Norway grants

National Science Centre Poland

Quantum Information at Colliders

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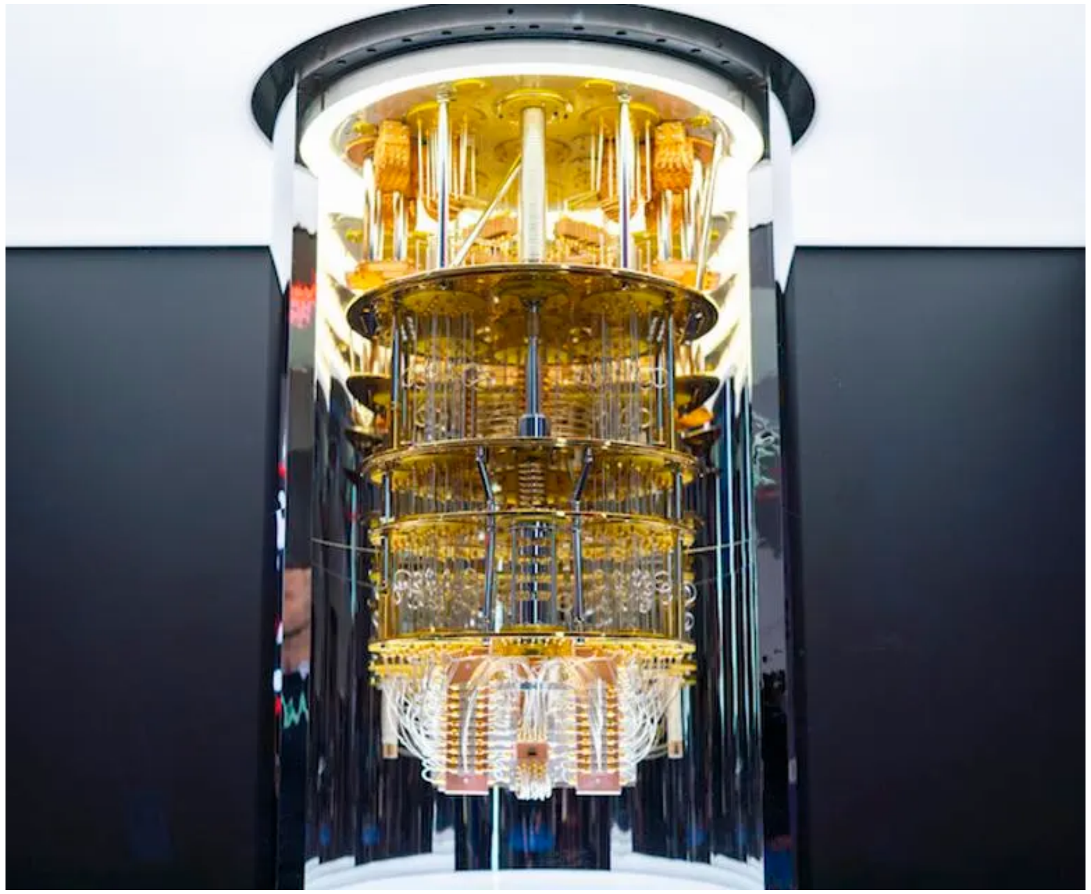
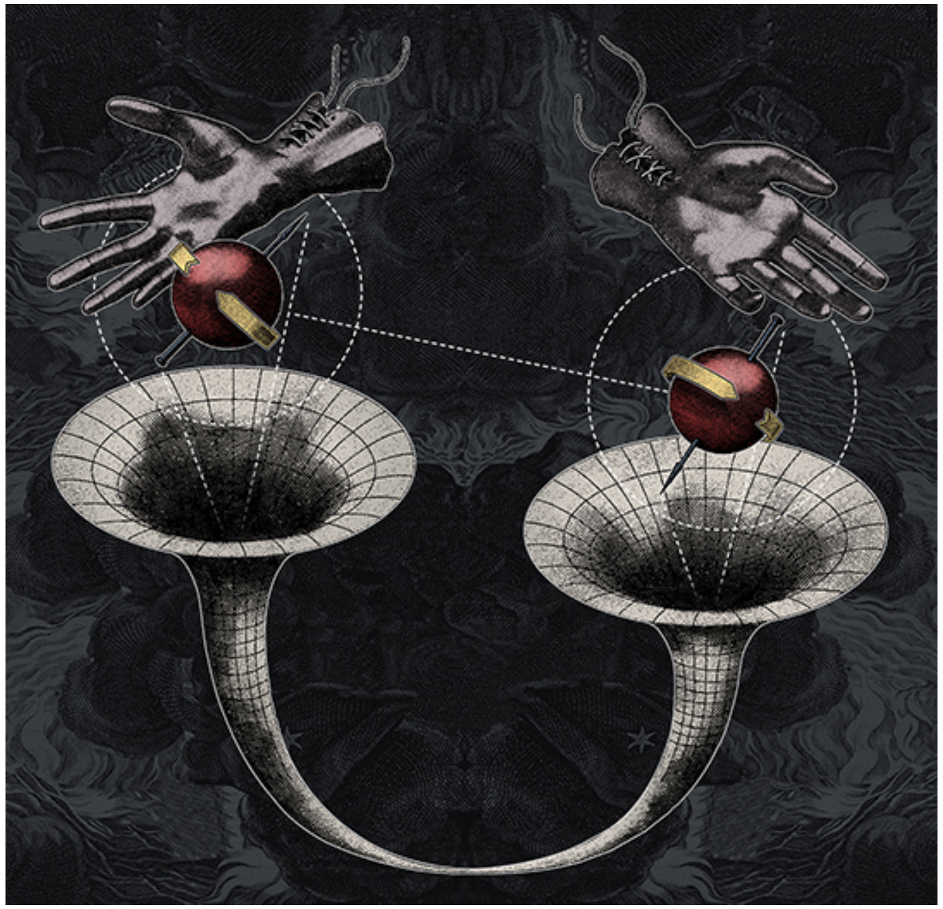
2024/3/15, Seminar @ Warsaw U

Entanglement and other quantum properties are crucial in:

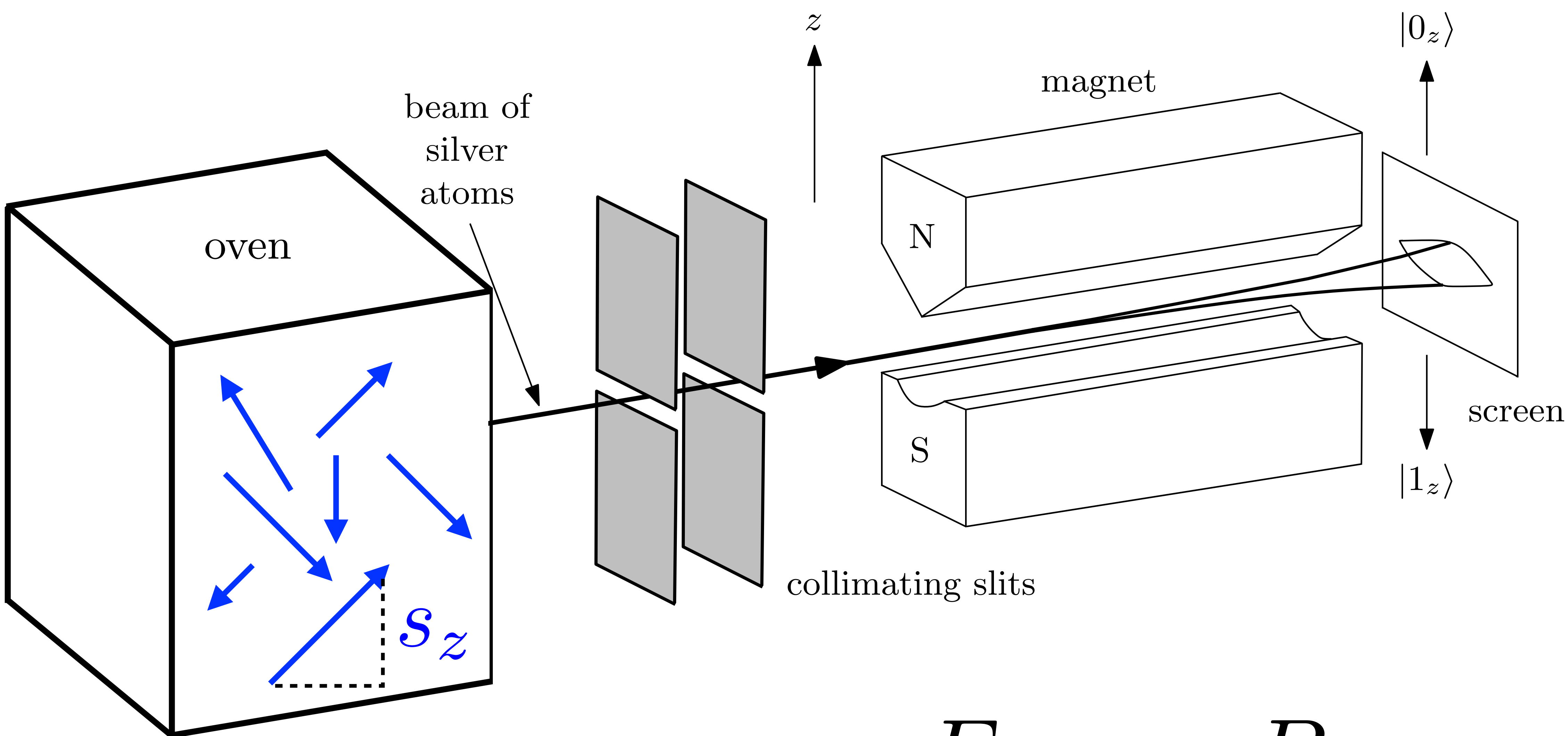
- developing **quantum technology/devices**
- understanding **QFT** and quantum **gravity**



IBM Q system



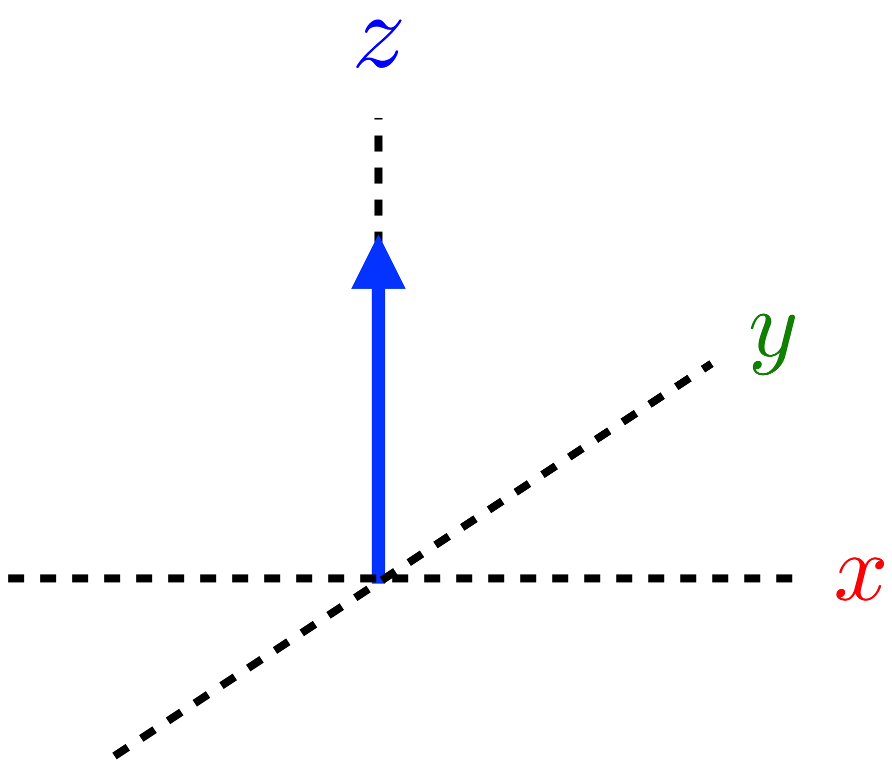
Spin is **weird**



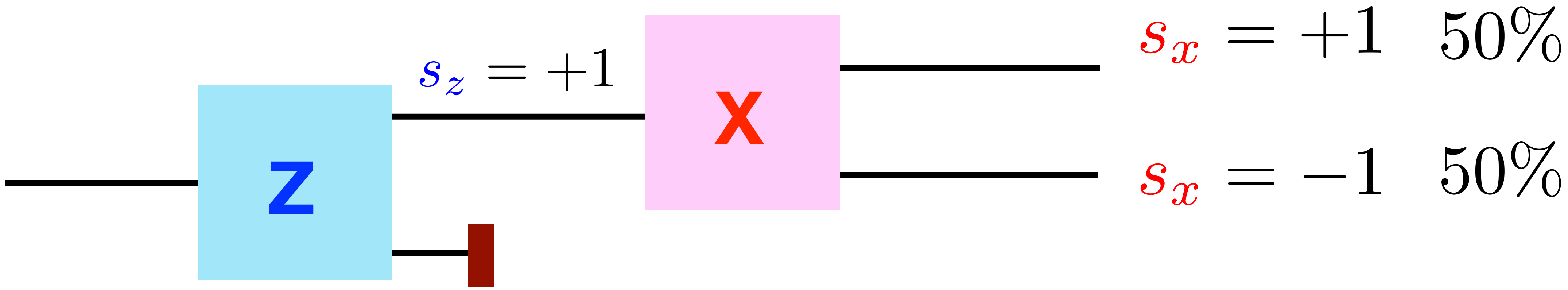
$$F_z \propto s_z B_z$$

- **Z-component** of spin has only two possible values: **+1** and **-1**

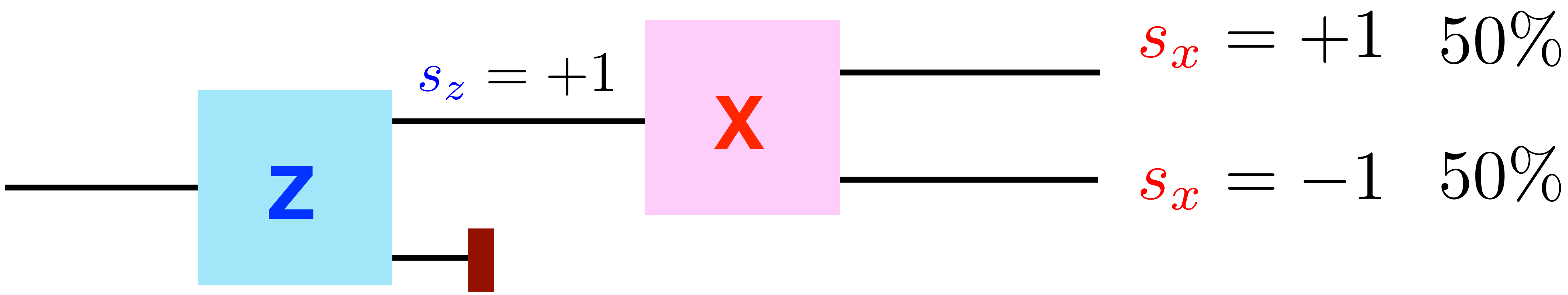
- If spin is in the **Z**-direction, **X**, **Y**-component should be zero! (?)

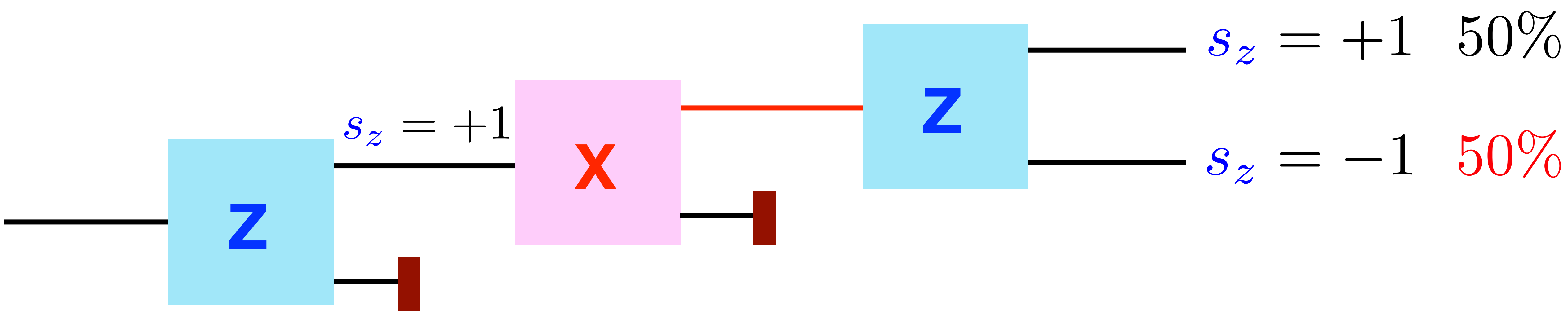


- If spin is in the **Z**-direction, **X**-component should be zero! (?)

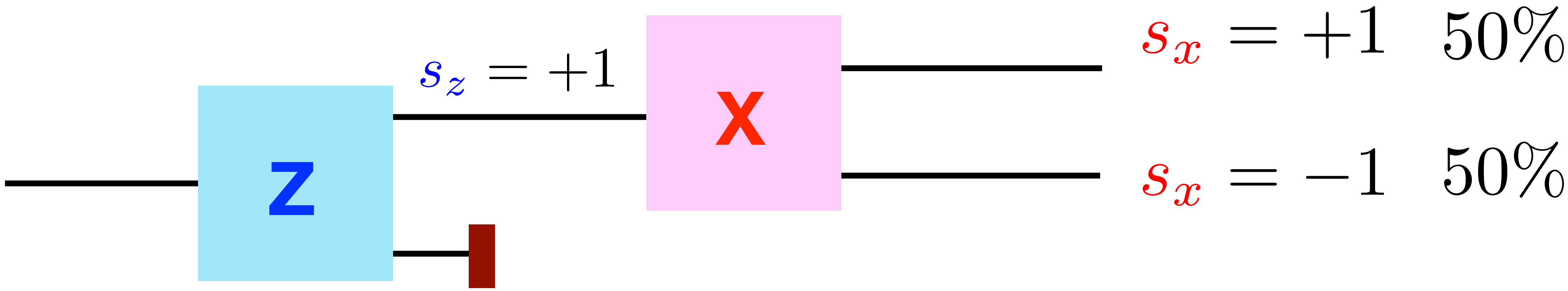


- If spin is in the **Z**-direction, **X**-component should be zero! (?)

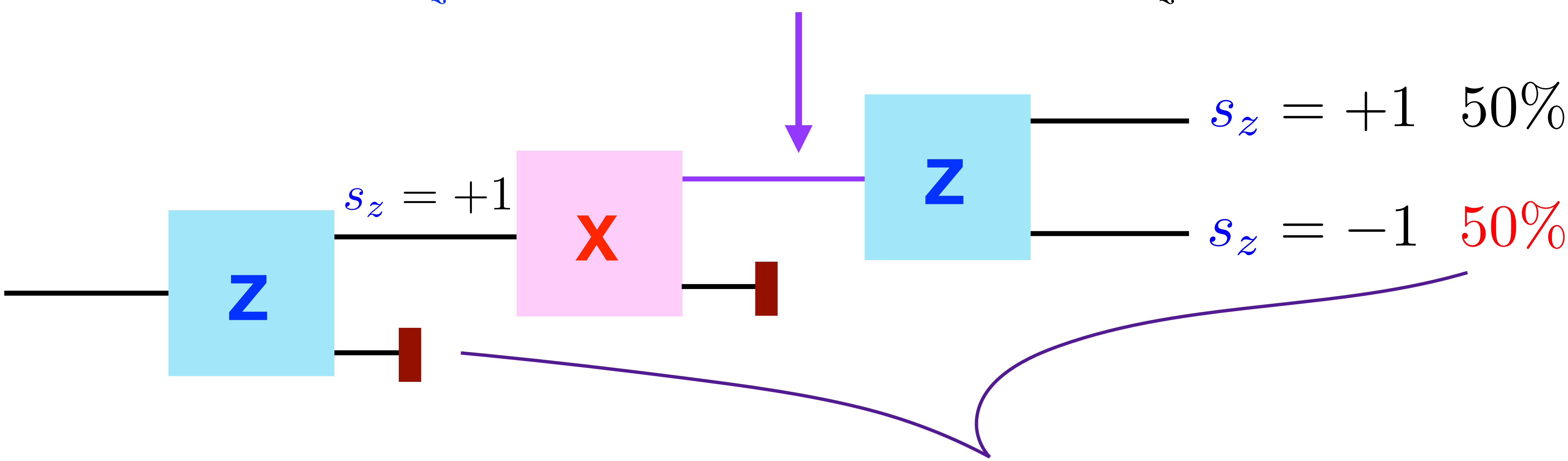




- If spin is in the **Z**-direction, **X**-component should be zero! (?)



