Problems in Theoretical Physics

(or rather a small subset of them)

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There are so many different kinds of problems in theoretical physics!

Mathematical problems

For instance, we do not know how to solve Navier-Stokes equations!

$$\left(\frac{\partial}{\partial t} + (\vec{v}\cdot\vec{\nabla}) - \nu\nabla^2\right)\vec{v} + \frac{1}{\rho}\vec{\nabla}p + \vec{f}(\vec{x},t) = 0$$

- Necessary to fully grasp motion of fluids!
- One of the Millennium Prize Problems: $1 million prize to solve!

What problems?

There are so many different kind of problems in theoretical Physics!

Philosophical problems

For instance, how should we interpret ψ in quantum mechanics?

- Copenhagen interpretation: give up determinism!
- Bohmian mechanics: try to save determinism by giving up locality
- Superdeterminism: try to preserve both by giving up statistical independence
- ...

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle$$

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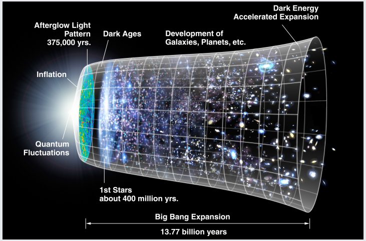
Conceptual problems

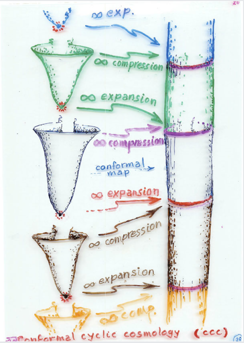
For instance, does the universe have a periodic or non-periodic history?

- Traditional Big Bang
- Conformal cyclic cosmology of Roger Penrose (2020 Nobel laureate)

Universe consists of periodic repetition of Big Bang singularities, connecting one aeon to another



vs.



Problems
Mathematical
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Conceptual
Technical
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Bad Omens
The Gist

What problems?

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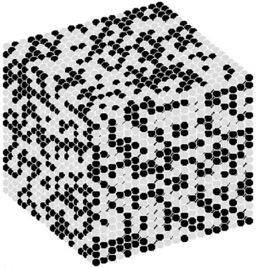
Technical problems

For instance, how can we solve $3d-$Ising model on a lattice?

- We know all the physics!
- The necessary mathematics is available!
- Yet, the problem is NP-complete! [Istrail, 2000]
- NP-complete problems require computational times superpolynomial in input size: the computation is *not feasible*!



What am I going to talk about?

Some of the major problems in field theories!

Obvius questions...

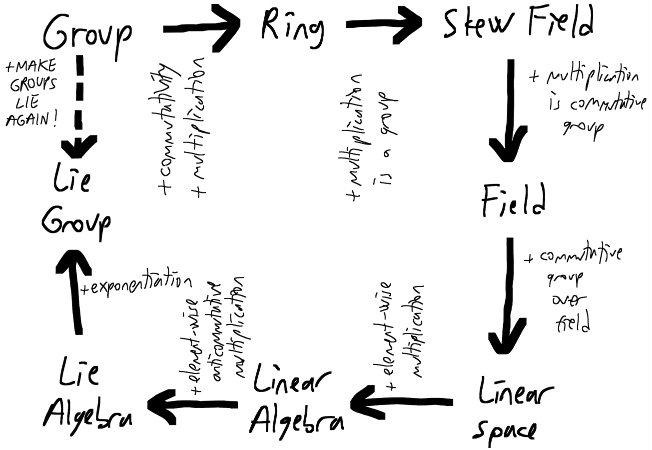
- What do we mean by *field theories*?
- Why should we care for them?
- What problems do they have?

What is a field in Mathematics?

