Theoretical Astroparticle & Neutrino Physics

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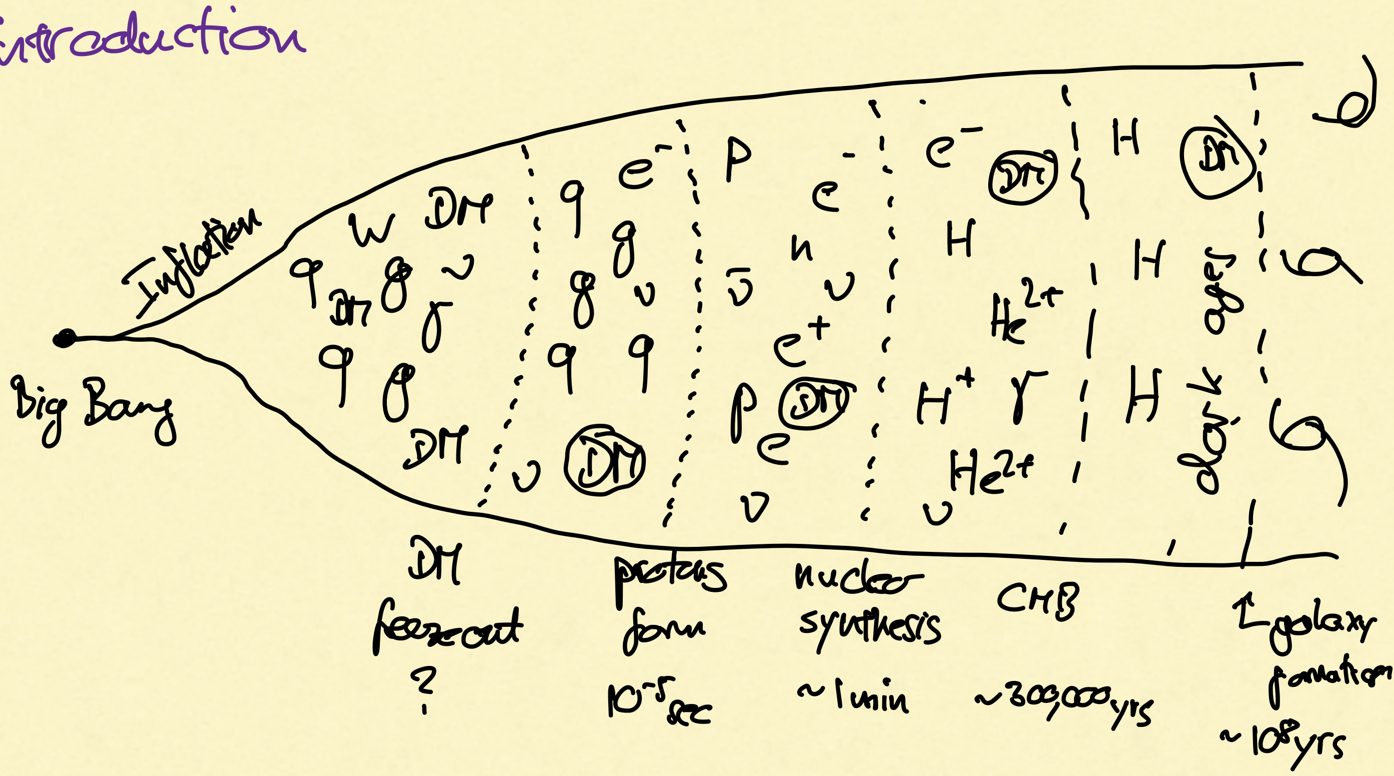
1. Introduction



2. Big Bang Theory

Einstein's equations

Ricci tensor, Ricci scalar $R = R^{\mu}{}_{\mu}$, metric, energy-momentum tensor

$$\underbrace{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu}}_{\text{geometry}} = \underbrace{8\pi G \, T_{\mu\nu}}_{\text{matter}}$$

cosmological constant

Metric: $g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} + \delta g_{\mu\nu}$