Is there a ***definite*** s_z **value** of **this particle** before the s_z measurement?



previously blacked $s_z = -1$

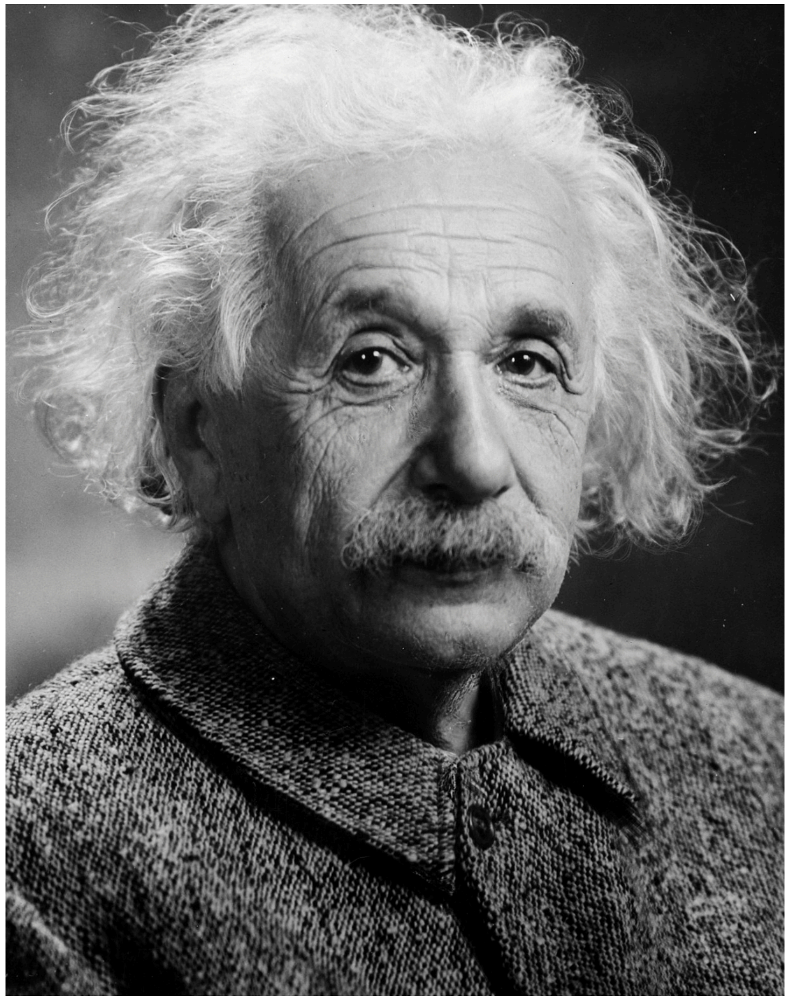
reappears!

Is probabilistic outcomes in experiments **fundamental** or **due to our ignorance** about the physical state/system?

Is probabilistic outcomes in experiments **fundamental** or **due to our ignorance** about the physical state/system?

Can't be fundamental!

God does not play dice!



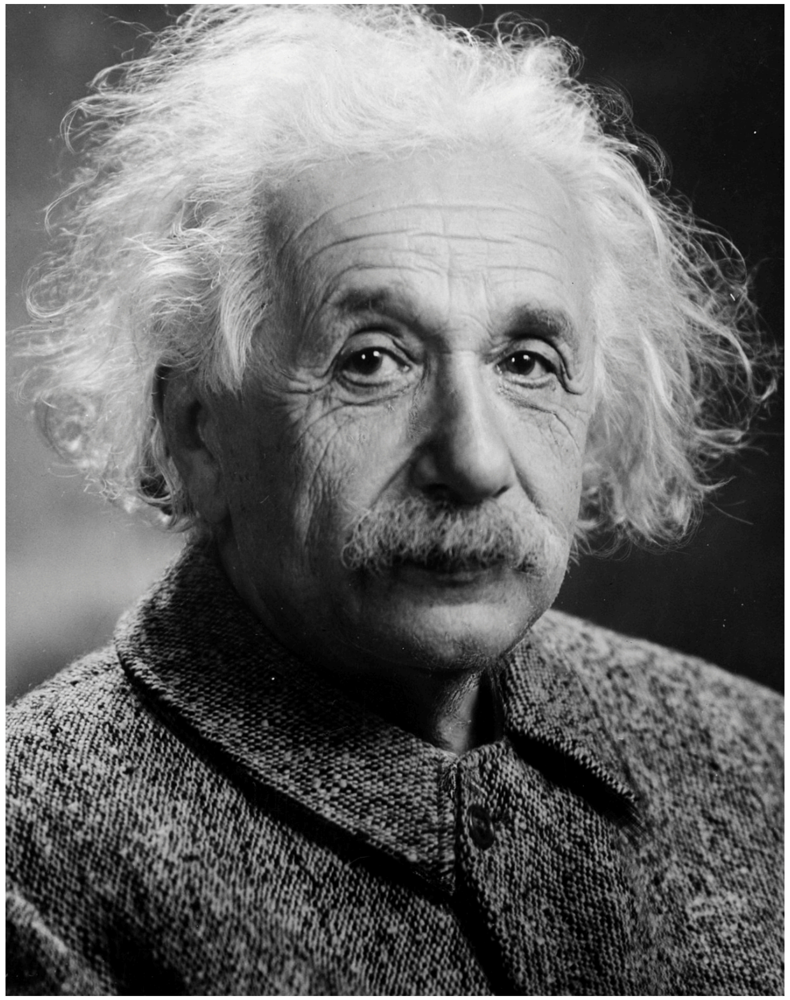
Albert Einstein

Is probabilistic outcomes in experiments **fundamental** or **due to our ignorance** about the physical state/system?

Can't be fundamental!

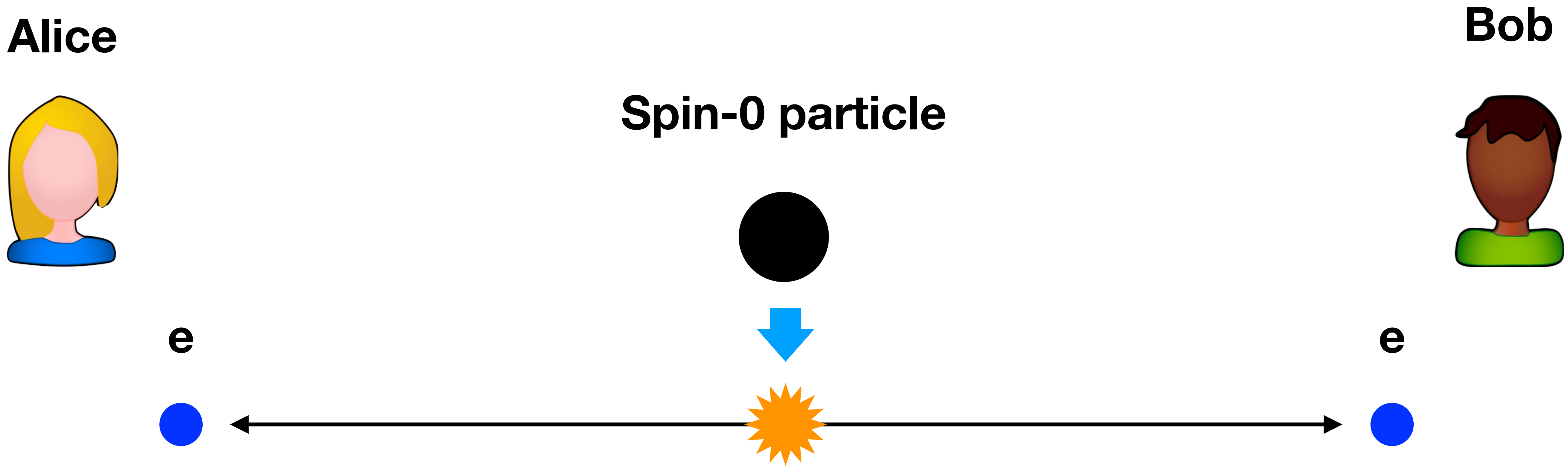
God does not play dice!

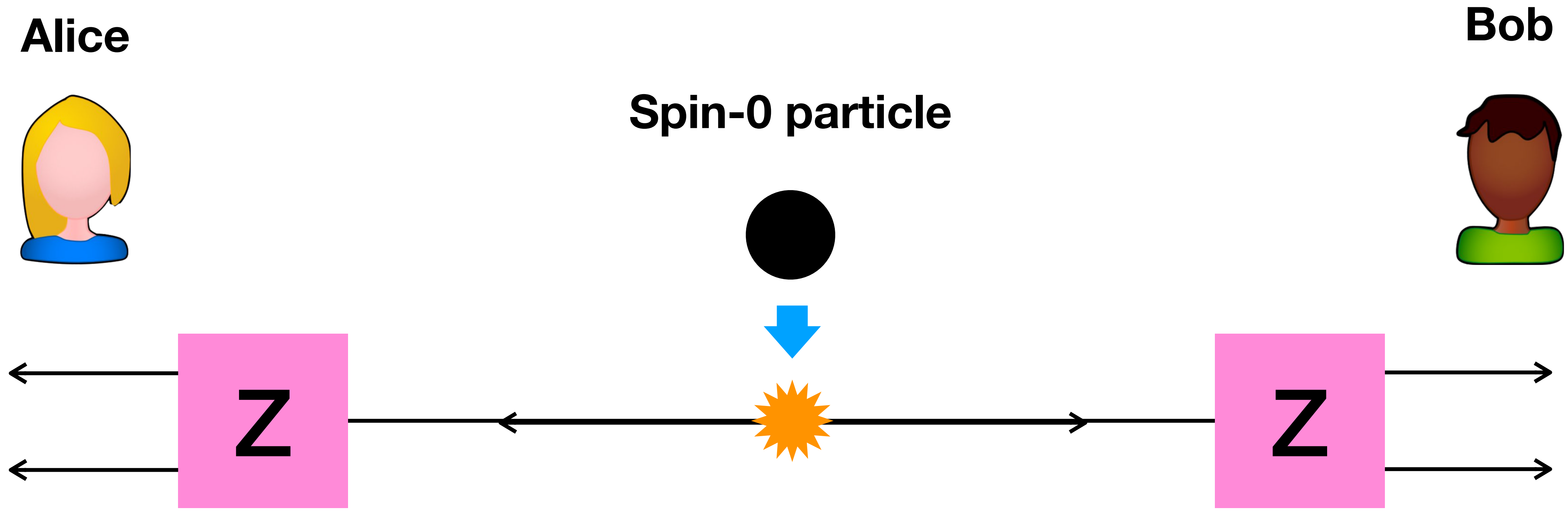
Can experimentally check :-)

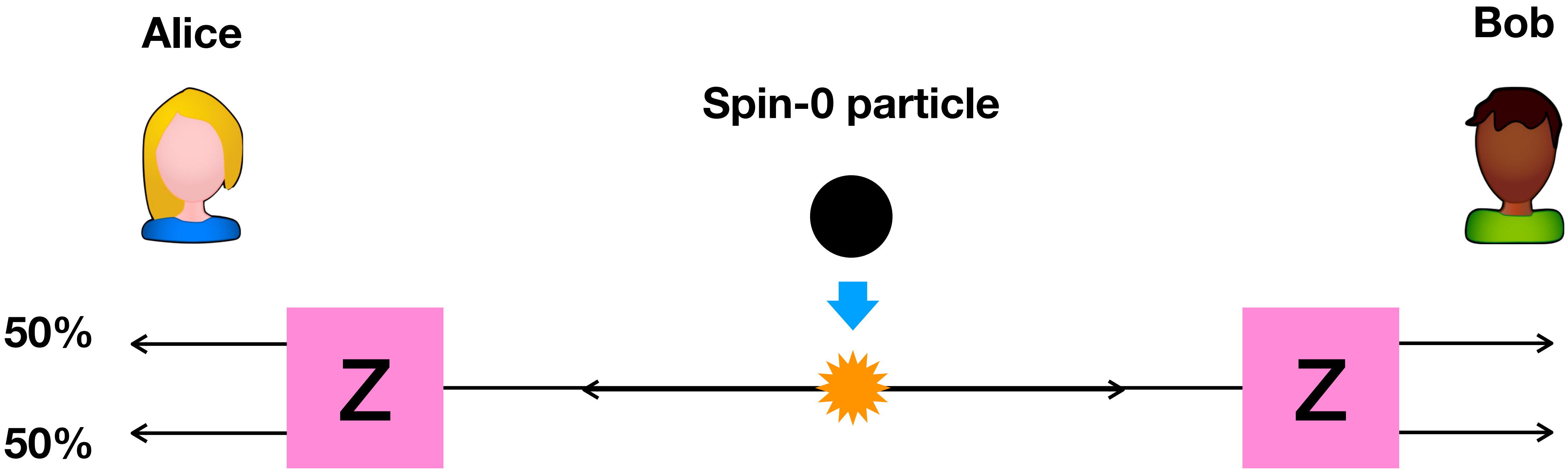


Albert Einstein

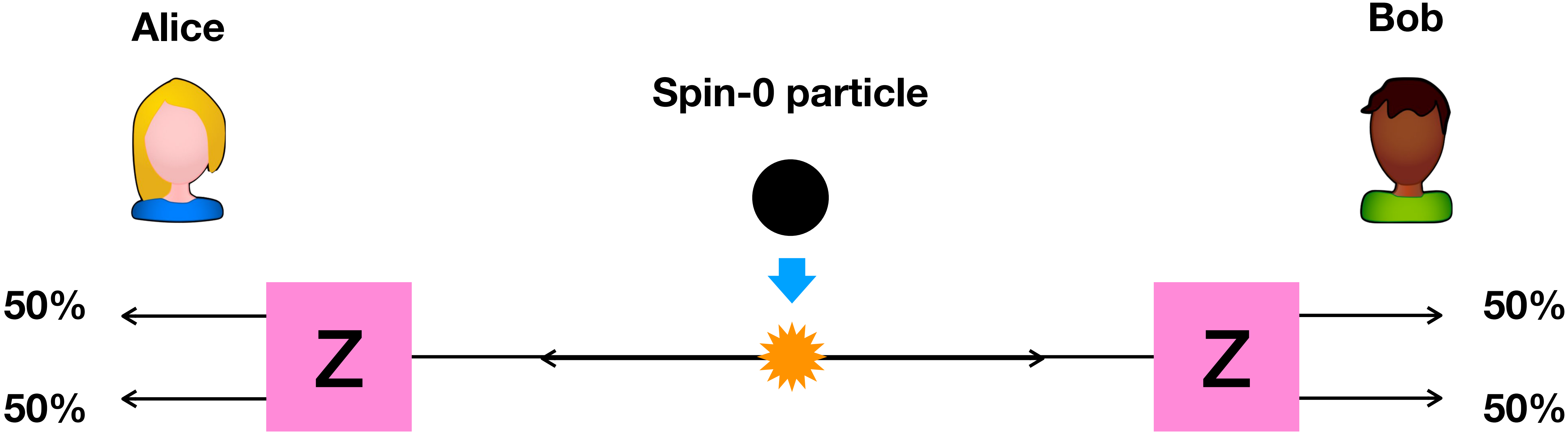
John Bell


Alice
Bob
Spin-0 particle
e
e

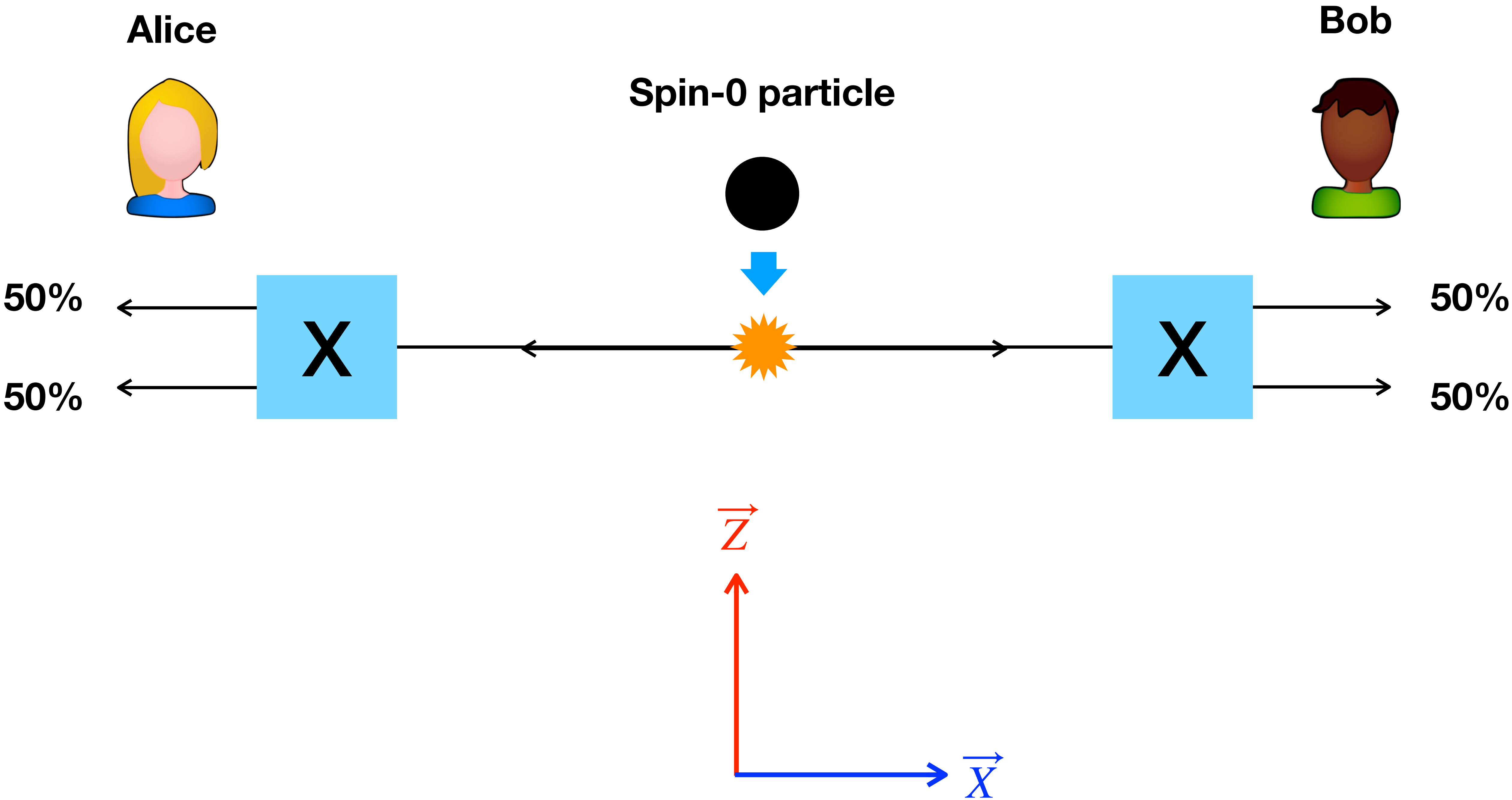

Alice
Bob
Spin-0 particle
Z
Z



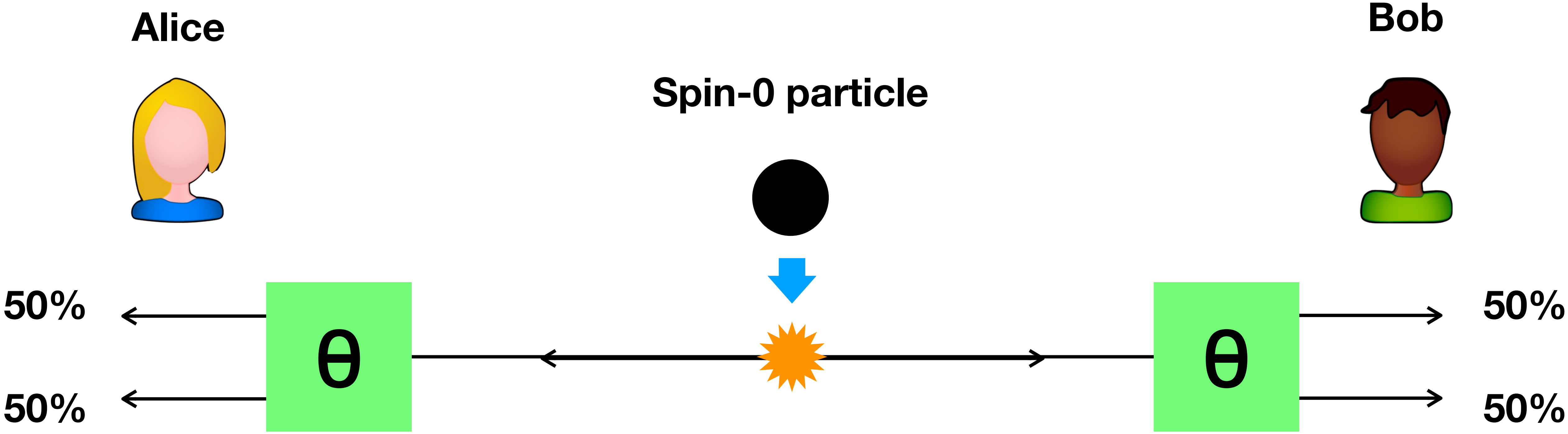
Alice	+	+	-	+	-	-	+	+	+	-	+	-

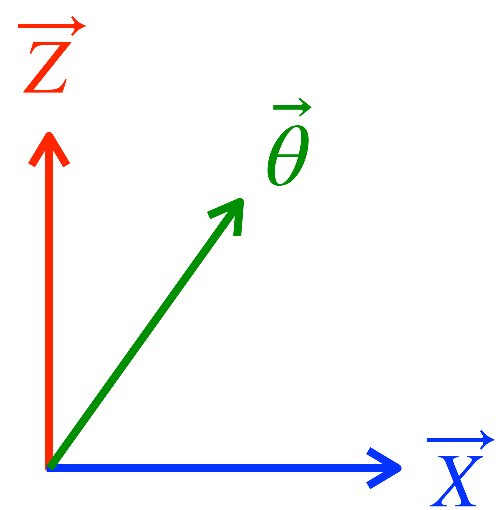


Alice	+	+	-	+	-	-	+	+	+	-	+	-
Bob	-	-	+	-	+	+	-	-	-	+	-	+

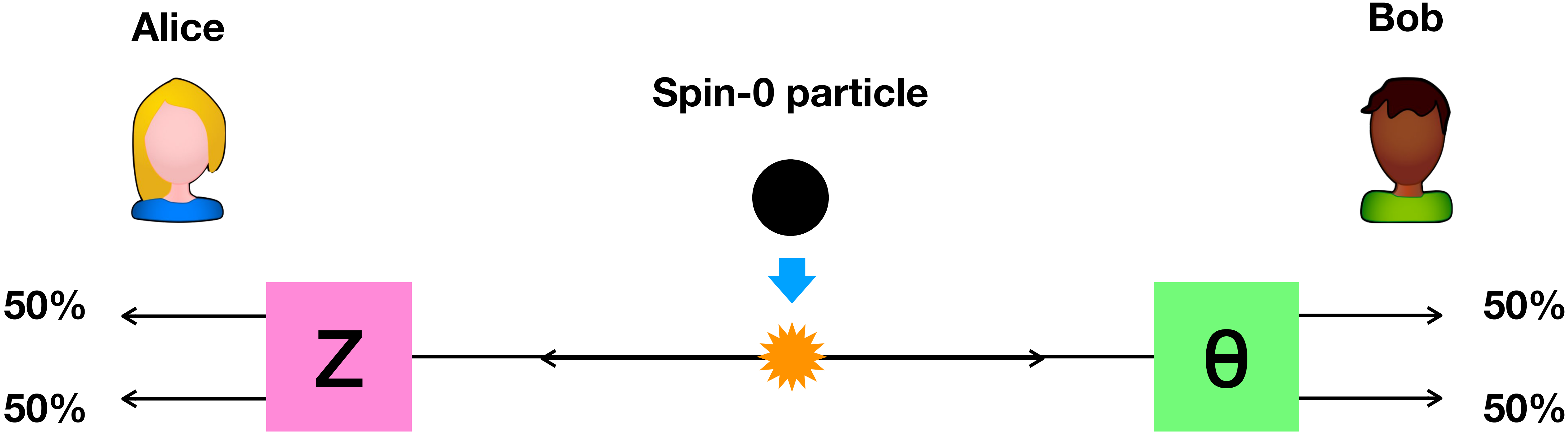


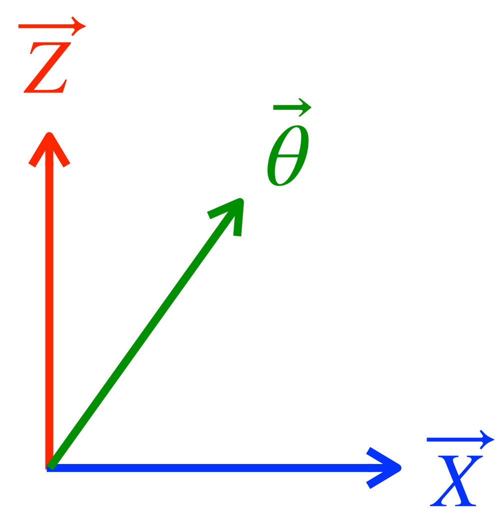
Alice	+	+	-	+	-	-	+	+	+	-	+	-
Bob	-	-	+	-	+	+	-	-	-	+	-	+