A field $\mathcal{F}$ is a set of elements with two operations $+$ and $\times$ such that

- $\mathcal{F}$ is a commutative group with respect to $+$, with identity element e
- $\mathcal{F} \setminus \{e\}$ is a commutative group with respect to $\times$

where a commutative group is a set of elements $g_i \in G$ with an operation $*$ such that $\exists\ e, \hat{g}_1$ s.t. $g_1 * e = e * g_1 = g_1$, $g_1 * \hat{g}_1 = \hat{g} * g_1 = e$, and $g_1 * (g_2 * g_3) = (g_1 * g_2) * g_3\ \forall g_i \in G$



What do we mean by a field in Physics?

Classical definition

A field is any physical quantity which takes on different values at different points in space.

[Feynman, 1964]

Field theories are descriptions of such fields, e.g. temperature field $T(x, y, z, t)$:

$$\frac{\partial}{\partial t} T(x, y, z, t) = \alpha \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) T(x, y, z, t) \qquad \text{(Heat Equation)}$$

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Modern definition

Field theories are descriptions s.t. objects of interest are mappings from *some space*!

- Traditional field theories: temperature, electromagnetic field, gravitational field, etc.
- Field theoretic interpretation of other theories, e.g. QM:
  1. object of interest does not fill space: $\left(i\hbar \frac{\mathrm{d}}{\mathrm{d}t} - \hat{H}\right) |\psi(t)\rangle = 0$
  2. interpret $t \in \mathbb{R}$ as a $1d$ space: QM $=$ $1d$ QFT
- New field theories, e.g. holographic description of a bulk theory:
  1. object of interest fills a $d - 1$ dimensional space
  2. extend the definition to any generic $n-$dimensional space!

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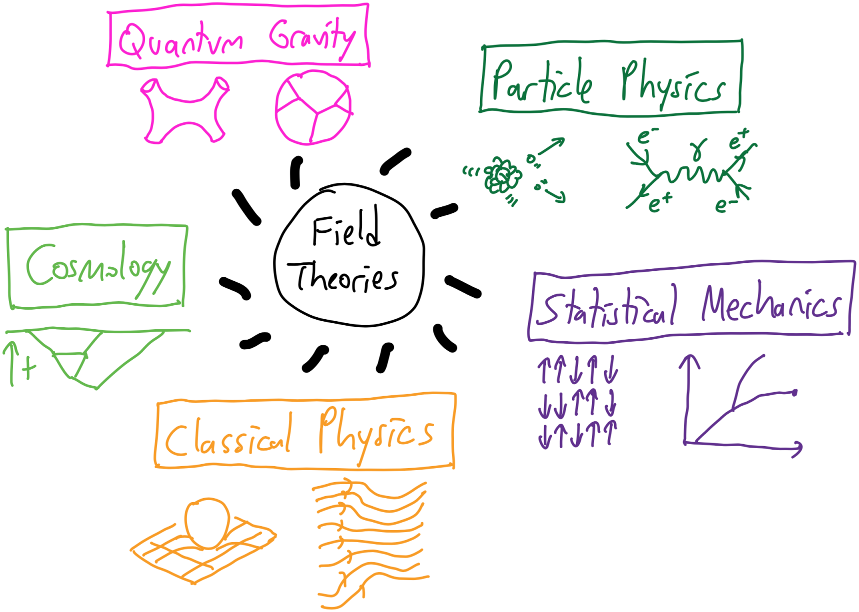

