt} = k\frac{m_1m_2}{2r^2} \cdot \dot{r}$$

- Comparing the two relations about $\frac{dE}{dt}$ above, we get the rate of orbital decay of the two masses, since while they rotate around each other, they emit gravitational radiation and lose energy:

$$\dot{r} = \frac{64k^3m_1m_2(m_1 + m_2)}{5c^5r^3}$$

3.2 Orbital Decay and Merger Time

- Integrating this formula, we find the expected time of the merger of these two rotating masses:

$$T = \frac{5}{256} \cdot \frac{c^5}{k^3} \cdot \frac{r^4}{m_1 m_2 (m_1 + m_2)} \tag{30}$$

- All these formulas above are correct as long as distance between the two masses of the binary system is much larger than their radius, in order to be able to think of them as massive points as we did.
- So, these formulas can't be applied for merger or ring-down, since then the distance between the masses are minimal.

3.3 Application of the Equation of Total Emitting Radiation on Physical Binary Systems

■ Hydrogen atom

- Its radius and the masses of the particles that compose it are:

$$\alpha_0 = 5,291 \cdot 10^{-9} cm, \quad m_e = 9,11 \cdot 10^{-28} g, \quad m_p = 1,67 \cdot 10^{-24} g$$

- The distance of the electron of the centre mass of the system proton-electron and the reduced mass as well are:

$$\alpha_0^* = \frac{m_e}{\mu}\alpha_0 = 5,295 \cdot 10^{-9} cm, \quad \mu = \frac{m_e \cdot m_p}{m_e + m_p}$$

- So, the gravitational radiation of the Hydrogen electron is:

$$\frac{dE}{dt} = \frac{64 k^4 m_e^5}{5 c^5 \alpha_0^{*5}} = 1,58 \cdot 10^{-180} \frac{g \cdot cm^2}{sec^3}: \quad \text{extremely small!}$$

3.3 Application of the Equation of Total Emitting Radiation on Physical Binary Systems

- So, since we know that the energy of the Hydrogen's electron in the ground state is 1 Ry:

$$1Ry = 2,18 \cdot 10^{-11} g \frac{cm^2}{sec^2},$$

the approximate time for the electron to collide into the nucleus will be:

$$t = \frac{1Ry}{\frac{dE}{dt}} \approx 10^{169} sec = 10^{153} billion \quad years!$$

practically never.

3.3 Application of the Equation of Total Emitting Radiation on Physical Binary Systems

Neutron stars binaries

- Let us take neutron stars of one solar mass which rotate around each other with radius r:

$$M_{neu} = M_{\odot} = 2 \cdot 10^{33} g, \quad r = 1,89 \cdot 10^{10} cm = 189.000 km$$

  - The energy they lose through gravitational waves is:

$$\frac{dE}{dt} = \frac{64 k^4 M_{neu}^5}{5 c^5 r^5} = 1,38 \cdot 10^{35} \frac{g \cdot cm^2}{sec^3}$$

  - The time of their merger is:

$$T = \frac{5}{256} \cdot \frac{c^5}{k^3} \cdot \frac{r^4}{M_{neu} M_{neu} (M_{neu} + M_{neu})} = 405.885 \quad years$$

  - Meanwhile, the period of their rotation is:

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{r^3}{k(M_{neu} + M_{neu})}} = 999,49 \quad sec = 16,7 minutes$$

3.3 Application of the Equation of Total Emitting Radiation on Physical Binary Systems

- When the radius is reduced to 1890 km or $r = 1,89 \cdot 10^8$ cm:
  - The energy the system loses is:

$$\frac{dE}{dt} = 1,38 \cdot 10^{45} \frac{g \cdot cm^2}{sec^3} :$$

    10 orders larger than before. The radius reduced by 2 orders and the energy increased by 10.
  - The collision merge time is:

$$T = 1,28 \cdot 10^5 sec \approx 36 \quad hours!$$

  - And the frequency of their rotation is:

$$\omega = \sqrt{\frac{2kM_{neu}}{r^3}} = 6,29s^{-1}, \quad f = \frac{\omega}{2\pi} = 1Hz$$

    The stars complete a rotation around each other in every 1 second and within 36 hours they merge.

3.3 Application of the Equation of Total Emitting Radiation on Physical Binary Systems

- So as the stars come closer $\rightarrow$ their rotation becomes more rapid$\rightarrow$ they lose much more energy through gravitational waves $\rightarrow$ thus the time of their merger is much smaller.