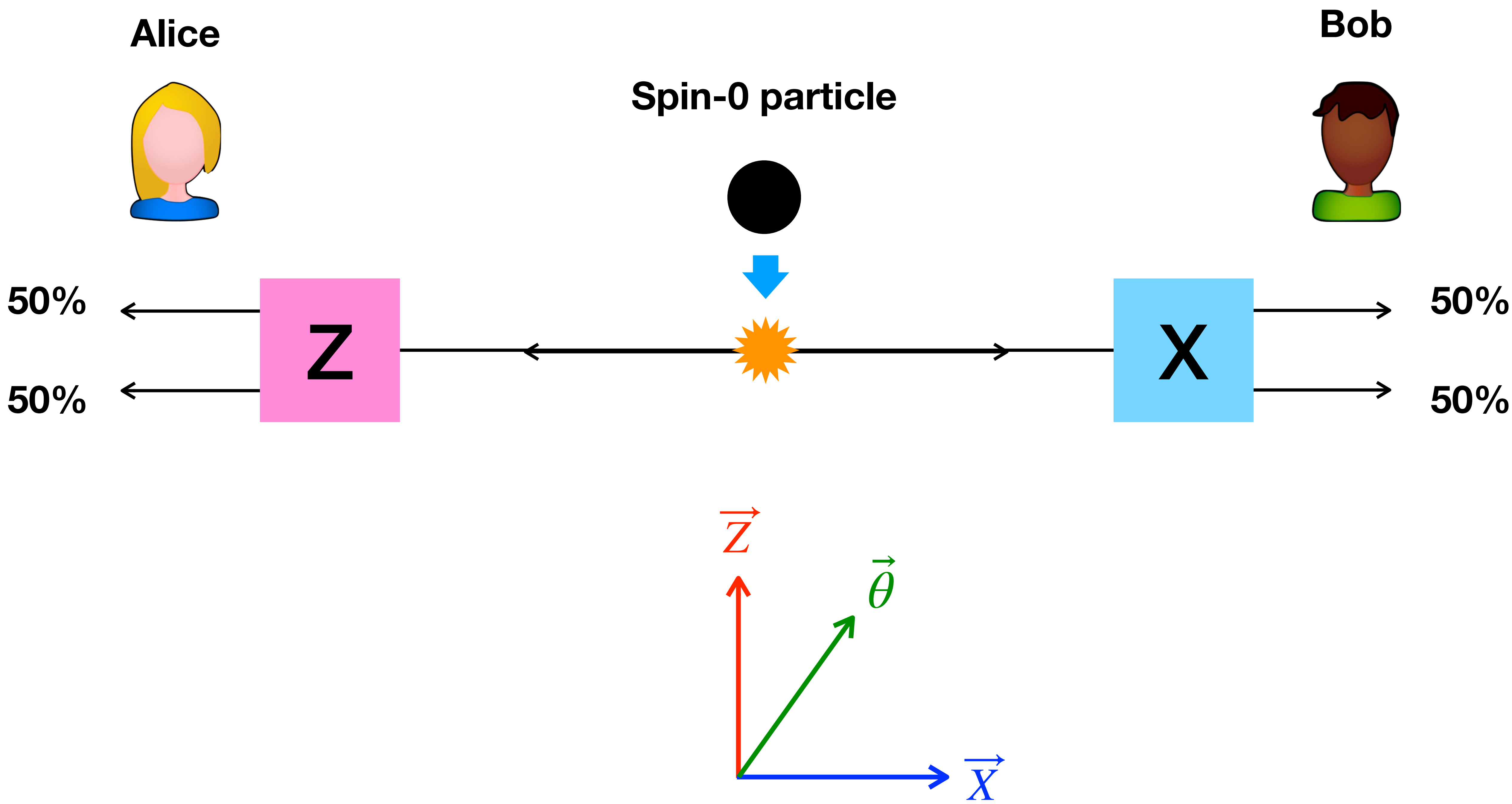
Alice	+	+	-	+	-	-	+	+	+	-	+	-
Bob	-	-	+	-	+	+	-	-	-	+	-	+



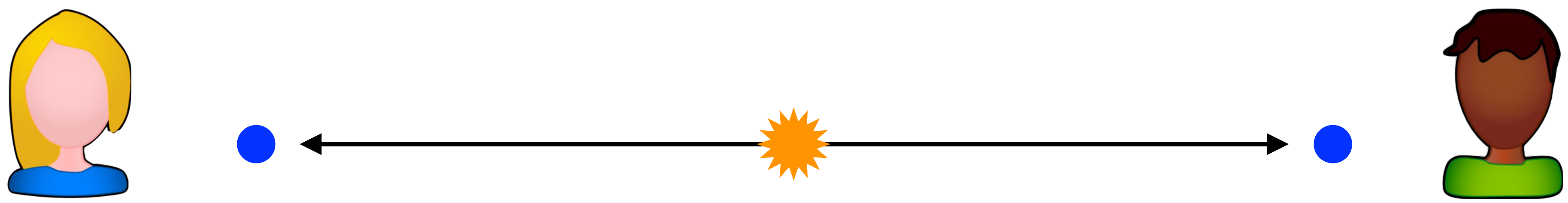


Alice	+	+	-	+	-	-	+	+	+	-	+	-
Bob	-	-	+	-	+	+	-	-	-	+	-	+

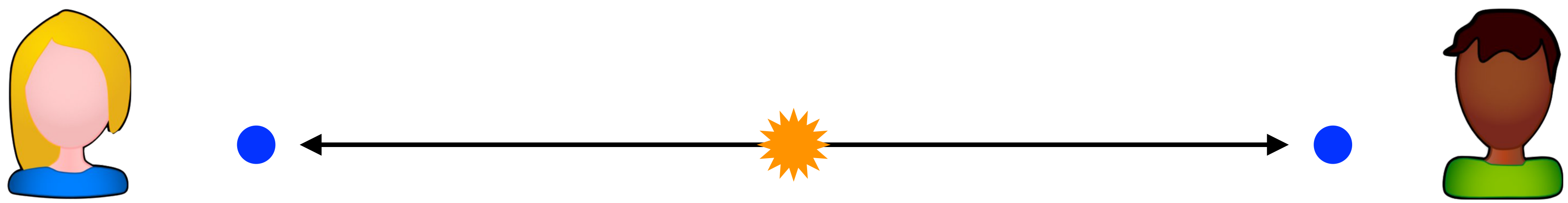
?



Alice	+	+	-	+	-	-	-	+	+	-	+	-
Bob	-	-	+	-	+	+	-	-	-	+	-	+



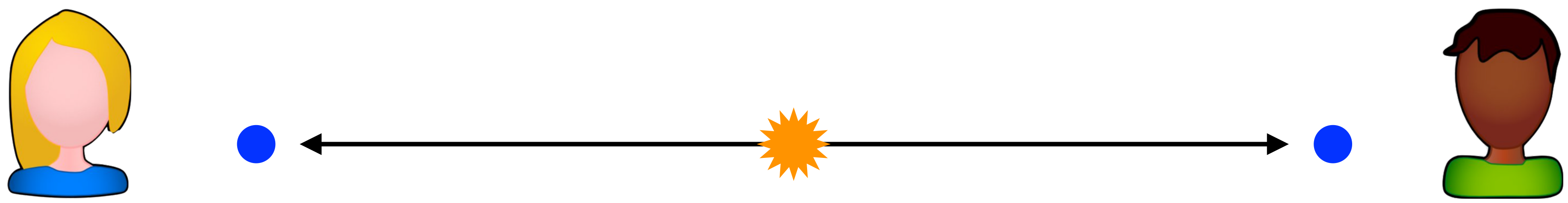
	Alice			Bob		
prob.	z	x	θ	z	x	θ
P_1	+	+	+	-	-	-
P_2	-	+	+	+	-	-
P_3	+	-	+	-	+	-
P_4	+	+	-	-	-	+
P_5	+	-	-	-	+	+
P_6	-	+	-	+	-	+
P_7	-	-	+	+	+	-
P_8	-	-	-	+	+	+



	Alice			Bob		
prob.	z	x	θ	z	x	θ
P_1	+	+	+	-	-	-
P_2	-	+	+	+	-	-
P_3	+	-	+	-	+	-
P_4	+	+	-	-	-	+
P_5	+	-	-	-	+	+
P_6	-	+	-	+	-	+
P_7	-	-	+	+	+	-
P_8	-	-	-	+	+	+

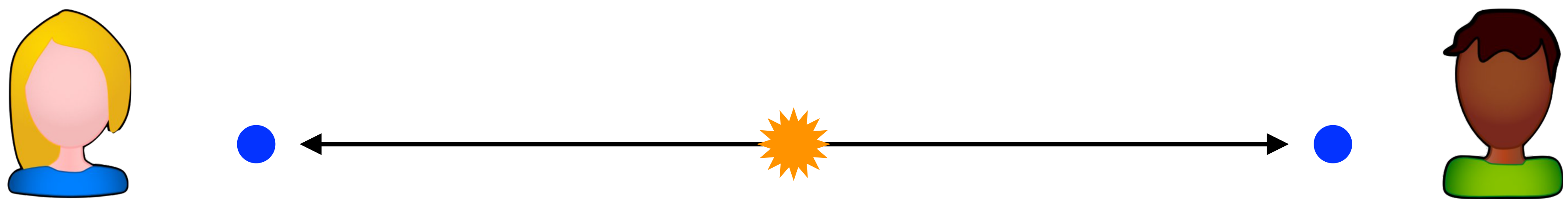
P(+z, any)

$$= P_1 + P_3 + P_4 + P_5 = 0.5$$



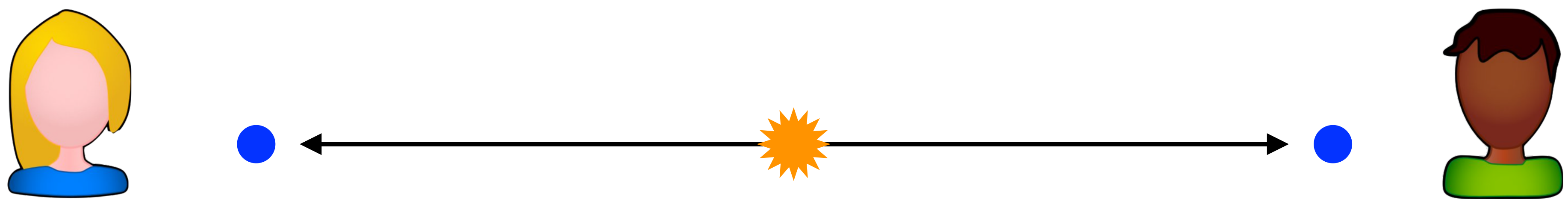
	Alice			Bob		
prob.	z	x	θ	z	x	θ
P_1	+	+	+	-	-	-
P_2	-	+	+	+	-	-
P_3	+	-	+	-	+	-
P_4	+	+	-	-	-	+
P_5	+	-	-	-	+	+
P_6	-	+	-	+	-	+
P_7	-	-	+	+	+	-
P_8	-	-	-	+	+	+

$P(+z, +x) =\ ?$



	Alice			Bob		
prob.	z	x	θ	z	x	θ
P_1	+	+	+	-	-	-
P_2	-	+	+	+	-	-
P_3	+	-	+	-	+	-
P_4	+	+	-	-	-	+
P_5	+	-	-	-	+	+
P_6	-	+	-	+	-	+
P_7	-	-	+	+	+	-
P_8	-	-	-	+	+	+

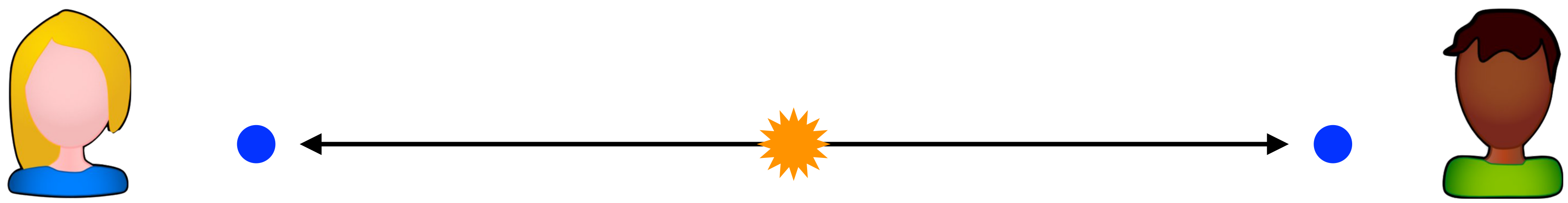
$$P(+z, +x) = P_3 + P_5$$



	Alice			Bob		
prob.	z	x	θ	z	x	θ
P_1	+	+	+	-	-	-
P_2	-	+	+	+	-	-
P_3	+	-	+	-	+	-
P_4	+	+	-	-	-	+
P_5	+	-	-	-	+	+
P_6	-	+	-	+	-	+
P_7	-	-	+	+	+	-
P_8	-	-	-	+	+	+

$P(+z, +x) = P_3 + P_5$

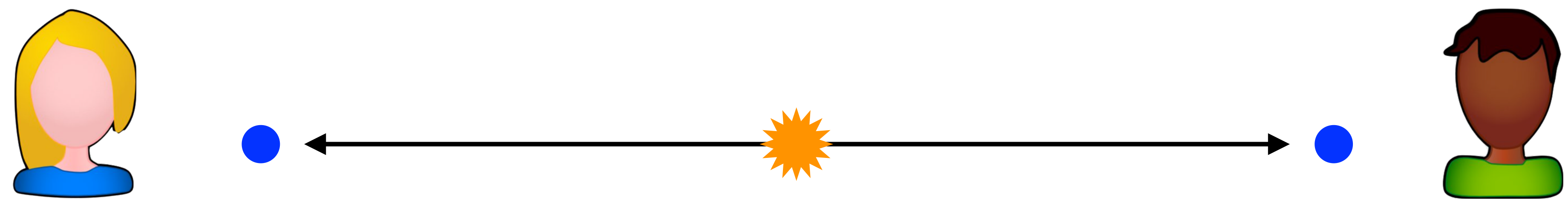
$P(+z, +\theta) = ?$



	Alice			Bob		
prob.	z	x	θ	z	x	θ
P_1	+	+	+	-	-	-
P_2	-	+	+	+	-	-
P_3	+	-	+	-	+	-
P_4	+	+	-	-	-	+
P_5	+	-	-	-	+	+
P_6	-	+	-	+	-	+
P_7	-	-	+	+	+	-
P_8	-	-	-	+	+	+

$$P(+z, +x) = P_3 + P_5$$

$$P(+z, +\theta) = P_4 + P_5$$

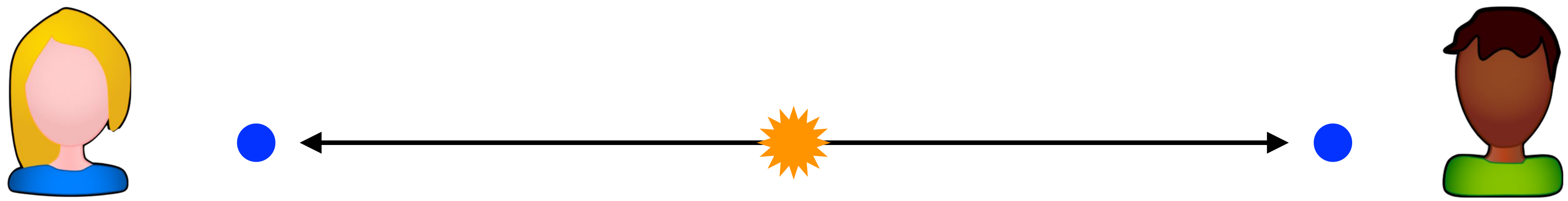


	Alice			Bob		
prob.	z	x	θ	z	x	θ
P_1	+	+	+	-	-	-
P_2	-	+	+	+	-	-
P_3	+	-	+	-	+	-
P_4	+	+	-	-	-	+
P_5	+	-	-	-	+	+
P_6	-	+	-	+	-	+
P_7	-	-	+	+	+	-
P_8	-	-	-	+	+	+

$P(+z, +x) = P_3 + P_5$

$P(+z, +\theta) = P_4 + P_5$

$P(+\theta, +x) = ?$

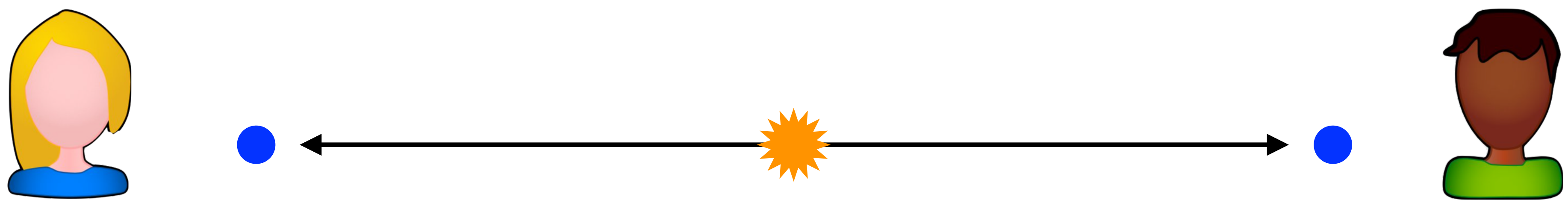


	Alice			Bob		
prob.	z	x	θ	z	x	θ
P_1	+	+	+	-	-	-
P_2	-	+	+	+	-	-
P_3	+	-	+	-	+	-
P_4	+	+	-	-	-	+
P_5	+	-	-	-	+	+
P_6	-	+	-	+	-	+
P_7	-	-	+	+	+	-
P_8	-	-	-	+	+	+

$$P(+z, +x) = P_3 + P_5$$

$$P(+z, +\theta) = P_4 + P_5$$

$$P(+\theta, +x) = P_3 + P_7$$

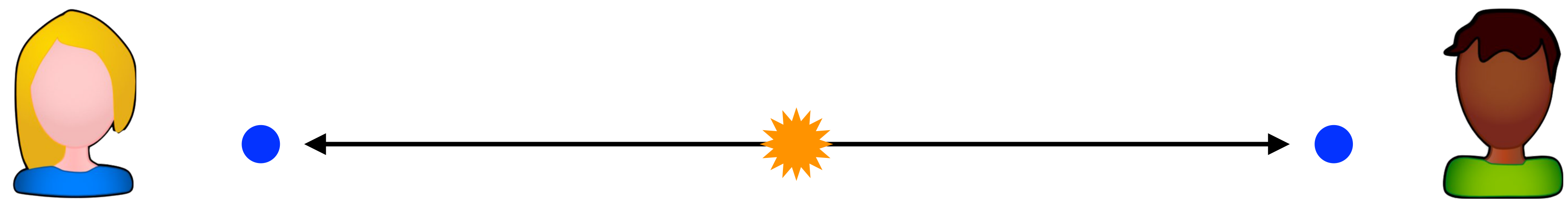


	Alice			Bob		
prob.	z	x	θ	z	x	θ
P_1	+	+	+	-	-	-
P_2	-	+	+	+	-	-
P_3	+	-	+	-	+	-
P_4	+	+	-	-	-	+
P_5	+	-	-	-	+	+
P_6	-	+	-	+	-	+
P_7	-	-	+	+	+	-
P_8	-	-	-	+	+	+

$$P(+z, +x) = P_3 + P_5$$

$$P(+z, +\theta) = P_4 + P_5$$

$$P(+\theta, +x) = P_3 + P_7$$



	Alice			Bob		
prob.	z	x	θ	z	x	θ
P_1	+	+	+	-	-	-
P_2	-	+	+	+	-	-
P_3	+	-	+	-	+	-
P_4	+	+	-	-	-	+
P_5	+	-	-	-	+	+
P_6	-	+	-	+	-	+
P_7	-	-	+	+	+	-
P_8	-	-	-	+	+	+

$$P(+z, +x) = P_3 + P_5$$

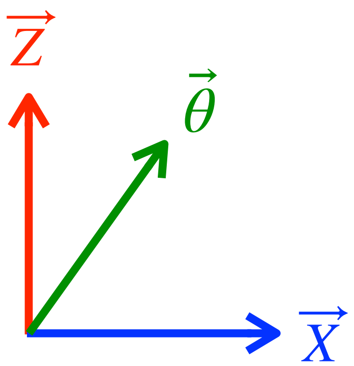
$$P(+z, +\theta) = P_4 + P_5$$

$$P(+\theta, +x) = P_3 + P_7$$

Bell inequality

$$P(+z, +\theta) + P(+\theta, +x) \geq P(+z, +x)$$

$$\mathbf{B}(\theta) = 4 \times [\, P(z+, \theta+) + P(\theta+, x+) - P(z+, x+) \,]$$