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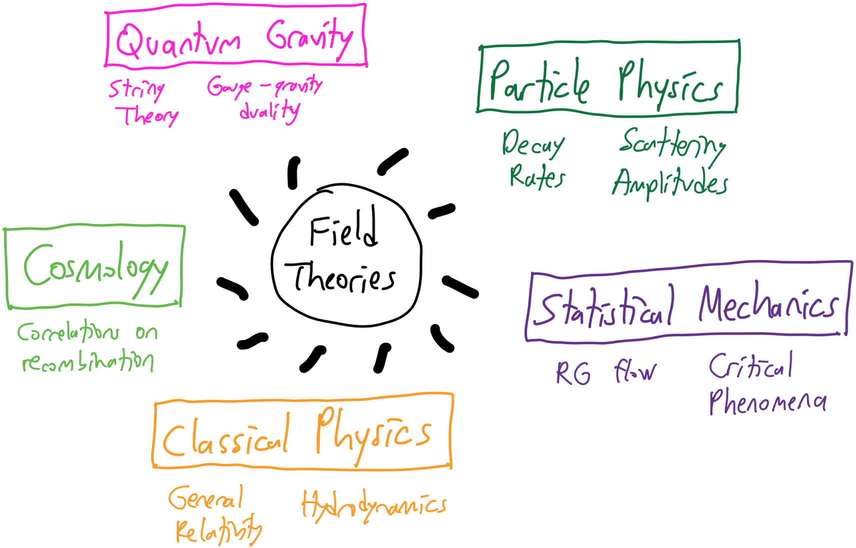

- In short: fields are *everywhere*! (pun intended)
- Despite that, they are plagued with problems!
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- We do not fully understand the quantum version of field theories in curved spacetimes!*

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†For instance, it is not even clear how to define gauge symmetries precisely, roughly because they are not required to act faithfully on the set of local operators [Harlow & Ooguri '19].

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WILL TALK ABOUT TODAY!

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Reminder: non-relativistic quantum mechanics

- Say you would like to compute the Larmor precession of a proton inside MRI:

$$|\text{in}\rangle = \left|S_x = -\frac{\hbar}{2}\right\rangle \;, \quad |\text{out}\rangle = \left|S_x = \frac{\hbar}{2}\right\rangle$$

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$$\langle\text{out}| \cdot \left(e^{-iH(t_f - t_i)} |\text{in}\rangle\right) \quad \rightarrow \quad \lim_{t\to\infty} e^{-2iEt} \langle\text{out}|\text{in}\rangle$$

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- Physical interpretation of Ω: convert from the Heisenberg to the interaction picture!

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Pretend you have never heard of Haag-Ruelle construction!

Introduction to intellectual ignorance

What we do:

- Ignore that charged particles cannot be isolated, that vacuum is not necessarily unique, that we have massless particles
- Assume LSZ reduction theorem holds anyway
- Learn to live with singular $S-$matrices obtained this way
  (a little bit divergence won't hurt anyone!)

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Utilies of this pragmatic approach

- Even though we have infrared divergences, they actually drop out once we sum over initial or final states
  (strong version of KLN theorem) [Frye et al. '19]
- Computations done this way do match the experimental data!

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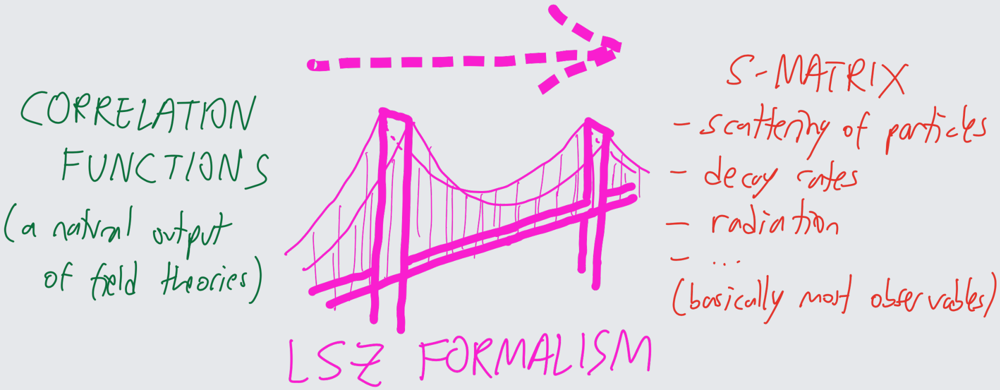