Riemann curvature tensor:

$$\frac{\partial g_{\mu\nu}}{\partial x^\alpha \partial x^\beta} = -\frac{1}{3}\left(R_{\mu\alpha\nu\beta} + R_{\mu\beta\nu\alpha}\right)$$

$$\Longleftrightarrow \quad g_{\mu\nu} \simeq \underbrace{\eta_{\mu\nu}}_{\text{Minkowski}} - \frac{1}{3} R_{\mu\alpha\nu\beta} \underbrace{x^\alpha x^\beta}_{\text{small}} + \ldots\ldots$$

Ricci tensor: $R_{\mu\nu} = R_{\mu\alpha\nu}$ $\quad$ [" $\hat{=}$ $\underset{\text{Laplace-op.}}{\Delta}$ $g_{\mu\nu}$ "]

In cosmology, we observe:

- Universe is homogeneous at large scales
- no preferred direction
- at large scales, only gravity matters

Motivates ansatz:

$$ds^2 = dt^2 - R^2(t)\left[dx^2 + dy^2 + dz^2\right]$$

To insert this metric into Einstein's equations, we need

$$\Gamma^\mu{}_{\nu\delta} = \frac{1}{2} g^{\mu\sigma}\left(g_{\sigma\nu,\delta} + g_{\sigma\delta,\nu} - g_{\nu\delta,\sigma}\right)$$

Einstein sum convention

$$\Gamma^0{}_{\underbrace{ij}_{\text{spatial indices}}} = -g_{ij}\frac{\dot{R}}{R} \; ; \quad \Gamma^i{}_{0j} = \Gamma^i{}_{j0} = -\frac{\dot{R}}{R}\delta^i{}_j$$

$$\Longrightarrow \quad R_{00} = -3\frac{\ddot{R}}{R}$$

$$R_{ij} = -\left[\frac{\ddot{R}}{R} + 2\frac{\dot{R}^2}{R}\right] g_{ij}$$

↳ insert into Einstein's equations (in the $\Lambda = 0$ case)

(00) - component :

$$-3\cancel{\frac{\ddot{R}}{R}} - \frac{1}{2}\left(-3\cancel{\frac{\ddot{R}}{R}} - \left[\cancel{\frac{\ddot{R}}{R}} + 2\frac{\dot{R}^2}{R^2}\right]\cdot 3\right) = 8\pi G T_{00}$$

$$\boxed{\frac{\dot{R}^2}{R^2} = \frac{8\pi G}{3}\cdot \underbrace{T_{00}}_{=\rho \text{ (energy density)}}} \quad (1)$$

Friedmann equation

(ii) - components :

$$\boxed{2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} = -8\pi G \underbrace{T_{ii}}_{=p \text{ (pressure density)}}} \quad (2)$$

$$\frac{(2)-(1)}{2} : \quad \boxed{\frac{\ddot{R}}{R} = -\frac{4}{3}\pi G(\rho + 3p)} \quad (3)$$

Friedmann - Lemaître equation

Here, we have used that the Universe at large scales behaves like an ideal fluid with

$$T^{\mu}{}_{\nu} = \mathrm{diag}\left(\rho(t), p(t), p(t), p(t)\right)$$

Behavior of $\rho(t)$ and $R(t)$

$$\frac{d}{dt}\left(R^2\cdot(1)\right) : \quad 2\dot{R}\ddot{R} = \frac{8\pi G}{3}\left(\dot{\rho}R^2 + 2\rho R\dot{R}\right)$$

use (3) $\quad : \quad \cancel{2}\dot{R}\cdot\left(-\cancel{\frac{4}{3}\pi G}\cancel{R}(\rho+3p)\right) = \cancel{\frac{8\pi G}{3}}\left(\dot{\rho}R^{\cancel{2}} + 2\rho\cancel{R}\dot{R}\right)$