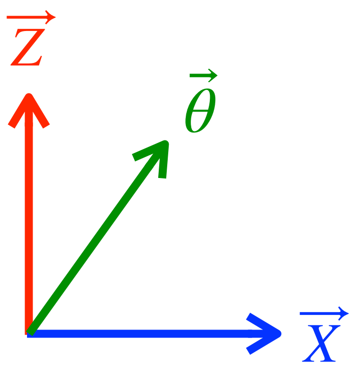


Bell inequality

$$\boxed{\mathbf{B}(\theta) \geq \mathbf{0}}$$

← **for any θ**

$$\mathbf{B}(\theta) = 4 \times [\, P(z+, \theta+) + P(\theta+, x+) - P(z+, x+)\,]$$



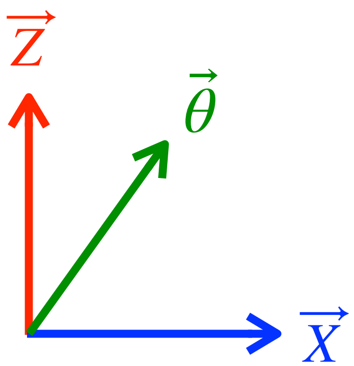
Bell inequality

$$\boxed{\mathbf{B}(\theta) \geq \mathbf{0}}$$ $\longleftarrow$ **for any θ**

QM

$$|\Psi^{(0,0)}\rangle = \frac{|+,-\rangle_z - |-,+\rangle_z}{\sqrt{2}} \quad \Longrightarrow \quad \mathbf{B}(\theta) = 1 - \sqrt{2}\cos(\theta - \pi/4)$$

$$\mathbf{B}(\theta) = 4 \times [\, P(z+, \theta+) + P(\theta+, x+) - P(z+, x+) \,]$$



Bell inequality

$$\boxed{\mathbf{B}(\theta) \geq 0}$$

for any θ

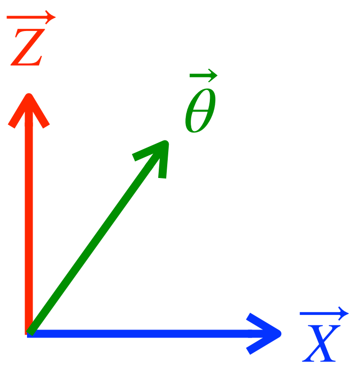
QM

$$|\Psi^{(0,0)}\rangle = \frac{|+,-\rangle_z - |-,+\rangle_z}{\sqrt{2}}$$

$$\mathbf{B}(\theta) = 1 - \sqrt{2}\cos(\theta - \pi/4)$$

$$\mathbf{B}(\theta = \pi/4) \simeq -0.414 < 0 \;!!$$

$$\mathbf{B}(\theta) = 4 \times [\, P(z+, \theta+) + P(\theta+, x+) - P(z+, x+) \,]$$



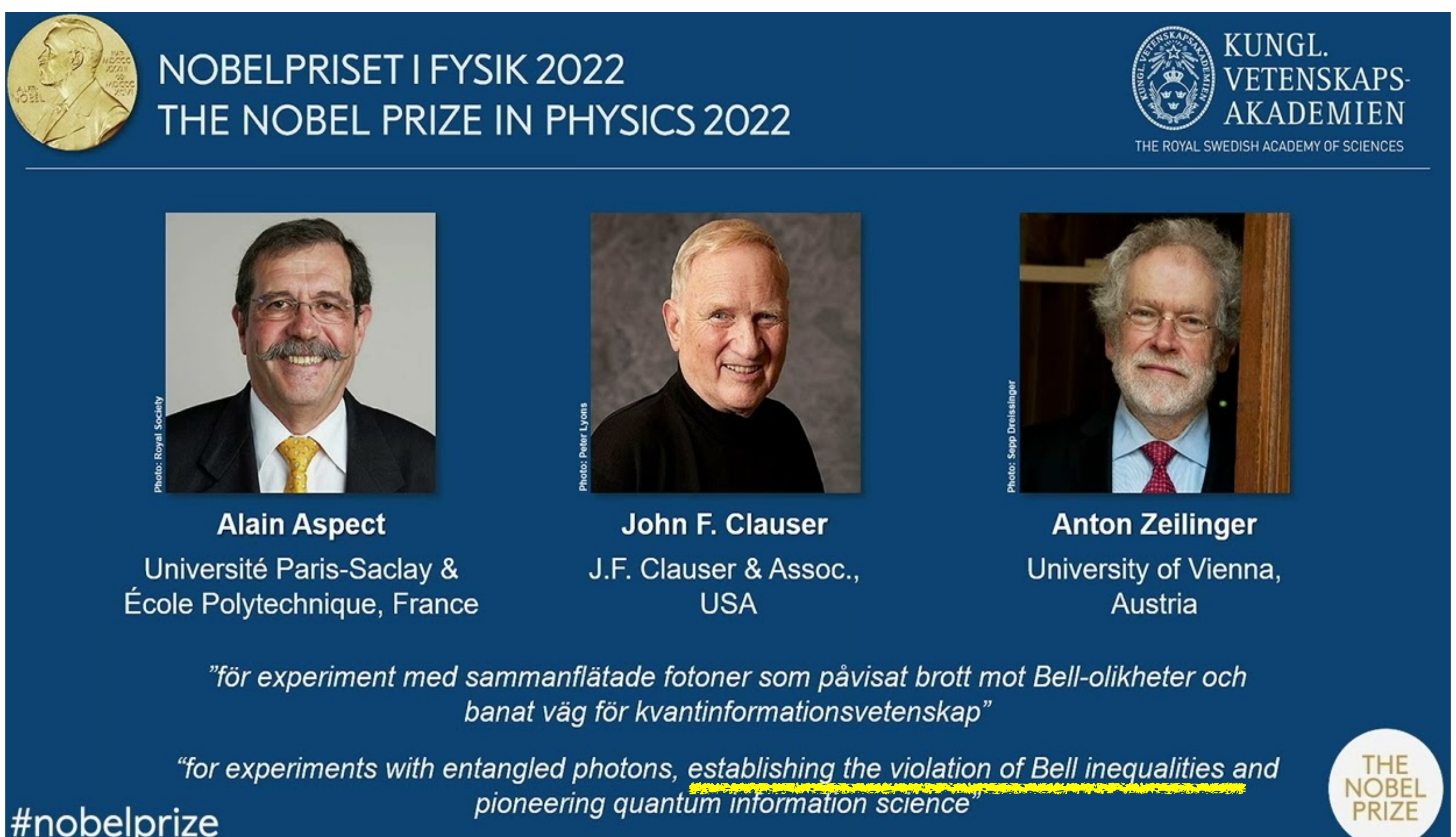
Bell inequality

$$\boxed{\mathbf{B}(\theta) \geq 0}$$

← **for any θ**

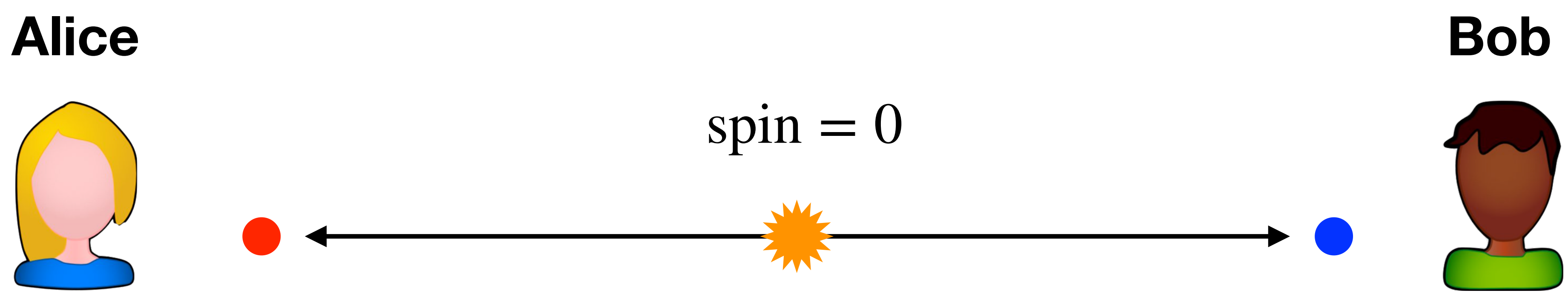
$$|\Psi^{(0,0)}\rangle = \frac{|+,-\rangle_z - |-,+\rangle_z}{\sqrt{2}} \xrightarrow{\text{QM}} \mathbf{B}(\theta) = 1 - \sqrt{2}\cos(\theta - \pi/4)$$

$$\mathbf{B}(\theta = \pi/4) \simeq -0.414 < \mathbf{0} \;!!$$



Violation of Bell inequality has been experimentally confirmed!!

Entanglement



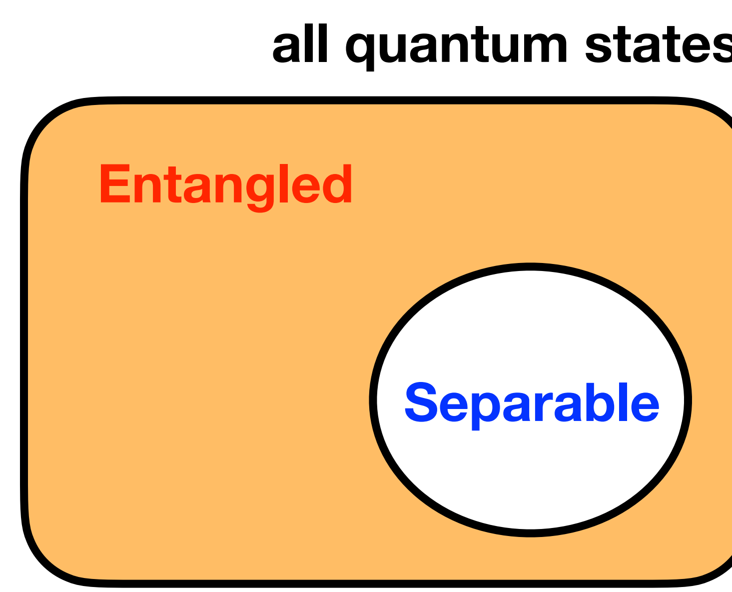
Alice Bob

$$|\Psi^{(0,0)}_{AB}\rangle \simeq |+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle$$

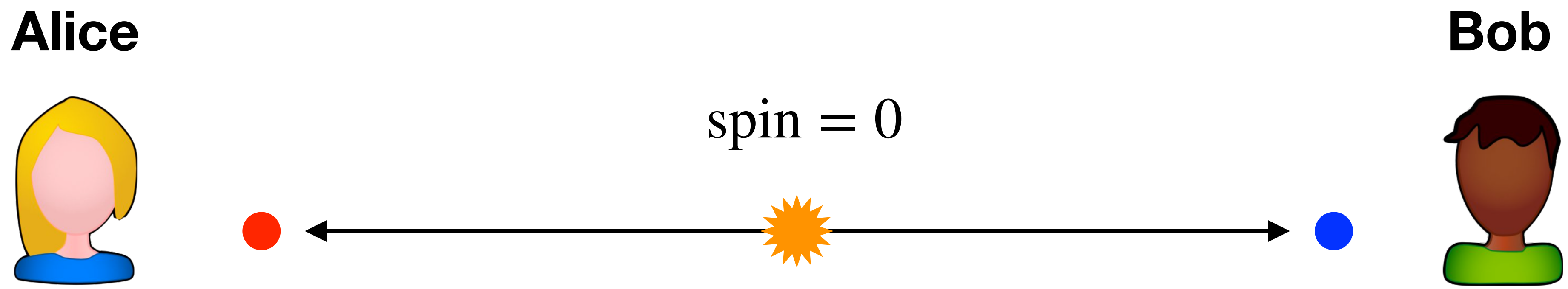
$$\neq |\Psi_A\rangle \otimes |\Psi_B\rangle \longleftarrow \text{entangled}$$

$$|\Psi^{\text{sep}}_{AB}\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle \longleftarrow \text{separable}$$

$$\text{Entanglement} = \overline{\text{Separable}}$$



Entanglement



Alice Bob

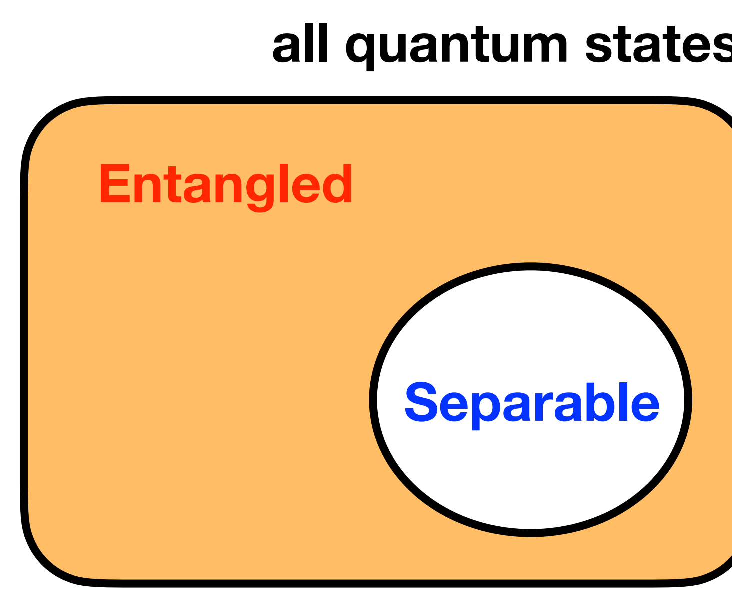
$$|\Psi_{AB}^{(0,0)}\rangle \simeq |+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle \xrightarrow{\text{Alice measures } s_z \text{ and found +}} |+,-\rangle$$

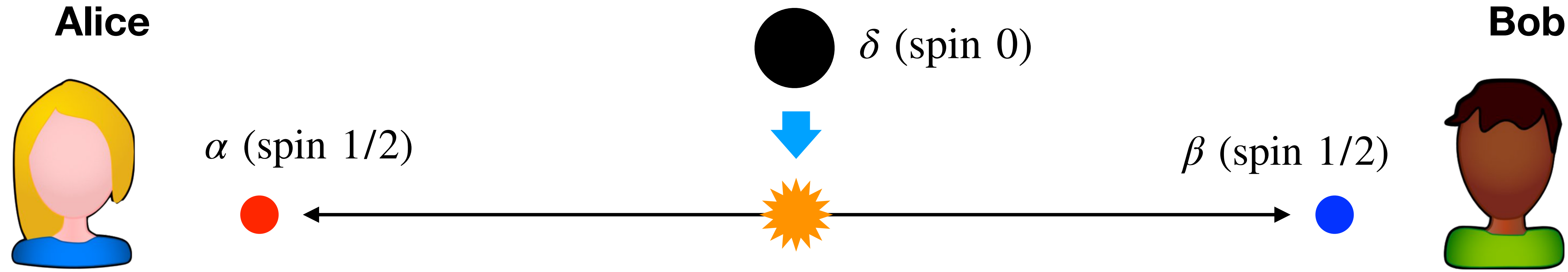
$$\neq |\Psi_A\rangle \otimes |\Psi_B\rangle \longleftarrow \textbf{entangled}$$

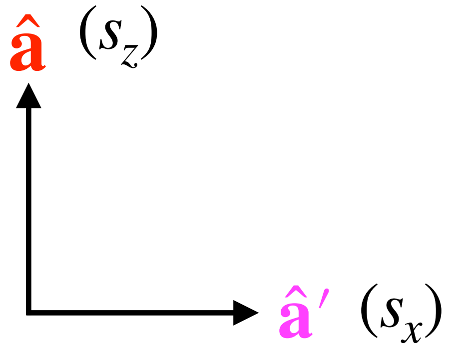
Alice's local measurement changes the global state involving (spacelike separated) Bob's particle.
~ **Nonlocality**

$$|\Psi_{AB}^{\text{sep}}\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle \longleftarrow \textbf{separable}$$

$$\text{Entanglement} = \overline{\text{Separable}}$$

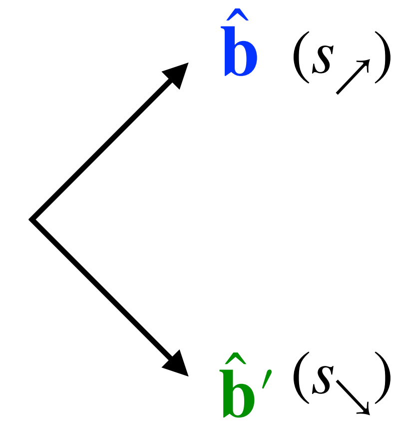






The experiment consists of 4 sessions:

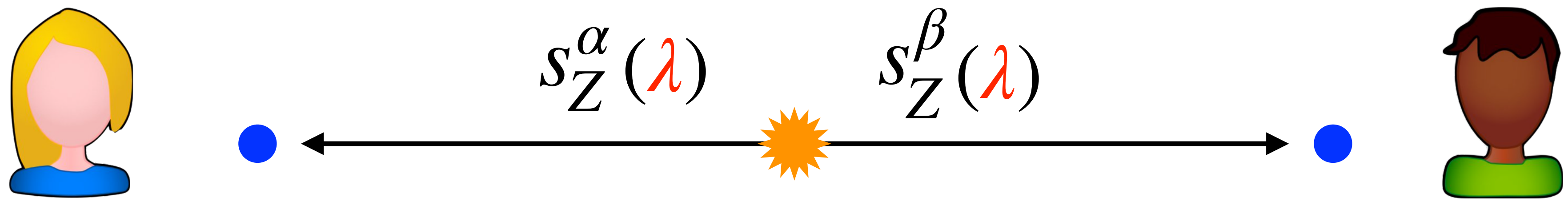
1) Alice and Bob measure s_a^α and s_b^β, respectively. Repeat the measurement many times and calculate $\langle s_a \cdot s_b \rangle$.
2) Repeat (1) for a and b'.
3) Repeat (1) for a' and b.
4) Repeat (1) for a' and b'.



