Computing neutrino interaction rates at high temperature

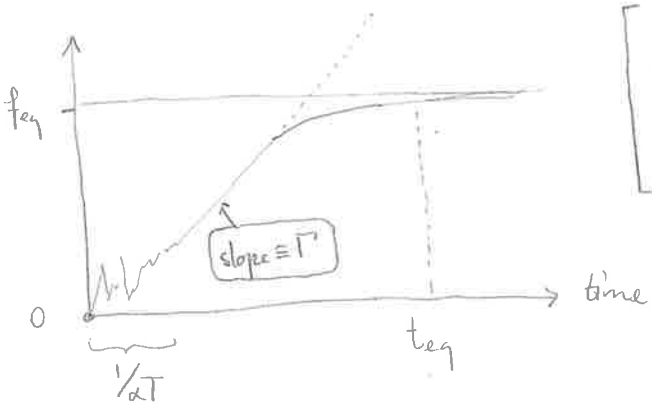
thermal field theory $\rightarrow$ static (bulk) properties

$\log Z \sim$  $+ \ldots$

$\rightarrow$ dynamical quantities (dissipation, transport coeffs)

$\langle \hat{\mathcal{O}}(t)\, \hat{\mathcal{O}}(0) \rangle \sim$ 

sketch of equilibration:



$$\left[\; f_{\underline{k}}(t,x) \equiv \frac{1}{V} \langle \hat{a}^{\dagger}_{\underline{k}} \hat{a}_{\underline{k}} \rangle \qquad \langle \ldots \rangle = \frac{1}{Z} \mathrm{tr}\left[e^{-\beta \hat{H}} \ldots \right] \;\right]$$

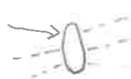
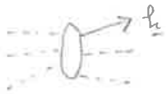
notation:

$\underline{k} = (k_x, k_y, k_z)$

$\dot{f} = (\partial_t - H \underline{k}\, \partial_{\underline{k}}) f$

E.g. time evolution of single particle distribution function

$$\dot{f}_{\underline{k}} = -\, \underbrace{f_{\underline{k}} \Gamma_d}_{\text{decay rate}} + \underbrace{(f_{\underline{k}} \overset{B}{\underset{F}{\pm}} 1)\, \Gamma_p}_{\text{production rate}} \tag{1}$$

"loss"   "gain"

Both Γ_d, Γ_p depend on f at t, so eq. (1) is non-linear.

After a long time, reach fixed point,

$$f_{\underline{k}} \xrightarrow[\text{equilib}]{\text{therm.}} \begin{cases} n_F(E) = \dfrac{1}{e^{E/T}+1} \\[2ex] n_B(E) = \dfrac{1}{e^{E/T}-1} \end{cases} \text{for} \quad \begin{matrix} F & \text{"}-\text{"} \\ B & \text{"}+\text{"} \end{matrix}$$

Detailed balance implies (for equilibrium) $\Gamma_d / \Gamma_p = e^{E/T}$.

situation # 1

Close to equilibrium, can replace $f_{\underline{k}} \rightarrow f_{eq}$ in Γ_d, Γ_p

Then eq. (1) can be simplified,

$$\dot{f}_{\underline{k}} = -\Gamma_{eq}(\underline{k}) \underbrace{\left[f_{\underline{k}} - f_{eq} \right]}_{\equiv \delta f} + \mathcal{O}(\delta f^2) \tag{2}$$

$$\Gamma_{eq} = \Gamma_d \overset{B}{\underset{F}{\mp}} \Gamma_p$$

$$\Rightarrow \quad f_{\underline{k}} \simeq f_{eq} + \delta f\, e^{-\Gamma t}$$

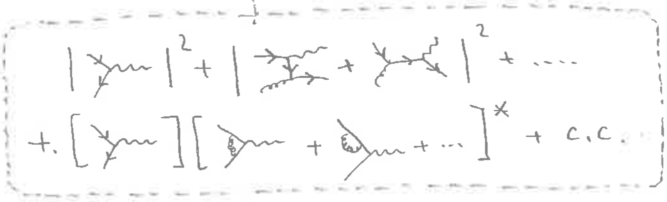
<u>situation # 2</u>

Very few particles, $f \ll 1$, but can be produced by thermal system. Then eq. (1) becomes

$$\dot{f}_{\underline{k}} = T_{prod} + \mathcal{O}(f) \qquad (3)$$

E.g. photon production by a quark-gluon plasma

$$T_{prod} \sim \int d\Omega \, |\mathcal{M}|^2 \, (f f \cdots)$$



Beware, T_{prod} is a <u>production</u> rate while T_{eq} describes an <u>equilibration</u> rate. If one sets $f_{\underline{k}} \approx 0$ in eq. (2) and compares with (3) – not justified! – but correct relation in general:

Weldon 1983
Bödeker et al. 2015

$$\boxed{T_{prod} = f_{eq} \cdot T_{eq}} \qquad \sum |\mathcal{M}|^2 = 2\,\mathrm{Im}[-\oslash-]$$

"optical theorem at finite-T"

<u>sketch of result</u>

Consider scalar field ϕ, coupled to a heat bath with

$$\hat{H}_{int} = \int_x \left(e\, \hat{\phi}^\dagger \hat{J} + \text{c.c.} \right)$$

Assume $\hat{\rho}(0) = \hat{\rho}_{bath} \otimes \underbrace{|0\rangle\langle 0|}_{\text{no } \phi\text{-particles initially}}$, $\quad \hat{\rho}_{bath} = \boxed{Z^{-1} e^{-\beta \hat{H}}}$

$$\dot{f}_{\underline{k}} = \frac{e^2}{2E} \underbrace{\int_X e^{iK\cdot X} \langle \hat{J}^\dagger(0) \hat{J}(X) \rangle}_{\text{Wightman function } \Pi^< = 2 n_B(E)\, \rho(K)}$$

Spectral function: $\rho(K)$

<u>a few comments</u>

(i) This equation is true to LO in e^2, but all orders in the plasma constituents (where Boltzmann eq. not valid).

(ii) Requires a separation of timescales, $\frac{1}{\alpha T} \ll t$ so that no Quantum oscillations, and $t \ll t_{eq}$ for pert. theory.

(iii) The spectral function ρ can be computed from a discontinuity of the Euclidean correlator:

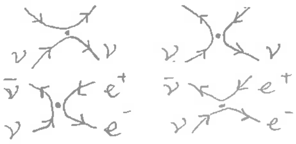
$$\text{bosons:} \quad \rho(E, \underline{k}) = \mathrm{Im}\left[\Pi(k_n, \underline{k}) \right]_{k_n \to -i(E + i0^+)}$$

$$\text{fermions:} \quad \rho(E, \underline{k}) = \mathrm{Tr}\left\{ \not{K}\, \mathrm{Im}\left[\Sigma(k_n, \underline{k}) \right] \right\}_{k_n \to \cdots} \qquad (4)$$

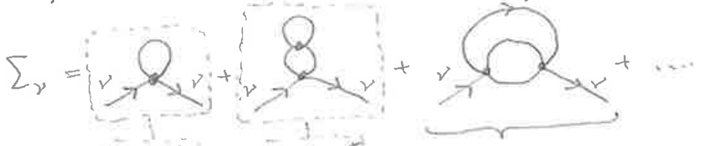
Ex: neutrinos @ decoupling

Weak interactions, $T \sim$ MeV can use Fermi model, vertex proportional to $G_F \sim 10^{-5}\,\mathrm{GeV}^{-2}$. An expansion in G_F, e should be treated consistently as an EFT.

- elastic scatterings (kinetic eq.)
- inelastic scat. (chemical eq.)



First, no QED corrections:



(5)

$G_F T^3$ $\quad$ $G_F^2 T^5$

local contributions w/ no mom. dep. have no discont.

imaginary part generates all allowed $2\to 2$ processes.

Now straight-forward to implement QED corrections:

(6)

1PI topologies. $\quad$ "plasmon resonance"

3-loop diagrams (scary!)

The last term represents an effective $\nu\bar\nu\gamma^*$ vertex, and requires the operator $\delta\mathcal{L} \sim e G_F \bar\nu_L \gamma_\mu \nu_L \partial_\nu F^{\mu\nu}$.

cf. Hill & Tomalak 2020 for computing $\delta\mathcal{L}$

After carrying out the loop integration in (6), and applying eq. (4), we find

$$\Gamma_\nu = G_F^2\, k \left(\int d\Omega^{(t)} - \int d\Omega^{(s)}\right) \overbrace{L_{\mu\nu}(K, K+P)}^{\text{(leptonic tensor)}}$$

$$\times \left[1 - n_F(k-p_0) + n_B(p_0)\right] \underbrace{\mathrm{Im}\,\Pi_{\mu\nu}(p_0, p)}_{\text{(spectral function)}} \tag{7}$$

hints to prove (7)

let $\mathcal{F}(K) \equiv \sum\!\!\!\!\!\!\!\int_P \Pi(P) \dfrac{\alpha(k_n+p_n)+\beta}{(K+P)^2}$