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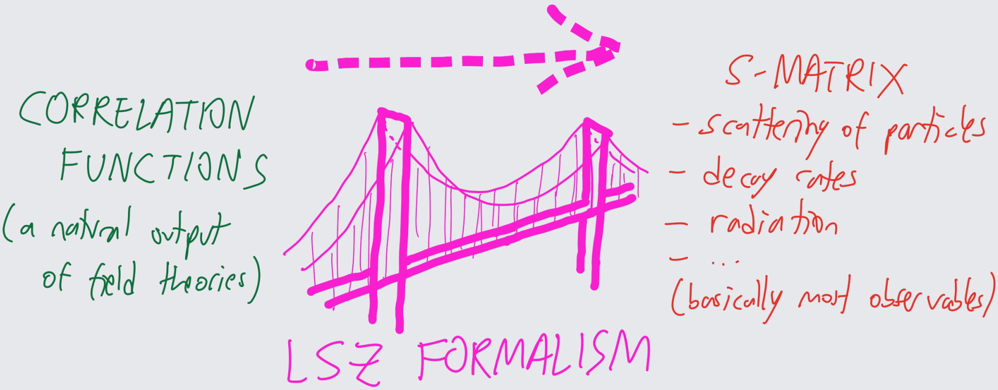

But can we compute the correlation functions in the first place?

this brings us to our second problem with field theories...

Review: correlation functions

correlation functions $=$ glorified expectation values

not of variables but of functions

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Ordinary expectation value

$$E = \int f(x_1, \ldots, x_n) P(x_1, \ldots, x_n) dx_1 \cdots dx_n$$

- x_i: random-variables
- $P(x_1, \ldots, x_n)$: joint probability distribution
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Correlation function

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle = \int \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \mathcal{P}(\mathcal{O}_1, \ldots, \mathcal{O}_n) [\mathcal{D}\mathcal{O}_1] \ldots [\mathcal{D}\mathcal{O}_n]$$

- $\mathcal{O}_i$: random-functions (operator valued fields)
- $P(\mathcal{O}_1, \ldots, \mathcal{O}_n)$: joint probability distribution – determined by the action S
- $[\mathcal{D}\mathcal{O}_i]$: measures on the function space

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for $\xi(\tau) \sim |\tau|^{-\nu}$. $\qquad (\tau \equiv T - T_{\text{Curie}})$

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for $\xi(\tau) \sim |\tau|^{-\nu}$. $\qquad (\tau \equiv T - T_{\text{Curie}})$

- η and ν are examples of critical exponents: many things can be computed in terms of them, for instance:

heat capacity $\sim \tau^{3\nu-2}$, compressibility $\sim \tau^{\nu(\eta-4)}$, and so on

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- How is this helpful?
  - *We can always do a perturbation expansion around a free theory if the coupling is not strong.*
  - In high energy physics, this is good enough for even precision computations of electroweak theory!
  - Basically, even though we cannot compute the necessary integrals, we can compute almost everything for each and every engineering!

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- In typical local Lagrangian theories, $\mathcal{P}$ is continuously related to $\mathcal{P}_{\text{free}}$ by bunch of parameters called coupling constants:

$$\begin{aligned}\mathcal{P} = \mathcal{P}_{\text{free}}\Big(1 &+ \lambda_1 g_{11}(\mathcal{O}_i) + {\lambda_1}^2 g_{12}(\mathcal{O}_i) + \ldots \\ &+ \lambda_2 g_{21}(\mathcal{O}_i) + {\lambda_2}^2 g_{22}(\mathcal{O}_i) + \ldots \\ &+ \ldots\Big)\end{aligned}$$

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We know how to compute free theory but not the general one that we need.