Finally, we calculate: $S_{\mathrm{CHSH}} \equiv \left| \langle s_a s_b \rangle - \langle s_a s_{b'} \rangle + \langle s_{a'} s_b \rangle + \langle s_{a'} s_{b'} \rangle \right|$

Local and **Real** $\Rightarrow$ $\boxed{S_{\mathrm{CHSH}} \leq 2}$ **CHSH inequality**

[Clauser, Horne, Shimony, Holt, 1969]



- Assuming the **reality**, Alice's result is predetermined before her measurement.
- The spin components of Bob's particle are also predetermined and not affected by Alice's measurement by the **locality** assumption.
- Without loss of generality, we can parametrise their spin components by a set of parameters λ, which appears with the probability $P(\lambda)$ in each decay.

$$\sum_\lambda P(\lambda) = 1$$

- The spin correlation is given by

$$\langle s_Z^\alpha \cdot s_Z^\beta \rangle = \sum_\lambda P(\lambda)\, s_Z^\alpha(\lambda)\, s_Z^\beta(\lambda) = -1$$

Let's derive

$$\boxed{S_{\mathrm{CHSH}} \equiv \left| \langle ab \rangle - \langle ab' \rangle + \langle a'b \rangle + \langle a'b' \rangle \right| \le 2}$$

$$\left| \langle ab \rangle - \langle ab' \rangle \right| = \left| \sum_{\lambda} abP - \sum_{\lambda} ab'P \right|$$

$$\pm aba'b'P - (\pm aba'b'P) = 0$$

$$= \left| \sum_{\lambda} \left[ab(1 \pm a'b')P - ab'(1 \pm a'b)P \right] \right|$$

$$\le \sum_{\lambda} \left[\left| ab \right| \left| 1 \pm a'b' \right| P + \left| ab' \right| \left| 1 \pm a'b \right| P \right]$$

$$|ab| = |ab'| = 1$$

$$|1 \pm a'b'|, |1 \pm a'b| \geq 0$$

$$= \sum_{\lambda} \left[\left(1 \pm a'b'\right) P + \left(1 \pm a'b\right) P \right]$$

$$= 2 \pm \left(\langle a'b' \rangle + \langle a'b \rangle \right)$$

$$\tilde{S}_{\mathrm{CHSH}} \equiv \frac{1}{2} \left(\left| \langle ab \rangle - \langle ab' \rangle \right| + \left| \langle a'b \rangle + \langle a'b' \rangle \right| \right) \le 2$$

$$\max_{(a,b,a',b')} R_{\mathrm{CHSH}} = \max_{(a,b,a',b')} \tilde{R}_{\mathrm{CHSH}}$$

$$\boxed{S_{\mathrm{CHSH}} \le 2}$$

The CHSH Bell-type inequality **can test** not only hidden local variable theories but also **Quantum Mechanics**!

$$S_{\mathrm{CHSH}} \leq \begin{cases} 2 & \textbf{Local theories} \quad \text{[CHSH(1969)]} \\ 2\sqrt{2} & \textbf{Quantum Mechanics} \quad \text{[Tsirelson(1987)]} \\ 4 & \textbf{No-signalling} \quad \text{[Popescu, Rohrlich(1994)]} \end{cases}$$

- Nonlocal states in QM does not violate causality

$$p(a|x,y) \equiv \sum_b p(a,b|x,y)$$

Condition for no causality violation: No-Signalling [Cirel'son(1980), Popescu, Rohrlich(1994)]

$$^\forall a,b,x,x',y,y' \begin{cases} p(a|x,y) = p(a|x,y') \\ p(b|x,y) = p(b|x',y) \end{cases}$$

- $p(a|x,y) = p(a|x,y')$ — **Alice's dist. is indep. of Bob's choice for meas. axis**
- $p(b|x,y) = p(b|x',y)$ — **Bob's dist. is indep. of Alice's choice for meas. axis**

No-signalling ⊃ Quantum ⊃ Local ⊃ Separable

✤ Violation of CHSH inequality has been observed at low energies $\ll$ TeV

- **Entangled photon pairs** (from decays of Calcium atoms)

  Clauser, Horne, Shimony, Holt (1969), Freedman and Clauser (1972), A. Aspect et. al. (1981, 1982), Y. H. Shih, C. O. Alley (1988), L. K. Shalm et al. (2015) [5σ]

- **Entangled proton pairs** (from decays of ^{2}He)

  M. M. Lamehi-Rachti, W. Mitting (1972), H. Sakai (2006)

- $K^0\overline{K^0}$, $B^0\overline{B^0}$ **flavour oscillation** CPLEAR (1999), Belle (2004, 2007)

- $B^0 \to J/\psi + K^*(892)^0$ **spin correlation,** $S_{\rm CGLMP} > 2$, **[36σ]**

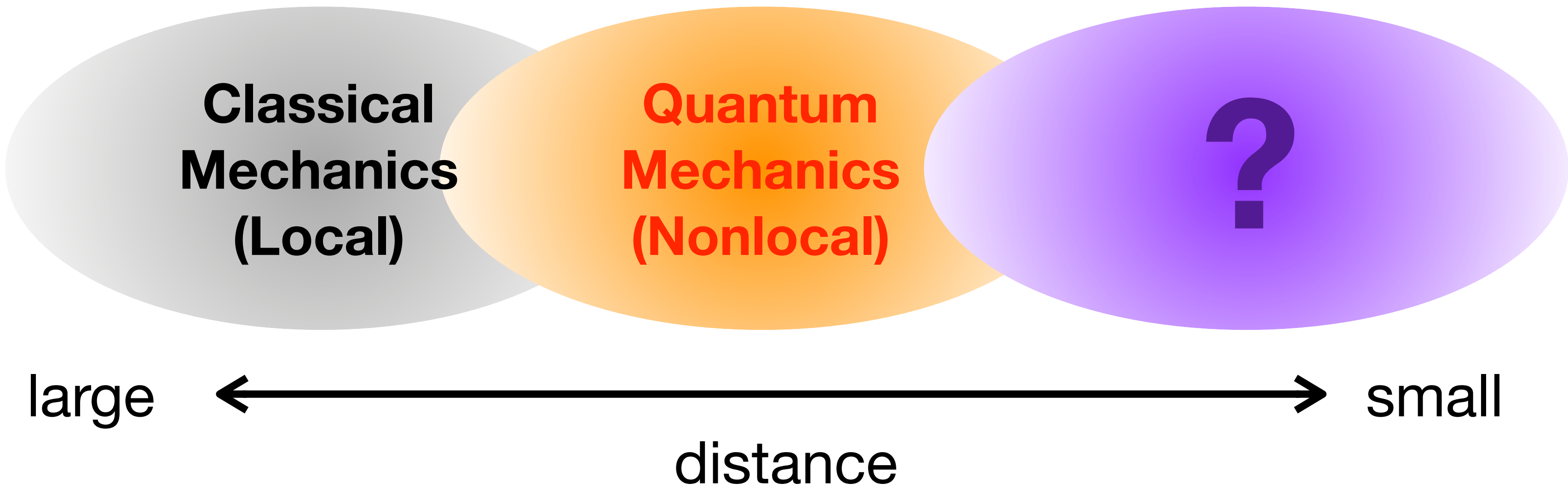
Fabbrichesi, Floreanini, Gabrielli, Marzola (2023)

Normalised helicity amplitude for $B^0 \to J/\psi + K^*(892)^0$

$$
\begin{aligned}
|A_{||}|^2 &= 0.227 \pm 0.004 \text{ (stat.)} \pm 0.011 \text{ (syst.)},\\
|A_{\perp}|^2 &= 0.201 \pm 0.004 \text{ (stat.)} \pm 0.008 \text{ (syst.)},\\
\delta_{||} \text{ [rad]} &= -2.94 \pm 0.02 \text{ (stat.)} \pm 0.03 \text{ (syst.)},\\
\delta_{\perp} \text{ [rad]} &= 2.94 \pm 0.02 \text{ (stat.)} \pm 0.02 \text{ (syst.)}.
\end{aligned}
$$

LHCb [1307.2782]

Testing QM at high energy colliders



Motivation

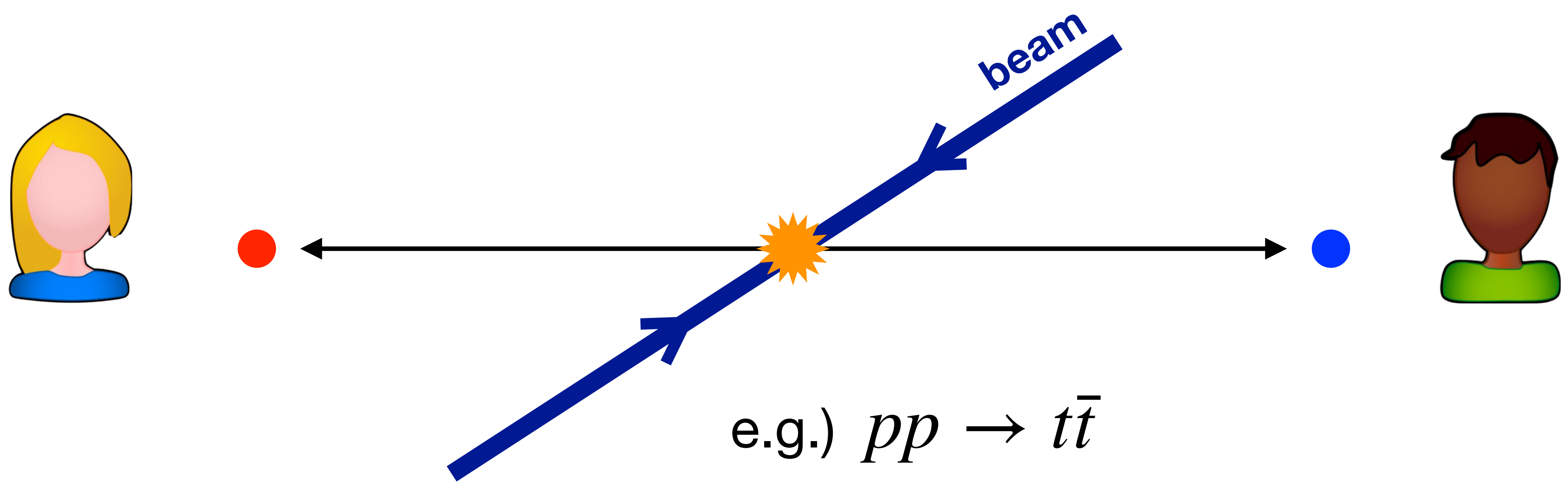
✤ Bell inequalities/Entanglement have not been tested at the TeV energy scale:

➡ LHC (and $FCC_{ee/hh}$) provides the unique opportunity for this test

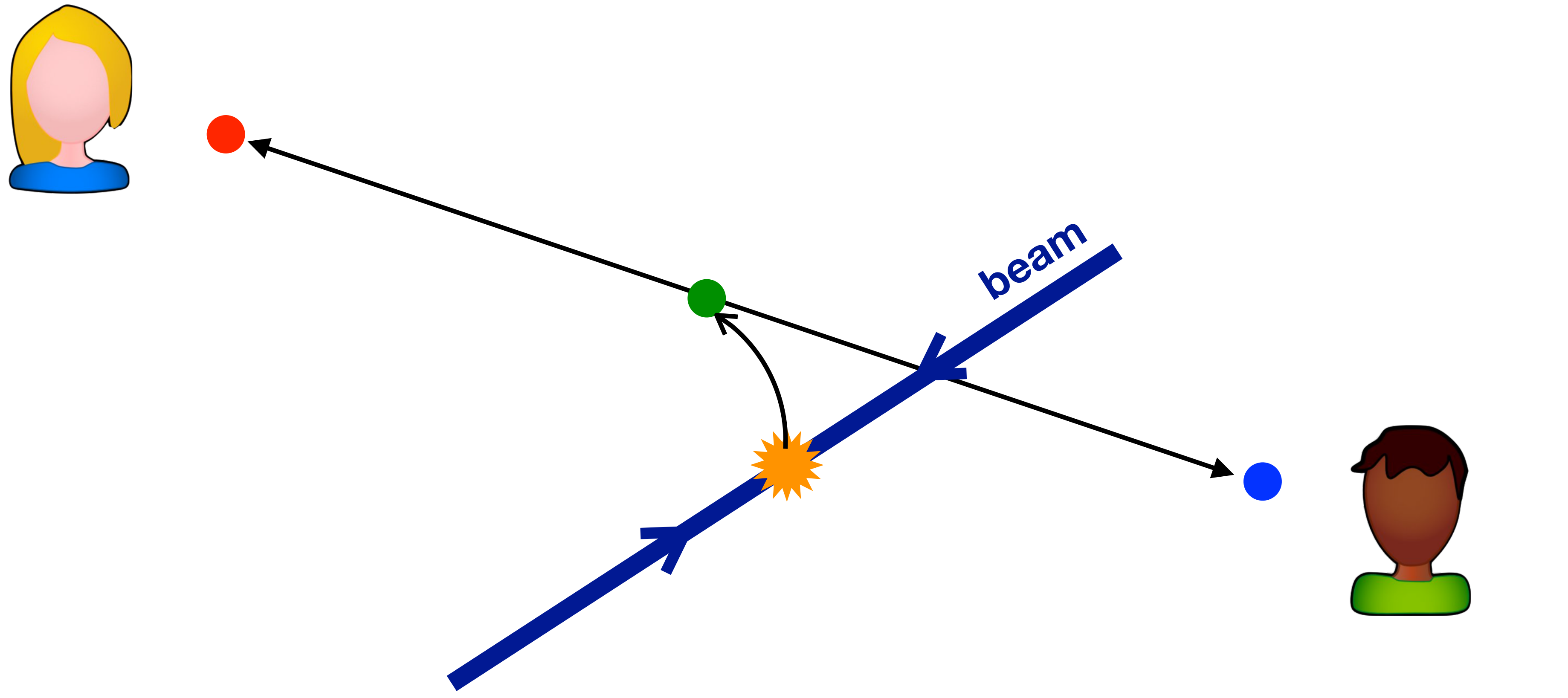
✤ Detection of Entanglement/Bell violation requires a detailed analysis of spin correlation:

➡ provides a very good test for the Standard Model (sensitive to BSM)

Entangled pairs at Colliders

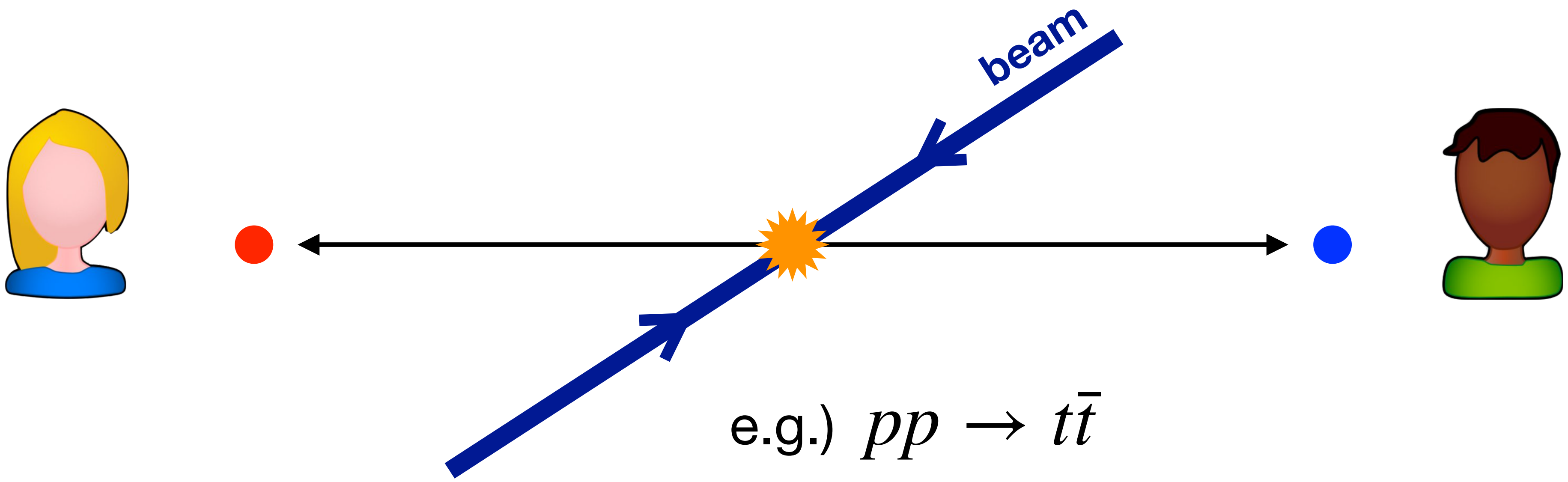


e.g.) $pp \to t\bar{t}$



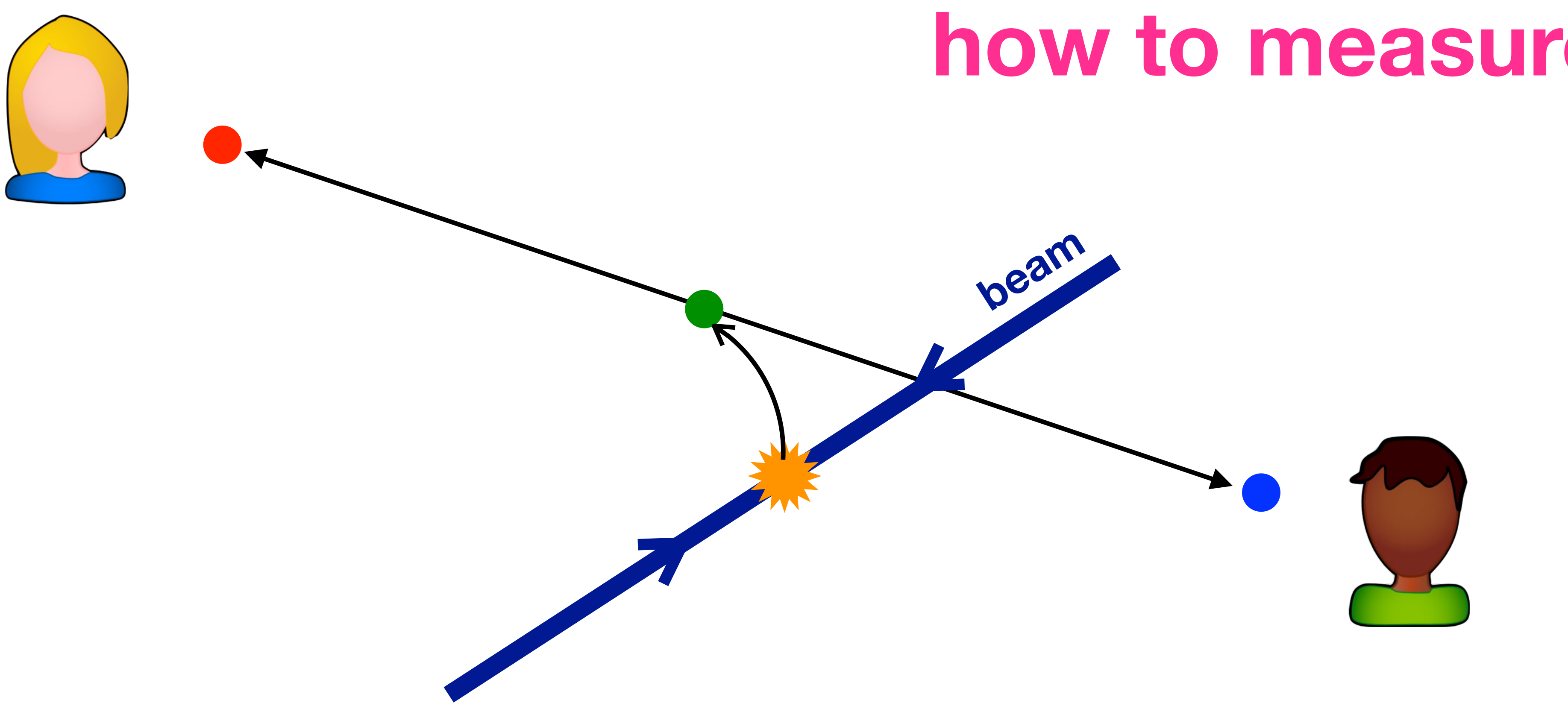
e.g.) $H \to \tau^+\tau^-, WW^*, ZZ^*$

Entangled pairs at Colliders



e.g.) $pp \to t\bar{t}$

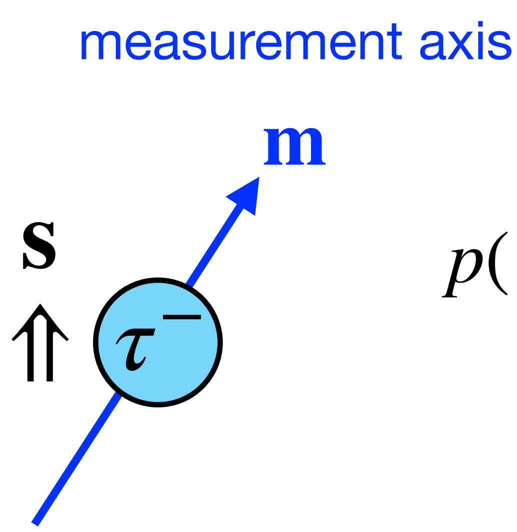
how to measure spin?



e.g.) $H \to \tau^+\tau^-, WW^*, ZZ^*$

Particles with weak decays are their own polarimeters

e.g.) For $\tau^- \to \pi^- + \nu_\tau$ (τ^- rest frame), the spin of τ^- is measured in the direction of π^- ($\vec{\pi}$) and the outcome is +1.