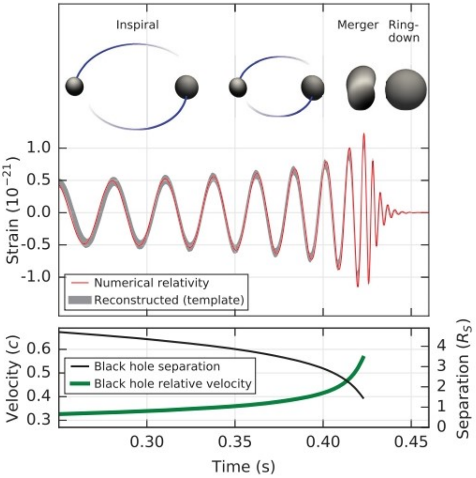
■ Binary Black Hole merger GW150914

- This system consisted of two rotating black holes whose masses were of about:

$$M_1 = 36M_\odot, \quad M_2 = 29M_\odot$$

- The event:
  - lasted approximately 0,2 sec: from a point of their inspiral to their merger
  - the signal: 35 Hz-250 Hz
  - a bigger black hole of about $M = 62M_\odot$ was formed.

3.3 Application of the Equation of Total Emitting Radiation on Physical Binary Systems



- For the merger only:
  - radius: $r = 350km = 35 \cdot 10^6 cm$,
  - time of the merger: $t_{merger} = 20\mu sec = 2 \cdot 10^{-5} sec$,
  - orbital frequency: $f = 75Hz$

3.3 Application of the Equation of Total Emitting Radiation on Physical Binary Systems

- We calculate these quantities through our formulas and compare them with the experimental data above:
  - $\omega = \sqrt{\dfrac{k(M_1 + M_2)}{r^3}} = 4,5 \cdot 10^2 sec^{-1}, \quad f = \dfrac{\omega}{2\pi} = 71,57Hz$
  - $T_{merger} = \dfrac{5}{256} \cdot \dfrac{c^5}{k^3} \cdot \dfrac{r^4}{M_1 M_2 (M_1 + M_2)} = 4,421 \cdot 10^{-3} sec$:

    2 orders of magnitude deviation! Expected!
- The Schwarschild radius: $r_g = \dfrac{2kM}{c^2}$ of the black holes above are:

$$r_{g1} = 106,72km, \quad r_{g2} = 85,97km, \quad r = 350km \approx r_{g1} + r_{g2}$$

3.3 Application of the Equation of Total Emitting Radiation on Physical Binary Systems

- We calculate the radius of the system for the time of merger that is depicted: $T_{merger} = 0,2sec$:

$$r = (\frac{256 \cdot k^3 M_1 M_2 (M_1 + M_2) T_{merger}}{5c^5})^{1/4} = 9,077 \cdot 10^7 cm = 907,7km$$

- Neutron stars: $r = 1,89 \cdot 10^7 cm, \quad T_{merger} = 12,8sec$
  Black holes: $r = 9,077 \cdot 10^7 cm, \quad T_{merger} = 0,2sec$
- The radiation the black holes emit per unit time when the radius is $r = 35 \cdot 10^6 cm$ is:

$$\frac{dE}{dt} = \frac{32k^4 M_1^2 M_2^2 (M_1 + M_2)}{5c^5 r^5} = 1,46 \cdot 10^{58} \frac{g \cdot cm^2}{sec^3}$$

3.3 Application of the Equation of Total Emitting Radiation on Physical Binary Systems

- For the corresponding radius $r = 18,9 \cdot 10^6 cm$ the neutron stars emit per unit time:

$$\frac{dE}{dt} = 1,38 \cdot 10^{50} \frac{g \cdot cm^2}{sec^3}$$

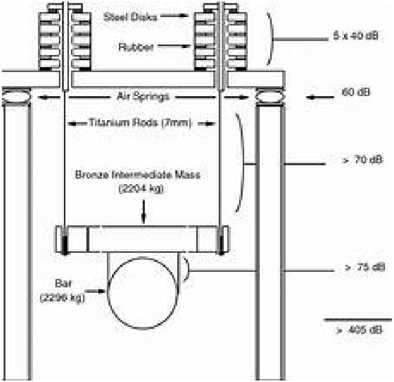
- Black Holes vs Neutron Stars:
  - more massive,
  - emit more energy and
  - they collapse faster.

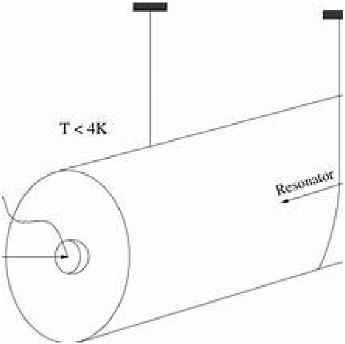
4.Running and Future GW Experiments

4.1 Cryogenic Resonant Bar Detectors

- The first GW detectors that were designed by Weber.
- The main part is a massive suspended cylinder of several tons which is vibrating at a characteristic resonance frequency.
- When a gravitational wave passes through them:
  - oscillates the cylinder
  - its resonance frequency is close to the frequency of gravitational radiation, around 75Hz
  - allow to detect the gravitational signal.
- In order to reduce the noise we:
  - suspend the cylinder
  - cool it in very low temperatures
  - increase its mass.
- Current experiments: ALLEGRO, AURIGA, EXPLORER, NAUTILUS, and NIOBI.