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- Example: scalar field (like Higgs or inflaton fields) with cubic coupling:

$$\begin{aligned}\langle \mathcal{O}_x \mathcal{O}_y \rangle = \langle \mathcal{O}_x \mathcal{O}_y \rangle_{\text{fr}} &+ \lambda^2 \int d^d\omega d^d z \, \langle \mathcal{O}_x \mathcal{O}_z \rangle_{\text{fr}} \langle \mathcal{O}_y \mathcal{O}_z \rangle_{\text{fr}} \langle \mathcal{O}_z \mathcal{O}_\omega \rangle_{\text{fr}} \langle \mathcal{O}_\omega \mathcal{O}_\omega \rangle_{\text{fr}} \\ &+ \lambda^2 \int d^d\omega d^d z \, \langle \mathcal{O}_x \mathcal{O}_z \rangle_{\text{fr}} \langle \mathcal{O}_y \mathcal{O}_\omega \rangle_{\text{fr}} \langle \mathcal{O}_z \mathcal{O}_\omega \rangle_{\text{fr}}^2 + \# \lambda^4 + \# \lambda^6 + \dots\end{aligned}$$

Review: perturbation theory in QFT

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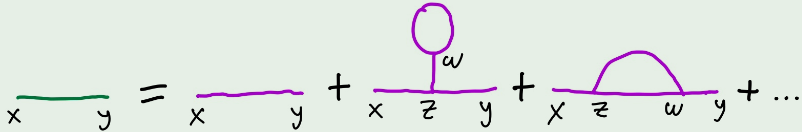
Physisists' way of dealing with this mess: Feynman diagrams



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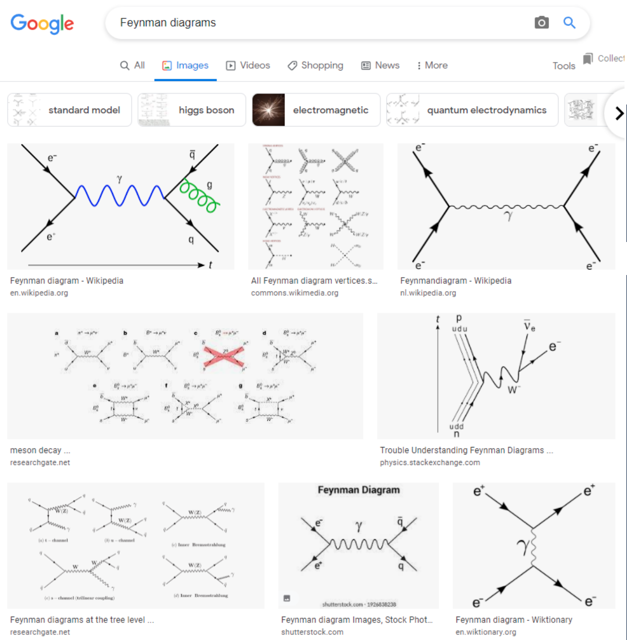
Bottomline: if you see a Feynman diagram *(almost everytime for anything to do with standard model of particle physics)*, it shows that we are doing perturbation theory!



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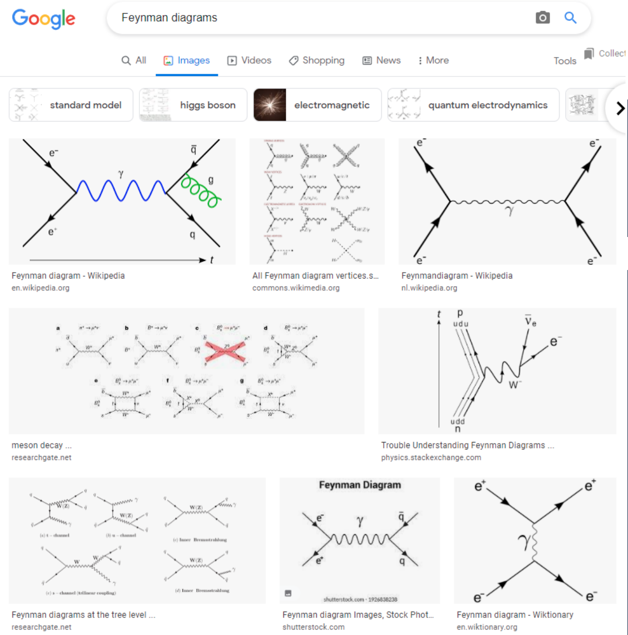