$$p(+|\mathbf{m}) = |\langle +_{\mathbf{m}} | +_{\mathbf{s}} \rangle|^2$$

$$= \frac{1 + (\mathbf{m} \cdot \mathbf{s})}{2}$$

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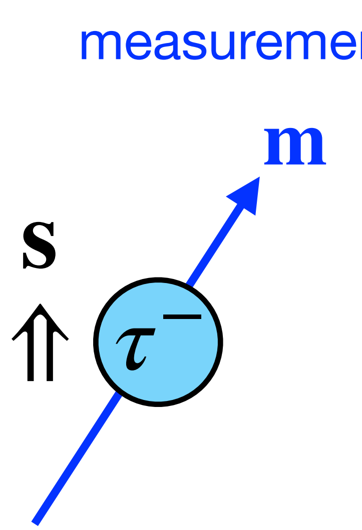
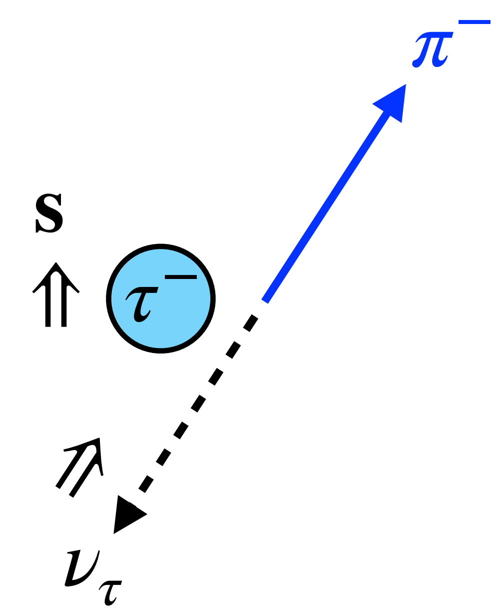


$$p(+\,|\,\mathbf{m}) = |\langle +_{\mathbf{m}} \,|\, +_{\mathbf{s}} \rangle|^2$$

$$= \frac{1 + (\mathbf{m} \cdot \mathbf{s})}{2}$$



$$\frac{d\Gamma}{d\Omega} = \frac{1 + (\vec{\pi} \cdot \mathbf{s})}{2}$$

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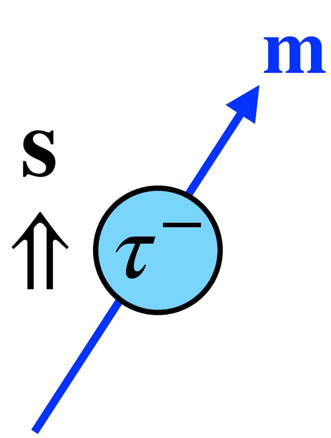

$$p(+\,|\,\mathbf{m}) = |\langle +_{\mathbf{m}} | +_{\mathbf{s}} \rangle|^2$$

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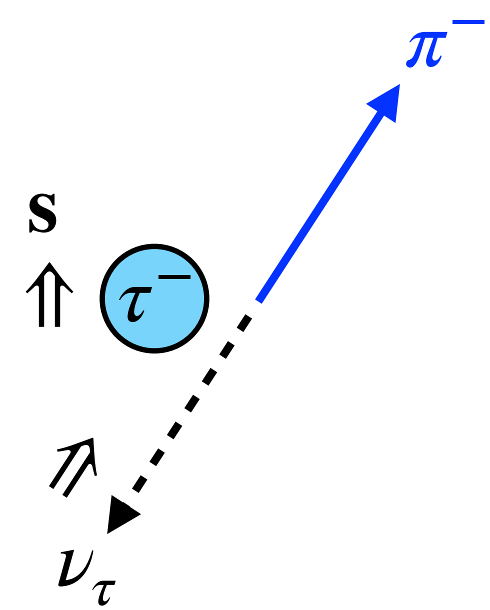


$$\frac{d\Gamma}{d\Omega} = \frac{1 + (\vec{\pi}\cdot\mathbf{s})}{2}$$

For spins to be measurable, one must focus on entangled pairs of weakly decaying particles

$$\tau,\ t,\ W^{\pm},\ Z^0$$

Particles with weak decays are their own polarimeters

More generally,

$$\frac{d\Gamma}{d\Omega} = \frac{1 + \alpha_x \cdot (\vec{x} \cdot \mathbf{s})}{2}$$

$\alpha_x \in [-1, +1]$: spin analyzing power

- tau decay

$\alpha_x = 1$ for ($x = \pi^-$ in $\tau^- \to \pi^- \nu$)

- top decay

decay product x	α_x
b	$-0.3925(6)$
W^+	$0.3925(6)$
ℓ^+ (from a W^+)	$0.999(1)$
$\bar{d}$, $\bar{s}$ (from a W^+)	$0.9664(7)$
u, c (from a W^+)	$-0.3167(6)$

Spin correlation:

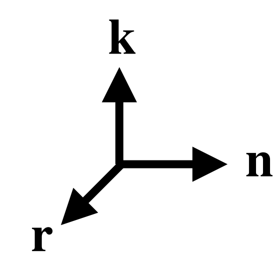
$\mathbf{n}, \mathbf{n}'$: spin measurement axes

$\vec{x}, \vec{y}$: direction of decay products

$$C_{\mathbf{n},\mathbf{n}'} \equiv \langle (\mathbf{s}_A \cdot \mathbf{n})(\mathbf{s}_B \cdot \mathbf{n}') \rangle = \frac{9}{\alpha_x \alpha_y} \langle (\vec{x} \cdot \mathbf{n})(\vec{y} \cdot \mathbf{n}') \rangle$$

$$S_{\text{CHSH}} \equiv C_{\mathbf{n}_1,\mathbf{n}_1'} + C_{\mathbf{n}_1,\mathbf{n}_2'} + C_{\mathbf{n}_2,\mathbf{n}_1'} - C_{\mathbf{n}_2,\mathbf{n}_2'} > 2$$ → Bell inequality violation

$$D \equiv \mathrm{Tr}[C]/3 < -\frac{1}{3}$$ → sufficient cond. for entanglement



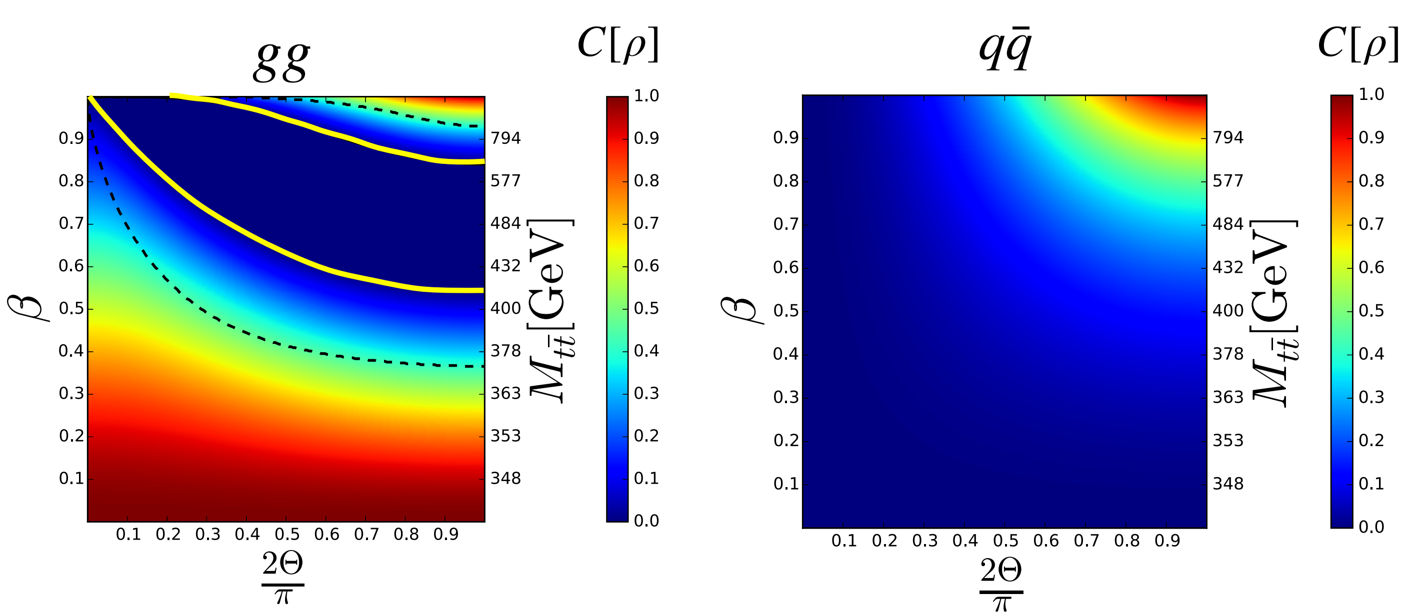
Recent activities to look into entanglements, etc. in HEP

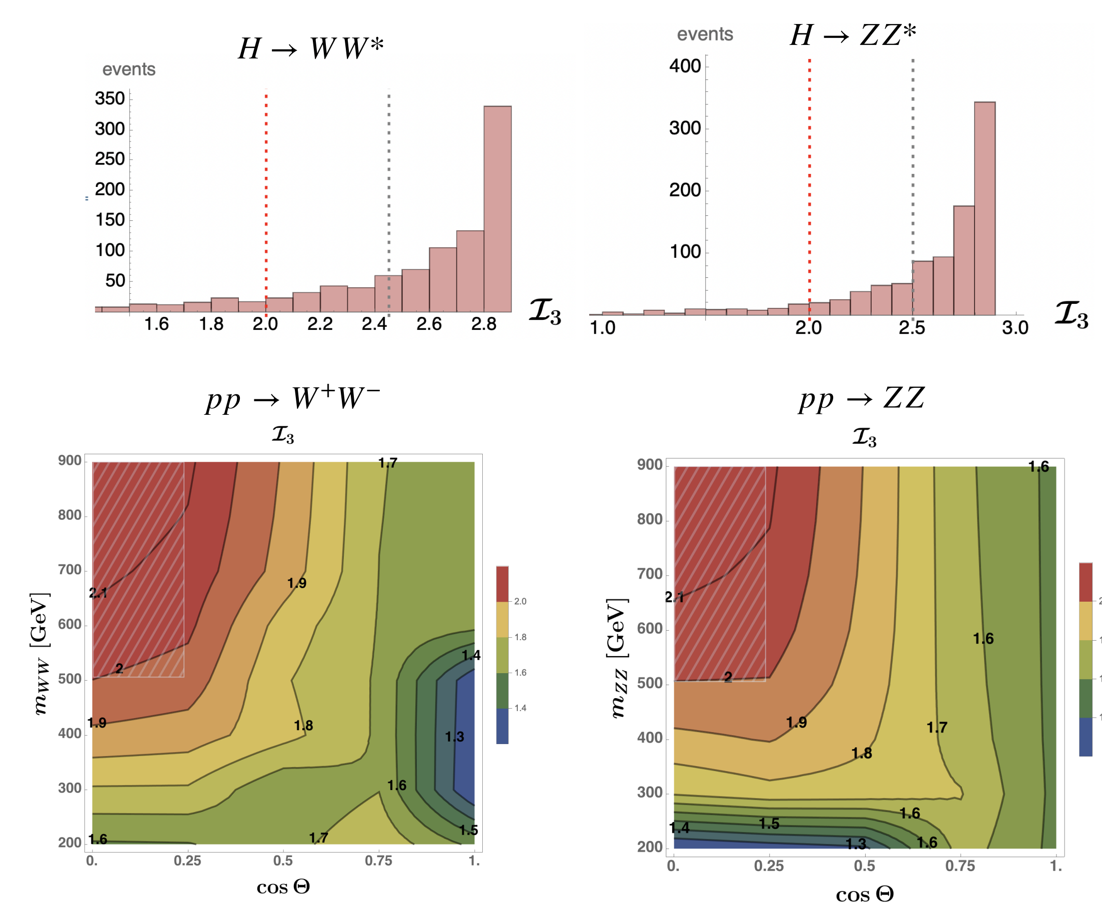
❖ Experimental observation of entanglement and Bell-ineq violation @ LHC

- $pp \to t\bar{t}$ — Y. Afik and J. R. M. de Nova ’21, ’22, M. Fabbrichesi, R. Floreanini, G. Panizzo ‘21, Z. Dong, D. Gonçalves, K. Kong, A. Navarro ‘23

- $H \to WW, ZZ$ — A. J. Barr ’21, J.A. Aguilar-Saavedra, A. Bernal, J.A. Casas, J.M. Moreno ’22, A. Bernal, P. Caban, J. Rembieliński ’23, M. Fabbrichesi, R. Floreanini, E. Gabrielli, Luca Marzola ‘23

- $H \to \tau^+\tau^-$ (@ e^+e^- colliders) — M. Fabbrichesi, R. Floreanini, E. Gabrielli 22, M. Altakach, P. Lamba, F. Maltoni, K. Mawatari, KS ’22, K. Ma, T. Li ‘23





• $pp \to t\bar{t}$

Observation of quantum entanglement in top-quark pairs using the ATLAS detector

The ATLAS Collaboration

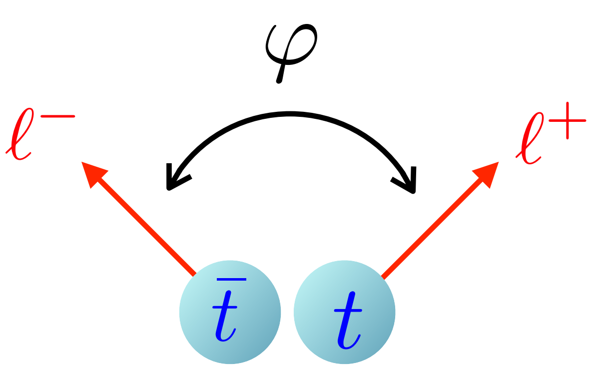
We report the highest-energy observation of entanglement, in top–antitop quark events produced at the Large Hadron Collider, using a proton–proton collision data set with a center-of-mass energy of $\sqrt{s}$ = 13 TeV and an integrated luminosity of 140 fb^{-1} recorded

$$D = \frac{\mathrm{tr}[\mathbf{C}]}{3} \quad < -\frac{1}{3}$$

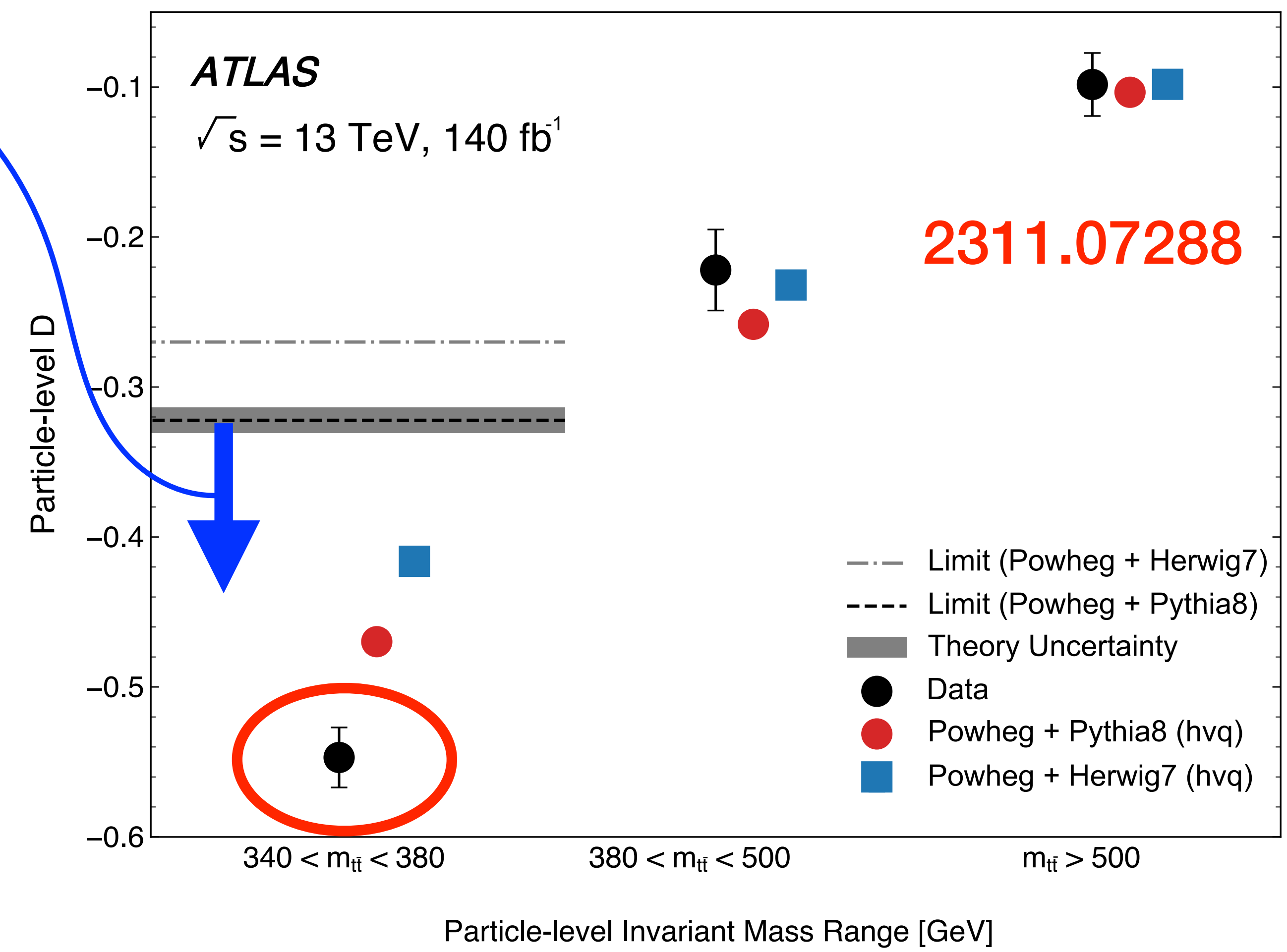
(sufficient cond. for entanglement)

$$\frac{1}{\sigma}\frac{d\sigma}{d\cos\varphi} = \frac{1}{2}(1 - D\cos\varphi)$$

$$\rightarrow \quad D = -3 \cdot \langle \cos\varphi \rangle$$

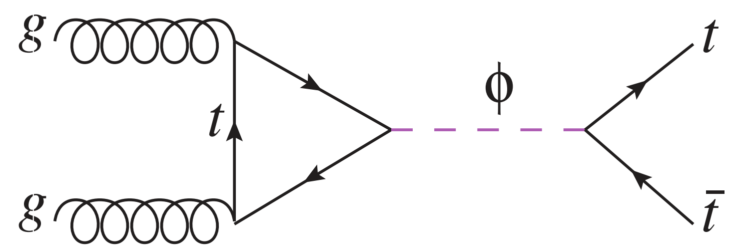


entangled

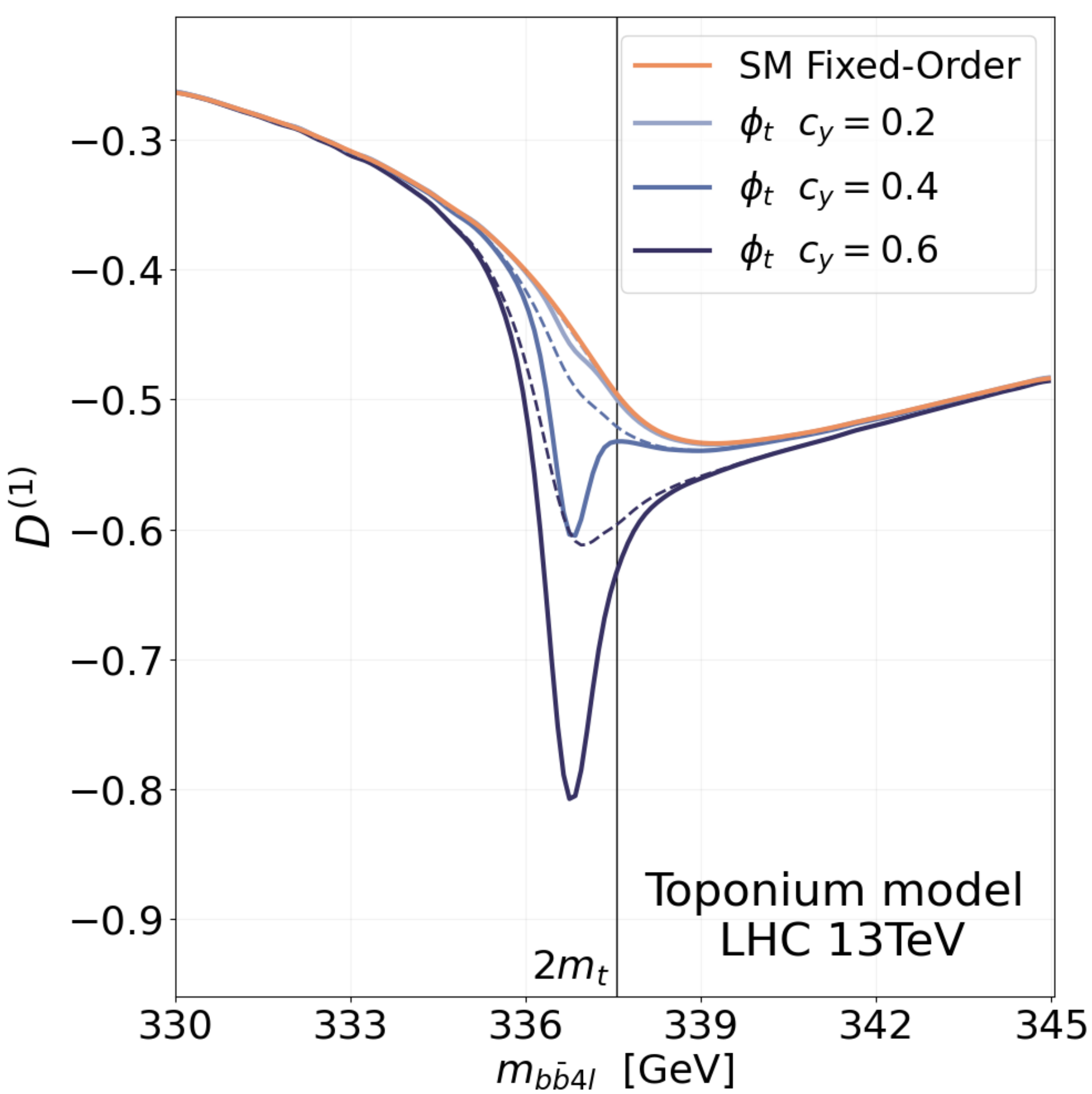


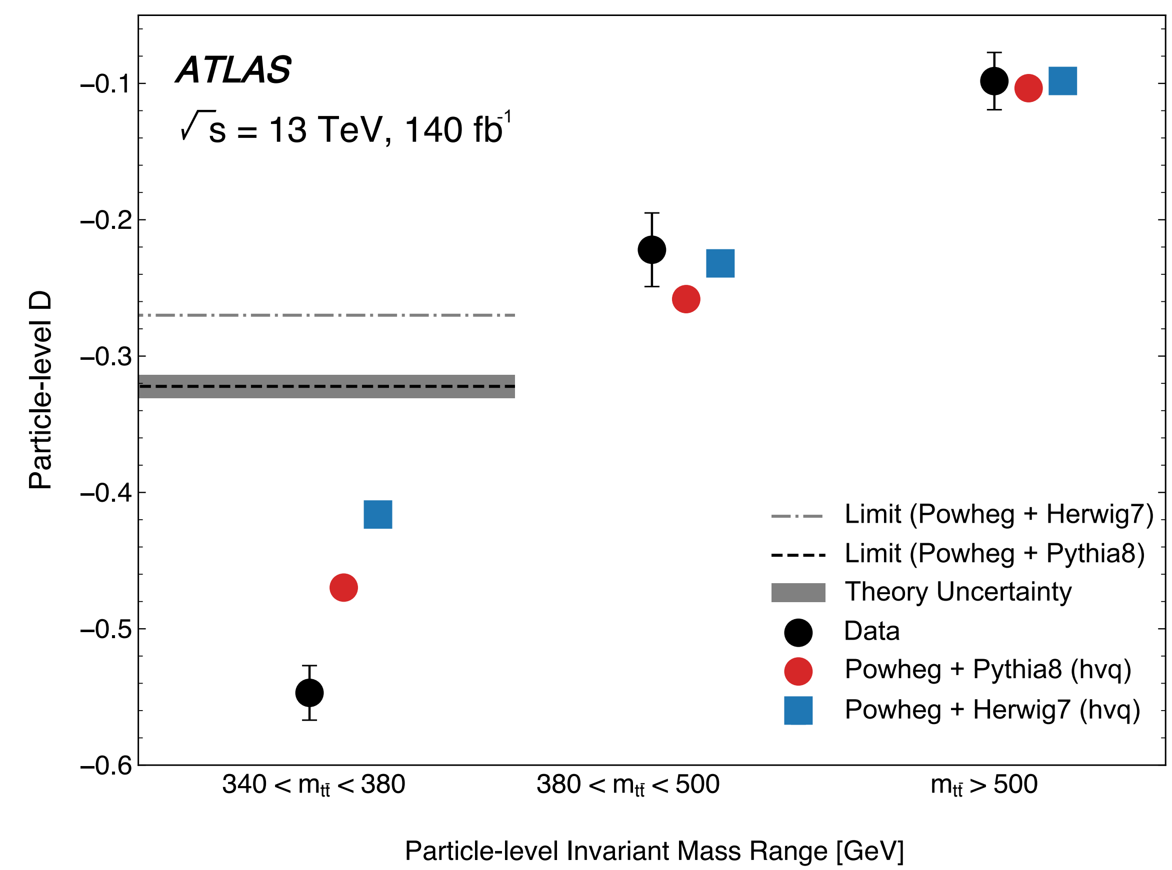
Maltoni, Severi, Tentori, Vryonidou [2401.08751]