4.1 Cryogenic Resonant Bar Detectors





Detector ALLEGRO

4.2 Michelson Laser Interferometers

- The mainly used GW detectors.
- They consist of:
  - two perpendicular vacuum optic paths
  - a laser
  - a beam splitter
  - mirrors and
  - a photodetector
- The beams from the laser:
  - at some point split through the beam splitter
  - they reflect on the mirrors
  - finally they combine again and end in the detector.
- When a gravitational wave passes through:
  - slightly stretches one arm and shortens the other, changing the lengths of the two paths
  - the frequencies of the interfered beams change
  - their interference too.

4.2 Michelson Laser Interferometers

- The fluctuation is very small: need for isolation from sources of noise.
- Current detectors: LIGO, VIRGO, KAGRA
- Future detectors:

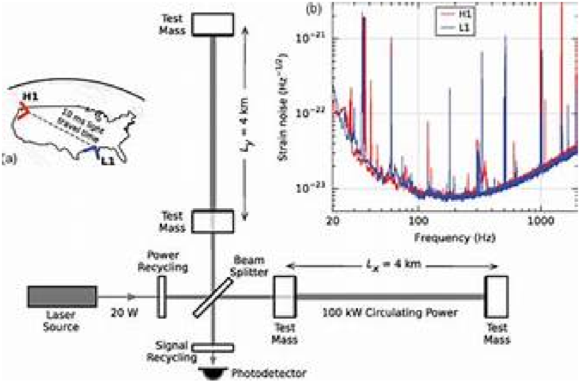
■ Einstein Telescope (ET)

- Underground detector limiting the effect of the seismic noise.
- Increase of the size of the interferometer: from the 3km arm length of the Virgo detector or 4 km of the LIGO to 10km, due to its shape.
- Cryogenic facilities that will cool down to 10–20K the mirrors to directly reduce the thermal vibration of the test masses
- New quantum technologies to reduce the fluctuations of the light
- A set of infrastructural and active noise-mitigation measures to reduce environmental perturbations

4.2 Michelson Laser Interferometers

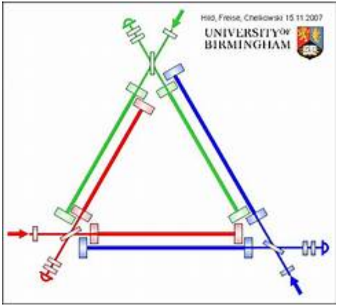
The big revolution in GW detectors: space-based interferometers$\rightarrow$ avoid earth originating noise, seismic, volcanos operation and others:

- LISA
  - Three spacecrafts arranged in an equilateral triangle
  - Sides 2.5 million kilometres long
  - Moving along an Earth-like heliocentric orbit
- DECIGO
  - Fills in the gap between the sensitive bands of frequency that LIGO and LISA detect.
- These future detectors are expected to operate around 2030 and 2035.

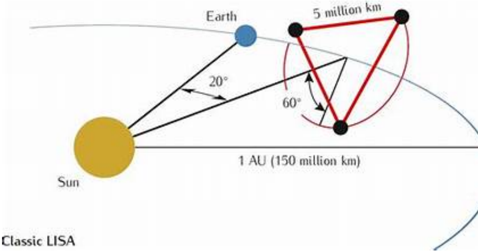


LIGO Detector

4.2 Michelson Laser Interferometers



Einstein Telescope Detector



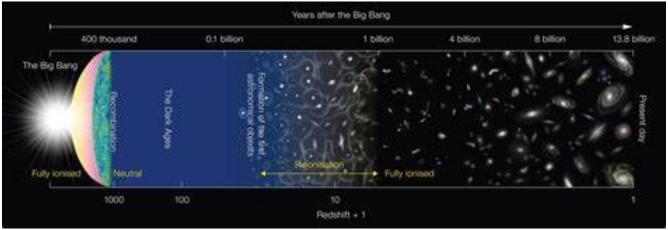
LISA Detiector

Conclusions

- We calculated the total gravitational energy radiation that the binary physical systems emits per unit time:
  - there must be acceleration of acceleration: in the formula there is third time derivative of the quadrupole momentum: $D_{ij} = \int \mu(3x^i x^j - \delta^{ij} x_\kappa^2) dV$
  - the gravitational radiation is proportional to $1/c^5$, in contrast to the electromagnetic radiation which is proportional to $1/c^3$: GW weaker than the EM waves.
- They propagate with the velocity of light
- There are only two polarizations perpendicular to the direction of the propagation
- Describe a massless particle, the graviton.
- The more massive the system is, the more violent the collision and bigger the emission of radiation is.

Conclusions

- In order to detect GW we need events of significantly large binary systems merging.
- Events that last for fractions of a second: we need detectors which can detect with accuracy such tricky signal.
- The bet for the years to come, is to construct the most efficient detectors in order to improve the studies on GW and retrieve the answers we expect we can get about the beginning of the universe.



Discussion

A follow up research in the study above would be calculating the formulas of merger time and total radiation for small distances of the system through the merger.

Thank you!