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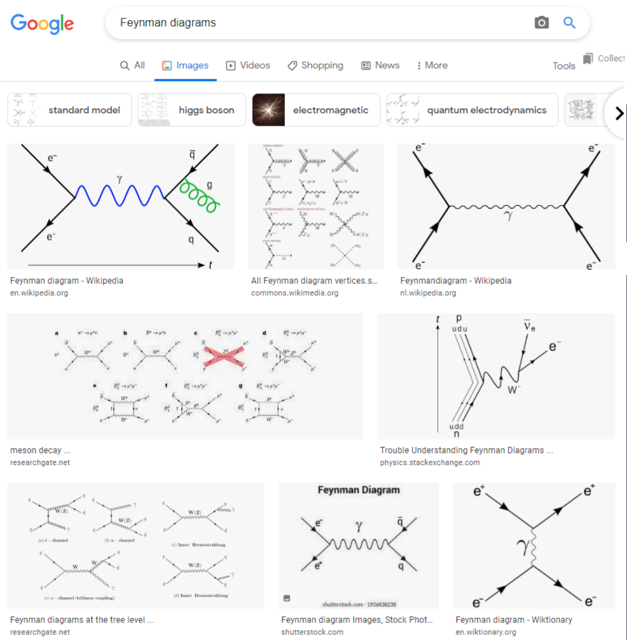
How good is this approximation?

$$\text{Actual thing} = \#\lambda + \#\lambda^2 + \cdots$$
$$\text{What we compute} = \#\lambda + \#\lambda^2$$

Smaller the λ, better the approximation!



Coupling constants in field theories

Summary

- We cannot compute the correlation functions in generic field theories, but approximate them in terms of correlation functions of free theories!
- This approximation is fine if the coupling constant is small!

Well, are coupling constants small in general?

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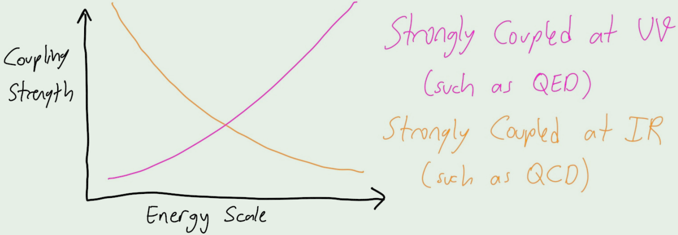
- λ change with E
- Analyzed by RG flow
- No generic theory where perturbative approach works everywhere



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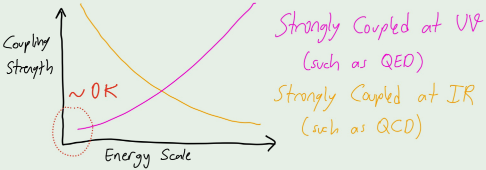

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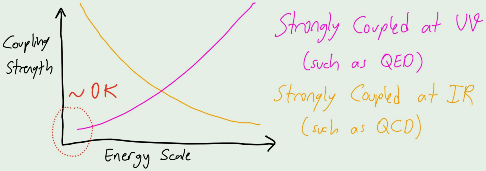

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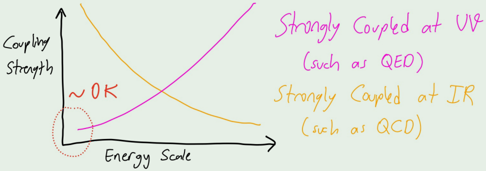

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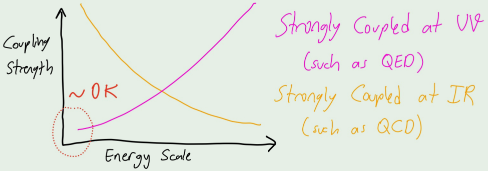

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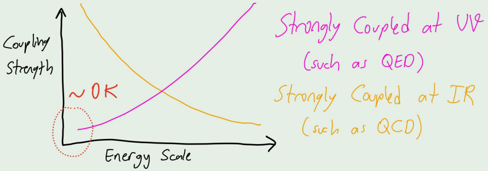