$$\mathcal{L} = \mathcal{L}_{\mathrm{SM}} - \frac{1}{2}\phi(\partial^2 + M_\phi^2)\phi + c_y \frac{y_t}{\sqrt{2}}\, \phi\, \bar{t}\,(\cos\alpha + i\gamma^5 \sin\alpha)\, t.$$



$M_\phi = 343.5\,\mathrm{GeV}, \quad \alpha = \pi/2$

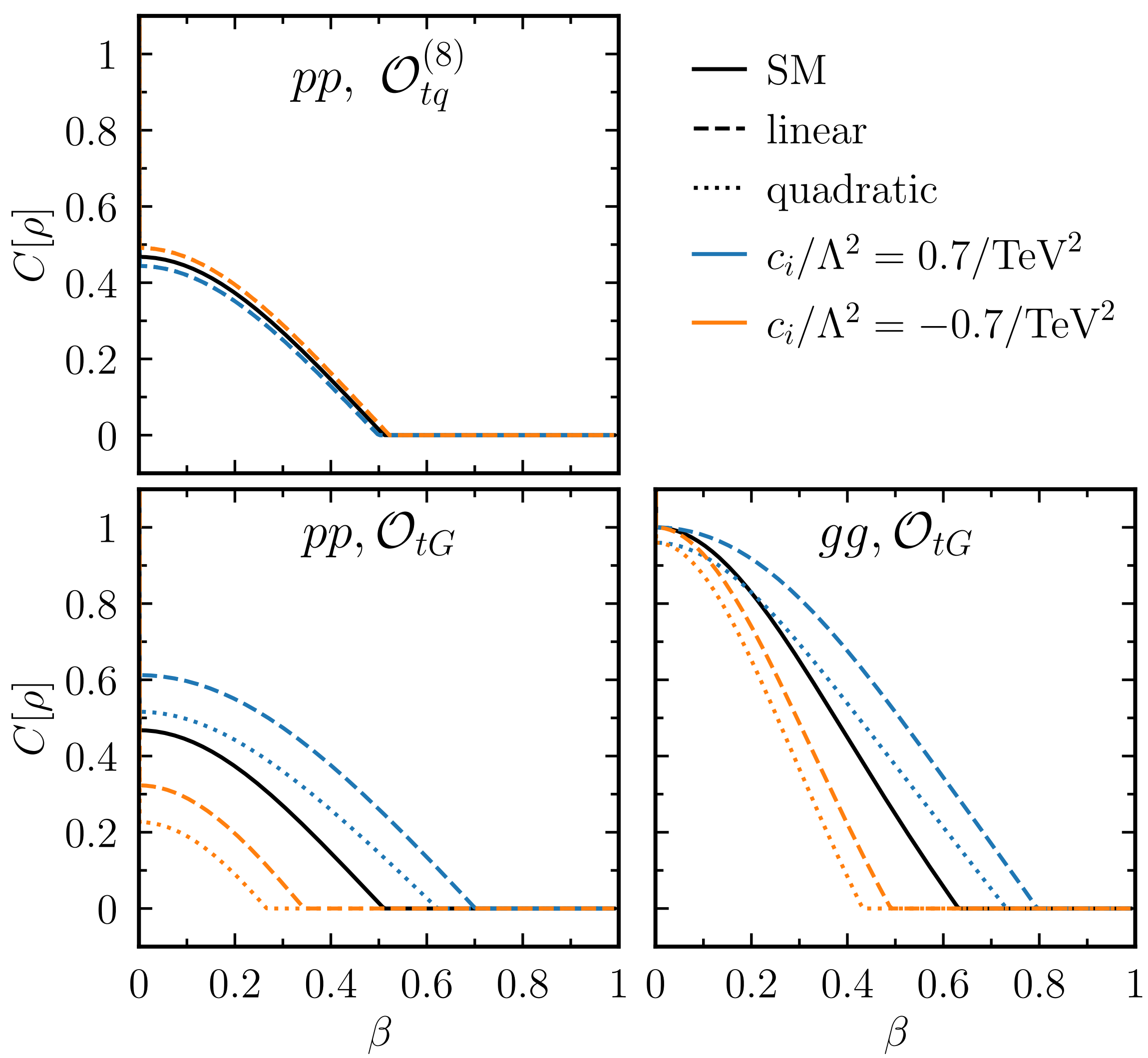




Effect of BSM $pp \to t\bar{t}$

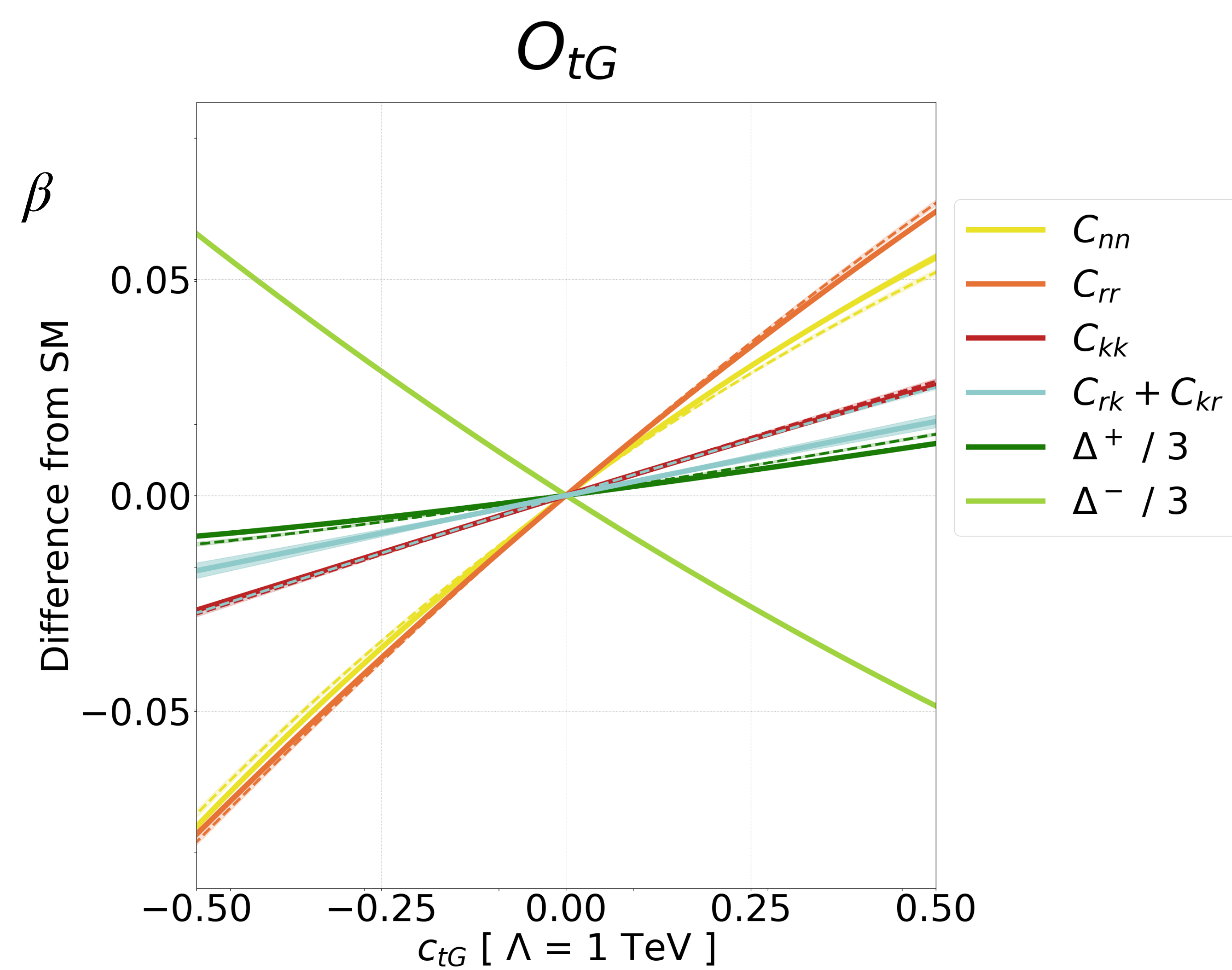
$$\mathcal{O}_{tG} = g_S\, \overline{Q} T_A \tilde{\varphi} \sigma^{\mu\nu} t\, G^A_{\mu\nu} \qquad \mathcal{O}^8_{tq} = \sum_{\mathrm{f}=1}^{2} (\bar{q}_{\mathrm{f}} \gamma_\mu T_A q_{\mathrm{f}})(\bar{t}\gamma^\mu T^A t)$$

[Aoude Madge Maltoni Mantani (2022)]



β: top velocity in the $t\bar{t}$ rest frame

[Severi Vryonidou (2023)]



$$\Delta^{\pm} = \pm(C_{kk} + C_{rr}) - C_{nn} - 1$$

$H \to \tau^+\tau^-$ @ e^+e^- colliders

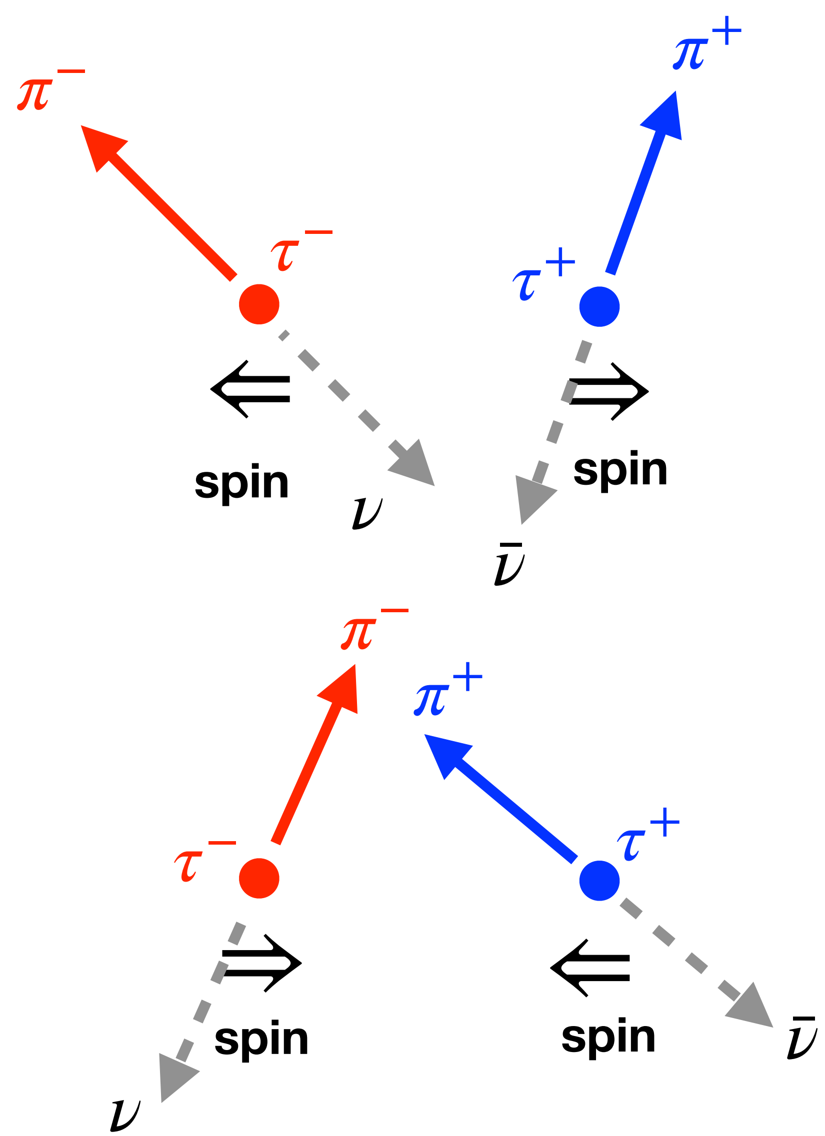
M. Altakach, F. Maltoni, K. Mawatari, P. Lamba, KS, *Phys.Rev.D* 107 (2023) 9, 093002 [2211.10513]

Experimental Challenge in $H \to \tau^+\tau^-$

Among weakly decaying particles, $\tau,\ t,\ W^{\pm},\ Z^0$, the tau-lepton is special because $m_\tau \ll m_H$

One has to measure the direction of pions at the rest frame of each tau.

→ Reconstruction of the tau rest frames (i.e. neutrino reconstruction) is necessary



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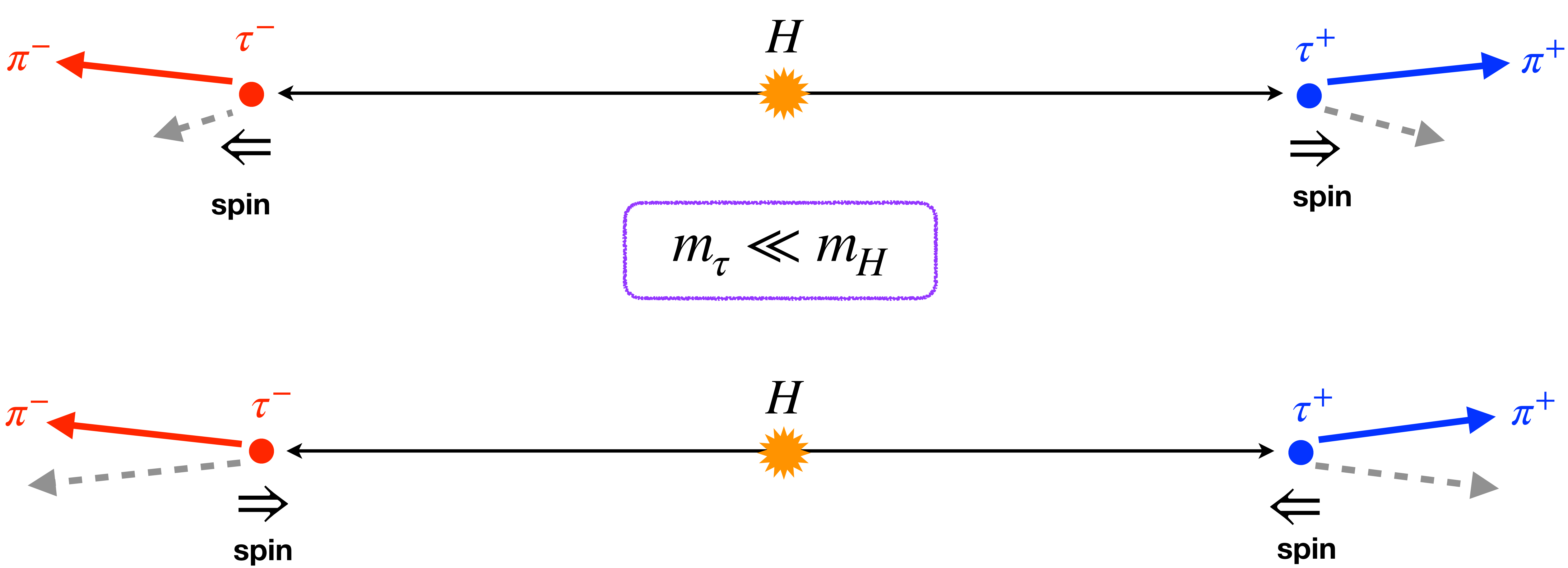
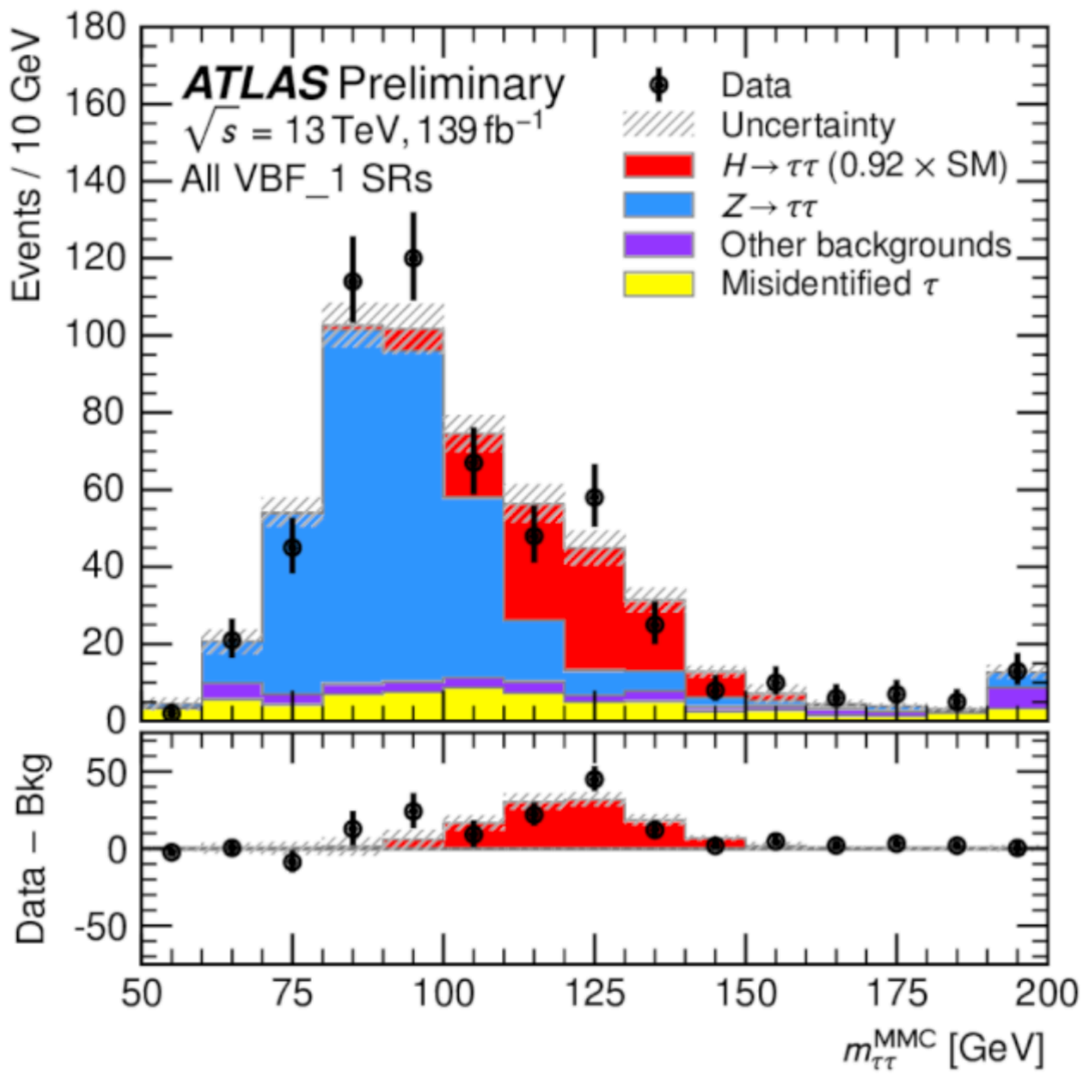