$\boxed{\text{insert } \Pi = \int_{-\infty}^{\infty} \frac{dp_0}{\pi} \frac{\mathrm{Im}\,\Pi(p_0,p)}{p_0 - i p_n}}$

notation

$\sum\!\!\!\!\!\!\!\int_Q = T\sum_{q_n} \int_q$

q_n = Matsubara frequency.

$Q^2 = q_n^2 + q^2$

$$\mathcal{F} = \int_p \int \frac{dp_0}{\pi} \mathrm{Im}\,\Pi(p_0,p)\; T\sum_{p_n} \frac{1}{p_0 - i p_n} \sum_{\{r_n\}} \frac{\alpha r_n + \beta}{r_n^2 + (k+p)^2}\, \delta_{r_n, k_n+p_n}$$

$\boxed{E_q = |k+p|}$ $\quad$ $T\int_0^{1/T} d\tau\, e^{i\tau(r_n - k_n - p_n)}$

$\Rightarrow$ Matsubara sums decouple.

useful identities: $T\sum_{p_n} \frac{e^{-ip_n\tau}}{p_0 - ip_n} = n_B(p^0)\, e^{(\beta-\tau)p_0}$

$$T\sum_{\{r_n\}} \frac{e^{i\tau r_n}}{r_n^2 + E_{k_p}^2} = \frac{n_F(E_{k_p})}{2E_{k_p}}\left[e^{(\beta-\tau)E_{k_p}} - e^{\tau E_{k_p}}\right]$$

$$\Rightarrow \mathcal{F} = \int_p \int \frac{dp_0}{\pi} \operatorname{Im}\Pi \, \frac{n_B(ip_0)\, n_F(E_{k_p})}{2E_{k_p}} \underbrace{\int_0^{1/T} d\tau\, e^{-ik_n\tau + (\beta-\tau)p_0} \left(-i d\partial_\tau + \beta\right)\left[\dots\right]}_{\text{creates energy denoms.}}$$

creates energy denoms. $\frac{\#}{ik_n + p_0 - E_{k_p}} + \frac{\#}{ik_n + p_0 + E_{k_p}}$

$$\downarrow \qquad\qquad \downarrow$$

taking $\operatorname{Im}\mathcal{F}$ gives δ-function $\longrightarrow$ $\delta(k_0 + p_0 - E_{k_p})$ $\quad$ $\delta(k_0 + p_0 + E_{k_p})$

then integrate angle of p w.r.t. $\underline{k}$.

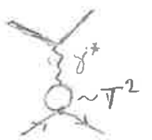
$|p_0| < p$ t-channel $\qquad$ $p_0 > p$ s-channel

Some details are a bit hidden in eq. (7), since the spectral function encodes a lot of information.

s-channel $\qquad$ t-channel.

$$\int d\Omega \sim \underbrace{\int \frac{dp_0\, dp}{k^2}}_{\text{2d-integ.}}$$

Everything is UV finite, but a danger lurks in some IR sensitive terms. For the t-channel plasmon term:



$$\int d\Omega^{(t)} |\mathcal{M}|^2 (\dots) \sim \underbrace{\int_0^{\# T^2} ds \int_{-s}^{0} dt \frac{1}{t}}_{\log T/m_D \,\sim\, \log 1/e}$$

resummed photon prop.

$$\Delta_P^{\mu\nu} = \underbrace{\mathbb{P}_T^{\mu\nu}\Delta_T + \mathbb{P}_L^{\mu\nu}\Delta_L}_{}$$

Hard Thermal Loop insertion

When all the pieces are put together, and the dust settles, the rate can be written:

(numerical functions of k, flavour dep.)

$$\Gamma_\nu(\underline{k}) = G_F^2\left\{ \# + e^2\left[\# + \#\log\frac{1}{e} + \#\log\frac{\bar{\mu}}{T}\right]\right\}$$

from HTL resum $\qquad$ from effective $\nu\bar{\nu}\gamma$ vertex, $\bar{\mu} \sim m_e$

for details, see 2312.07015

Relative QED corrections are $\sim -(0\dots2)\%$