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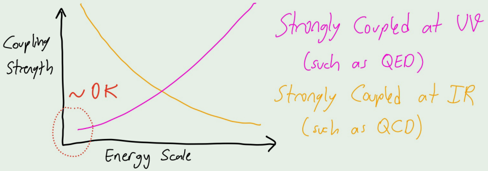

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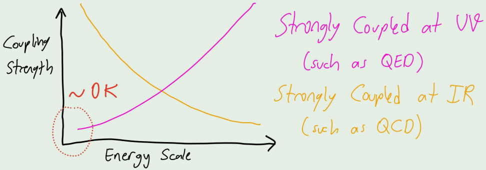

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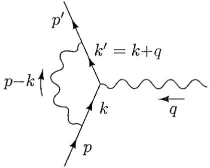

$$= 2ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{\bar{u}(p')\left[\not{k}'\gamma^\mu \not{k} + m^2\gamma^\mu - 2m(k+k')^\mu\right]u(p)}{((k-p)^2 + i\epsilon)(k'^2 - m^2 + i\epsilon)(k^2 - m^2 + i\epsilon)}.$$

[Peskin & Schröder]

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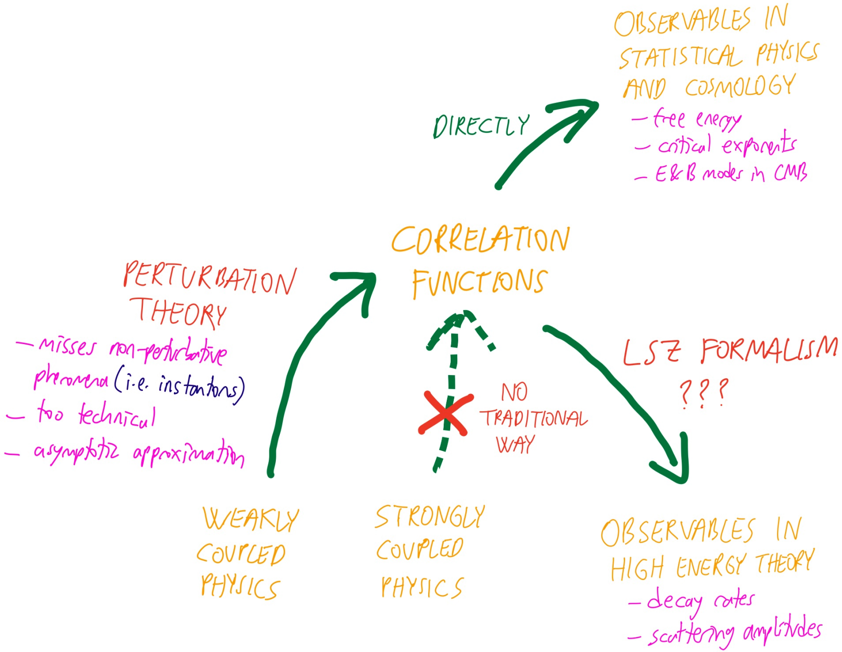
E.g.: $\int_{-\infty}^{\infty} \exp\left(-x^2 - \lambda x^4\right) dx = \text{well-defined if Re } \lambda \geq 0 \quad \Rightarrow \quad \lambda = 0 \text{ is bndry of convrgnce}$

Thus: $\int_{-\infty}^{\infty} \exp\left(-x^2\right) dx + \lambda \int_{-\infty}^{\infty} x^4 \exp\left(-x^2\right) dx + \ldots$ is not a convergent sum!

Summary of the conundrums!