$$\Leftrightarrow -\dot{R}(\rho + 3p) = \dot{\rho}R + 2\rho\dot{R}$$

$$\Leftrightarrow 3\rho\dot{R} + \dot{\rho}R + 3\dot{R}p = 0$$

We write $\boxed{p(t) = w\,\rho(t)}$

equation-of-state parameter

$\hookrightarrow \quad 3\dot{R}\rho(1+w) = -R\dot{\rho}$

$\Leftrightarrow \quad \frac{\dot{\rho}}{\rho} = -3\frac{\dot{R}}{R}(1+w)$

$\Leftrightarrow \quad \frac{d}{dt}\log\rho = -3(1+w)\frac{d}{dt}\log R$

$\int dt$; exp. $\Leftrightarrow \quad \boxed{\rho \sim R^{-3(1+w)}}$

Insert into Friedmann eq:

$$\frac{\dot{R}^2}{R^2} = \frac{8\pi}{3}G \cdot c\, R^{-3(1+w)}$$

$$R^{3(1+w)-2}\dot{R}^2 = \frac{8\pi G}{3}c$$

$$R^{\frac{3w+1}{2}}\,dR = \sqrt{\frac{8\pi G c}{3}}\,dt$$

$$\boxed{R \sim t^{\frac{2}{3(1+w)}}}$$

<u>Special cases:</u>

$w = 0$: "matter-dominated Universe" $(p=0)$

$R \sim t^{\frac{2}{3}}$; $\rho \sim R^{-3}$

$w = \frac{1}{3}$: "radiation-dominated Universe" $(p = \frac{1}{3}\rho)$

$R \sim t^{\frac{1}{2}}$; $\rho \sim R^{-4}$

$w = -1$ "vacuum-dominated Universe" ($p = -\rho$)

$$R \sim e^{Ht} \quad \text{(see later)}$$

$$\rho = \text{const}$$

Caveats:

- curvature:

  A more general ansatz for the metric is

$$ds^2 = dt^2 - R^2 \left(\frac{1}{1-kr^2} dr^2 + r^2 d\theta + r^2 \sin^2\theta \, d\phi^2 \right)$$

curvature parameter

↳ Friedmann eq. becomes:

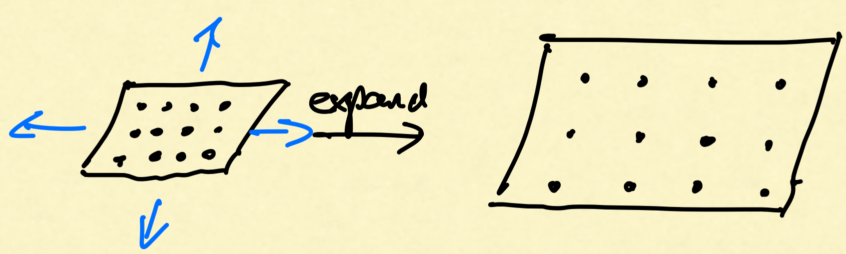
$$\left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2} = \frac{8\pi G}{3} \rho(t)$$

k controls the overall geometry of the Universe (open, closed, flat)

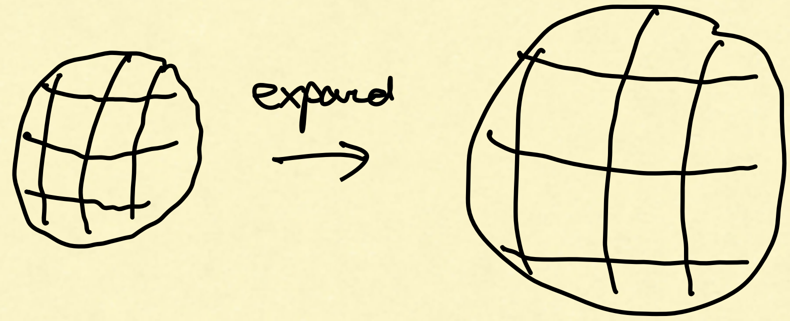
All observational evidence points towards a flat Universe.

Intuition: imagine a (2+1)-dim Universe on a rubber sheet

flat:



closed:



- cosmological constant

  can be viewed as part of $T_{\mu\nu}$

  $$T_{\mu\nu} \to T_{\mu\nu} + \frac{\Lambda}{8\pi G} g_{\mu\nu}$$

  This implies Λ behaves like a form of "energy"

  with $p = -\rho$ $\quad(\to w = -1)$

  As the Universe expands, $\rho = \text{const}$