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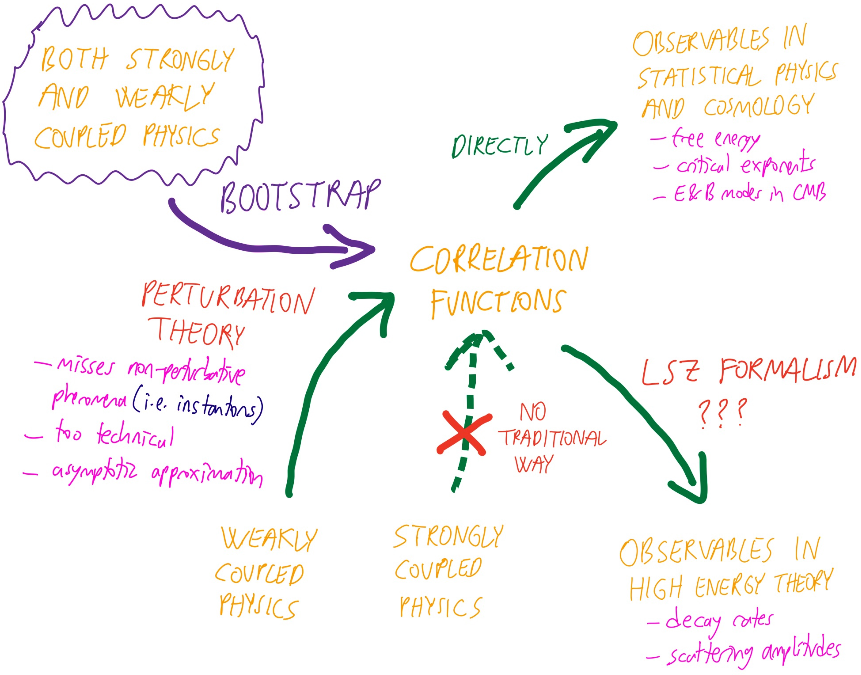

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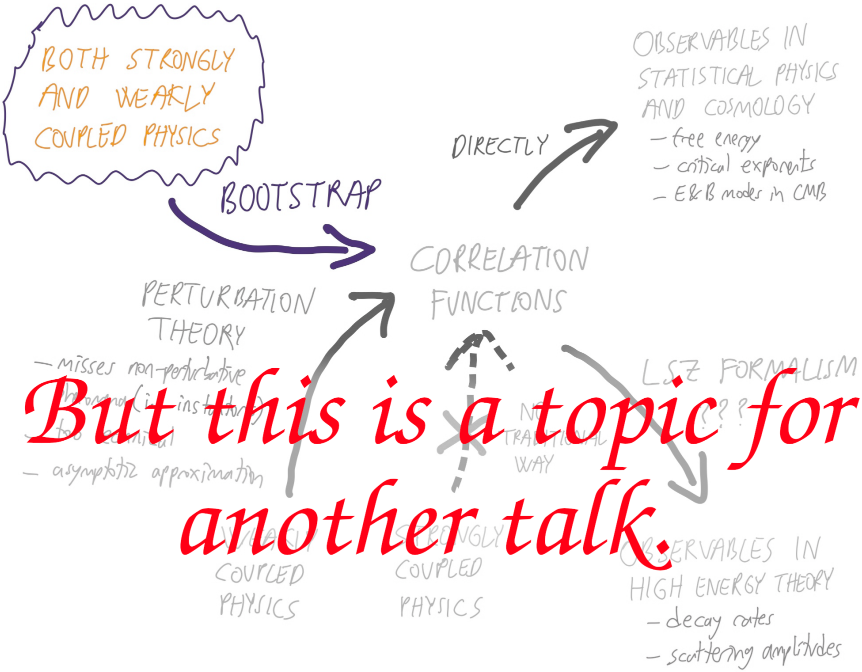

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- Mainstream approaches in high energy physics & cosmology:

Feynman diagrams: expand action around free action

Old-fashioned perturbation theory: expand hamiltonian around free hamiltonian

On-shell diagrams: use recursion relations, double copy, & similar methods

Positive geometry: use abstract mathematical structures such as polytopes

Bootstrap: start with physical properties and try to construct bottom-up

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