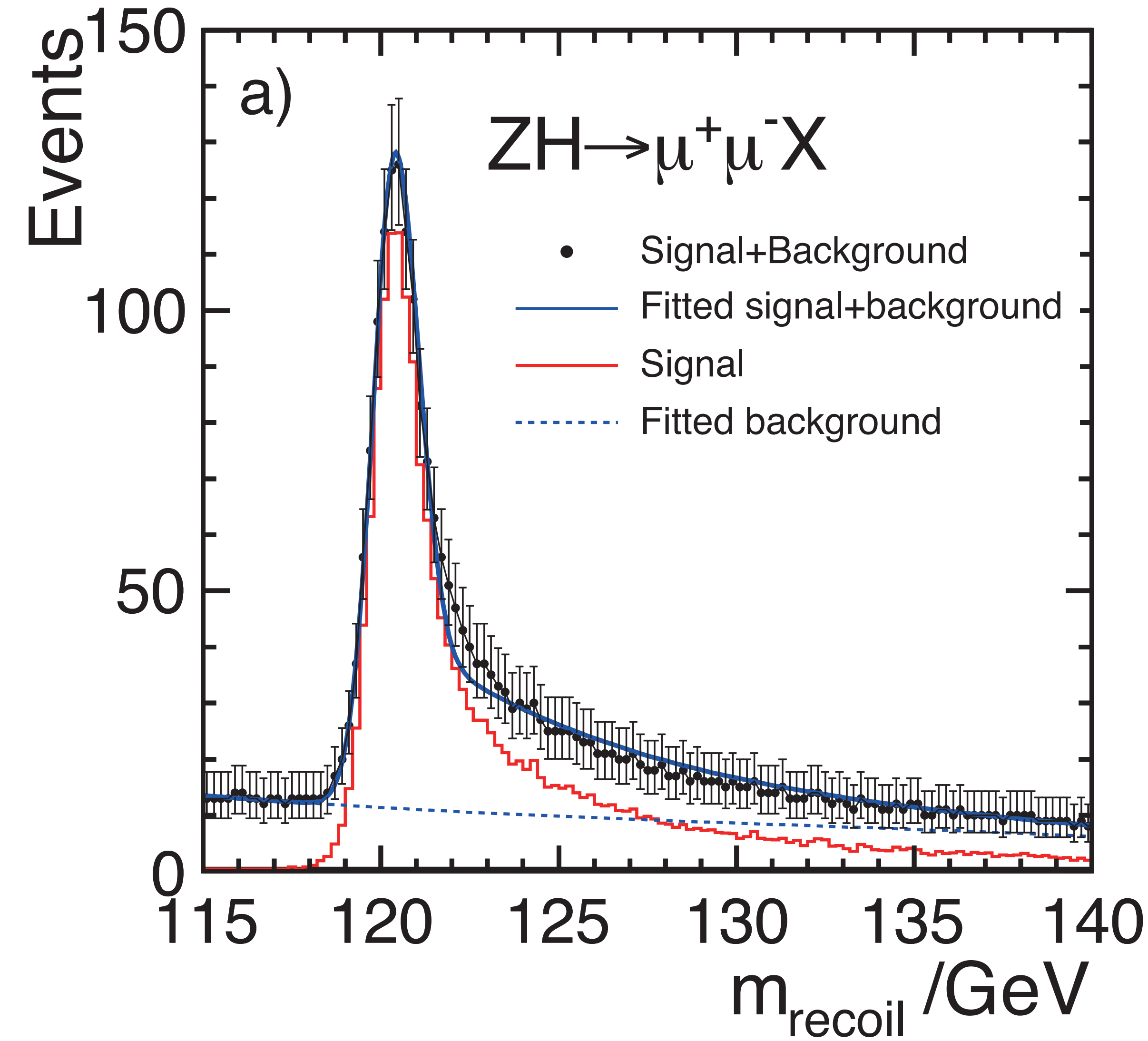
$H \to \tau^+\tau^-$ @ lepton colliders

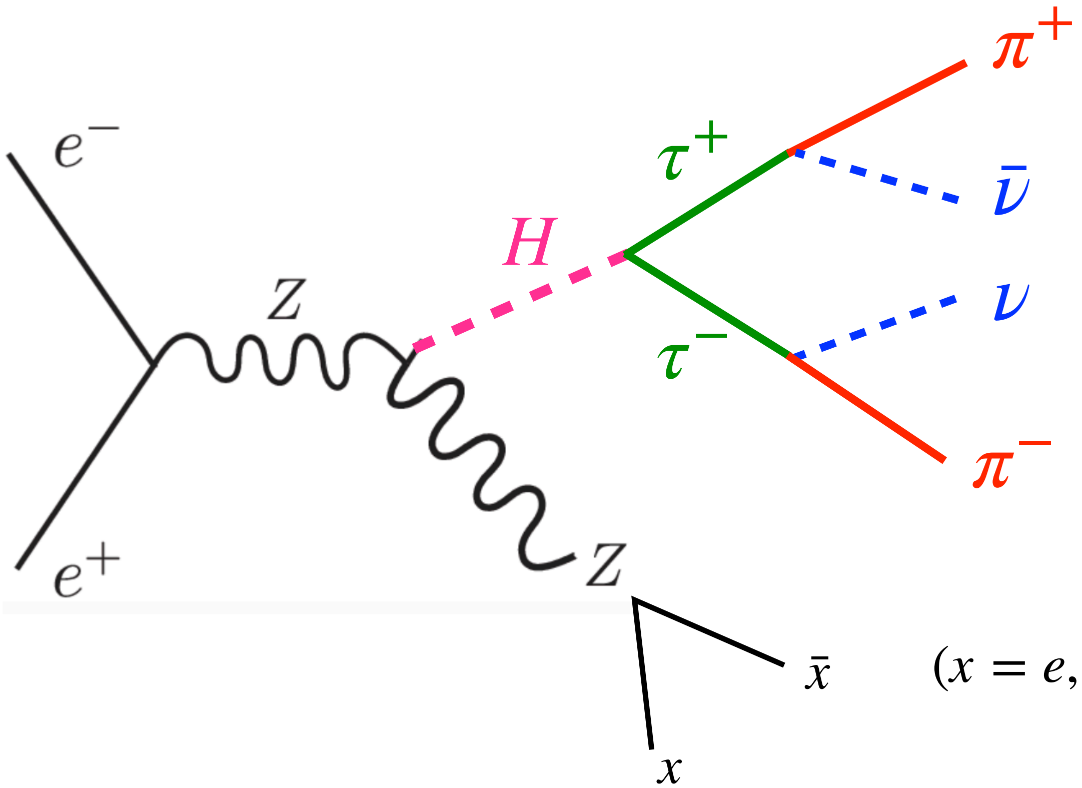
- For precise event reconstruction and for much smaller background, we consider lepton colliders.

LHC



ILC





$(x = e, \mu, j)$

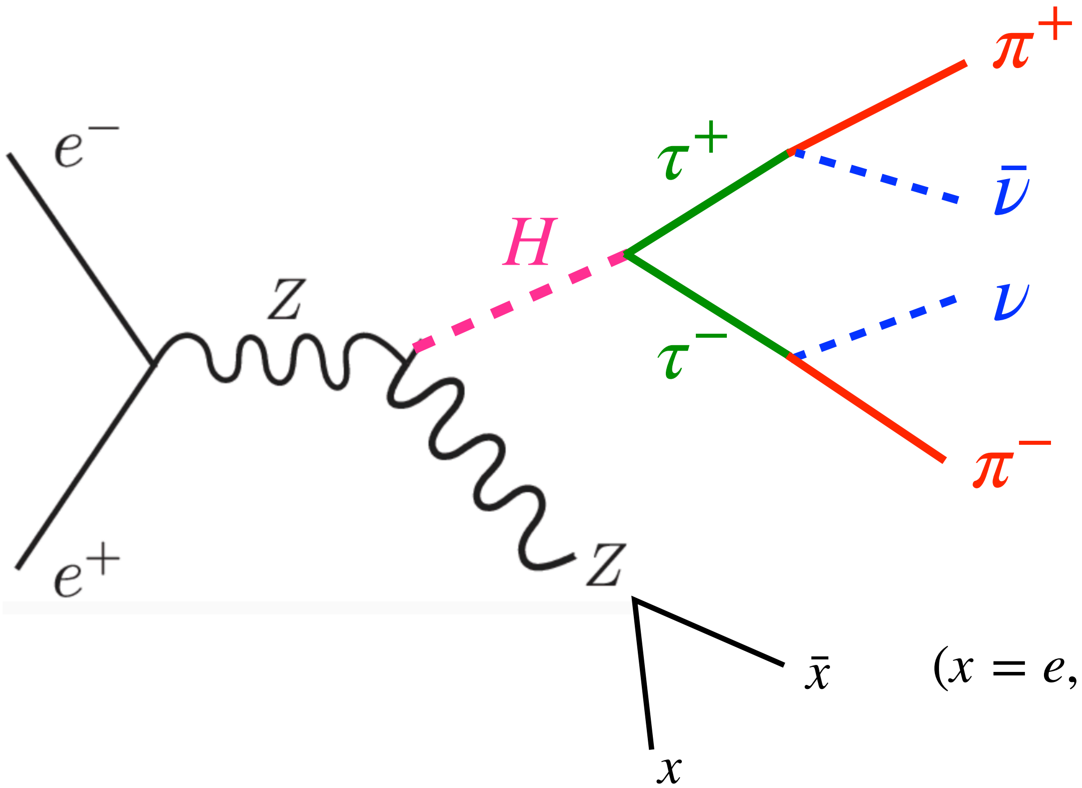
$$(P_H^{\text{reco}})^\mu \equiv P^\mu_{e^+e^-} - P^\mu_{Z\to x\bar{x}} \qquad M^2_{\text{recoil}} \equiv (P_H^{\text{reco}})^2$$

Event selection: $|M_{\text{recoil}} - 125\,\text{GeV}| < 5\,\text{GeV}$

$$e^+e^- \to Z + (Z^*/\gamma^*) \to f\bar{f} + \tau^+\tau^-$$

	ILC	FCC-ee
energy (GeV)	250	240
luminosity (ab^{-1})	3	5
beam resolution e^+ (%)	0.18	0.83×10^{-4}
beam resolution e^- (%)	0.27	0.83×10^{-4}
$\sigma(e^+e^- \to HZ)$ (fb)	240.1	240.3
# of signal $(\sigma \cdot \text{BR} \cdot L \cdot \epsilon)$	385	663
# of background $(\sigma \cdot \text{BR} \cdot L \cdot \epsilon)$	20	36

- Generate the SM events $(\kappa, \delta) = (1,0)$ with **MadGraph5.**
- **100 pseudo-experiments** to estimate the statistical uncertainties



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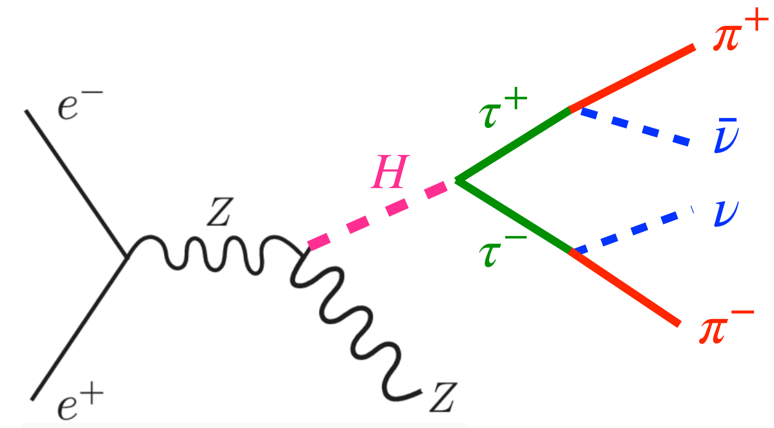
- Generate the SM events $(\kappa, \delta) = (1{,}0)$ with **MadGraph5.**
- **100 pseudo-experiments** to estimate the statistical uncertainties

- To determine the tau momenta, we have to reconstruct the unobserved neutrino momenta $(p_x^\nu, p_y^\nu, p_z^\nu)$, $(p_x^{\bar\nu}, p_y^{\bar\nu}, p_z^{\bar\nu})$.



- **6** unknowns can be constrained by **2** mass-shell conditions and **4** energy-momentum conservation.

$$m_\tau^2 = (p_{\tau^+})^2 = (p_{\pi^+} + p_{\bar\nu})^2$$

$$m_\tau^2 = (p_{\tau^-})^2 = (p_{\pi^-} + p_{\nu})^2$$

$$(p_{ee} - p_Z)^\mu = p_H^\mu = \left[(p_{\pi^-} + p_\nu) + (p_{\pi^+} + p_{\bar\nu})\right]^\mu$$

=> 2-fold solutions.

$$C_{ij}^{\rm SM} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

$$C_{\rm SM}[\rho] = 1$$

$$S_{\rm CHSH} = 2\sqrt{2}$$

$\Leftarrow$ **reproduced very accurately in the simulation**

→ we found that false solutions also give the correct correlations! (?)

Effect of momentum mismeasurement

$$E_i^{\rm true} \to E_i^{\rm obs} = (1 + \sigma_E \cdot \omega) \cdot E_i^{\rm true} \qquad \sigma_E = 0.03 \qquad (i = \pi^\pm, e^\pm, \mu^\pm, j)$$

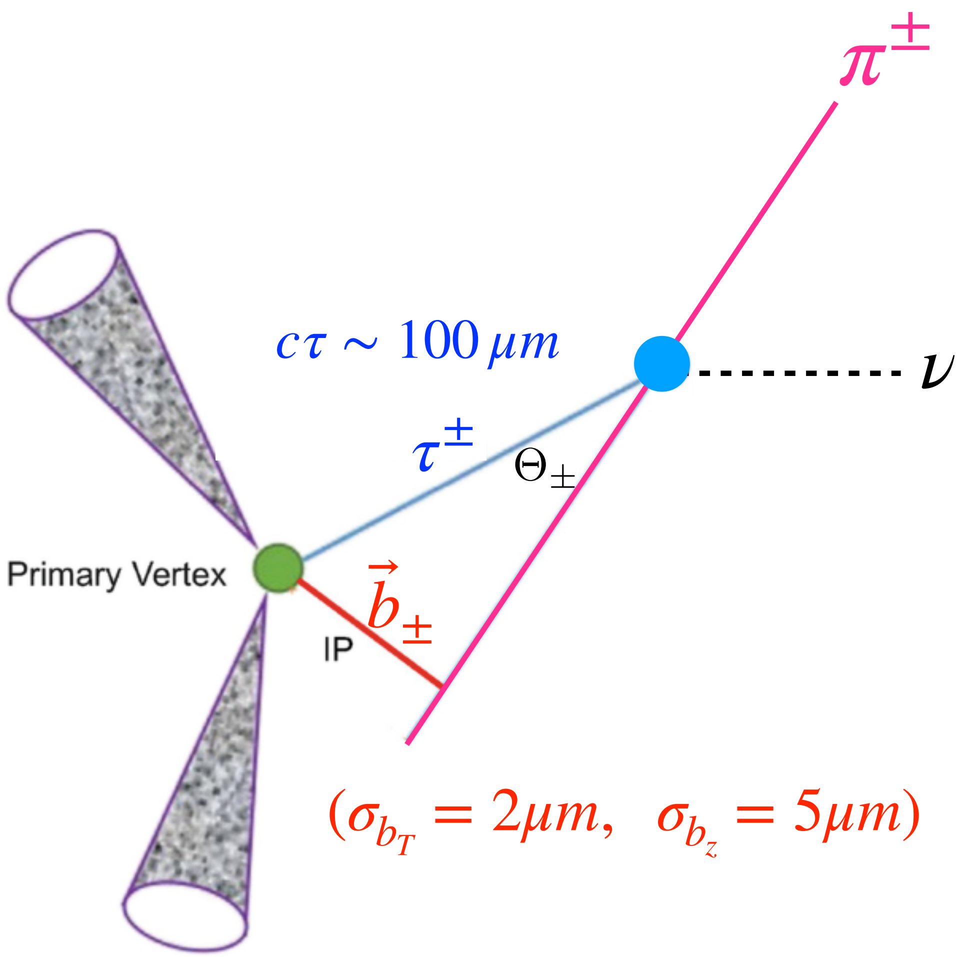
random number drawn from the normal distribution

	ILC	FCC-ee
C_{ij}	$\begin{pmatrix} -0.600 \pm 0.210 & 0.003 \pm 0.125 & 0.020 \pm 0.149 \\ 0.003 \pm 0.125 & -0.494 \pm 0.190 & 0.007 \pm 0.128 \\ 0.048 \pm 0.174 & 0.0007 \pm 0.156 & 0.487 \pm 0.193 \end{pmatrix}$	$\begin{pmatrix} -0.559 \pm 0.143 & -0.010 \pm 0.095 & -0.014 \pm 0.122 \\ -0.010 \pm 0.095 & -0.494 \pm 0.139 & -0.002 \pm 0.111 \\ 0.012 \pm 0.124 & 0.020 \pm 0.105 & 0.434 \pm 0.134 \end{pmatrix}$
$\mathcal{C}[\rho]$	0.030 ± 0.071	0.005 ± 0.023
$S_{\rm CHSH}/2$	0.769 ± 0.189	0.703 ± 0.134

$$C_{ij}^{\rm SM} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \qquad C_{\rm SM}[\rho] = 1 \qquad S_{\rm CHSH}^{\rm SM}/2 = \sqrt{2}$$

Momentum smearing spoils the previous good result…

Use impact parameter information



Goal:

$$E_i^{\rm true} \to E_i^{\rm obs} \to E_i^{\rm true} \quad (i = \pi^{\pm}, e^{\pm}, \mu^{\pm}, j)$$

What we do:

- modify $E_i^{\rm obs}$ for some amount by δ

$$E_i^{\rm obs} \to E_i(\delta_i) = (1 + \delta_i\,\sigma_E) \cdot E_i^{\rm obs}$$

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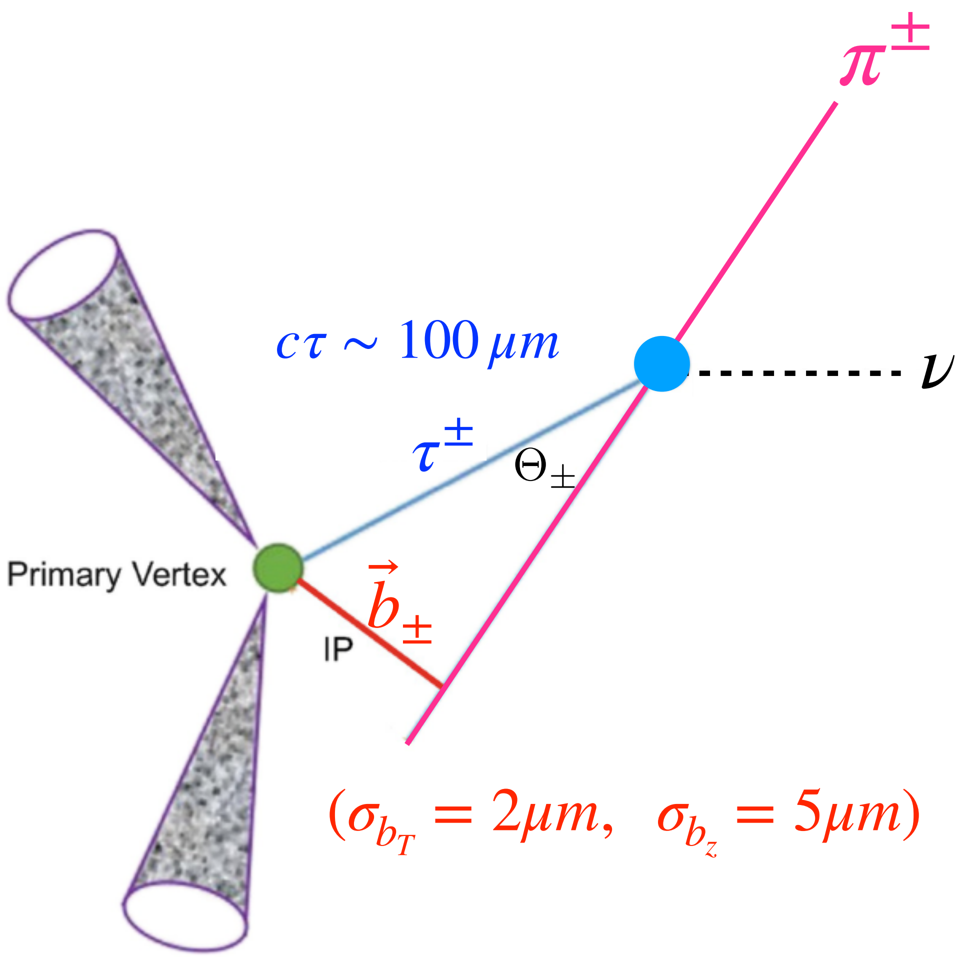

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→ lets us calculate $\vec{b}_{\pm}$ as functions of δ

$$\vec{b}_{\pm}^{\rm reco}\,(\mathbf{e}_{\tau^{\pm}}) \;=\; |\vec{b}_{\pm}| \cdot \left[\mathbf{e}_{\tau^{\pm}} \cdot \sin^{-1}\Theta_{\pm} \;-\; \mathbf{e}_{\pi^{\pm}} \cdot \tan^{-1}\Theta_{\pm}\right]$$

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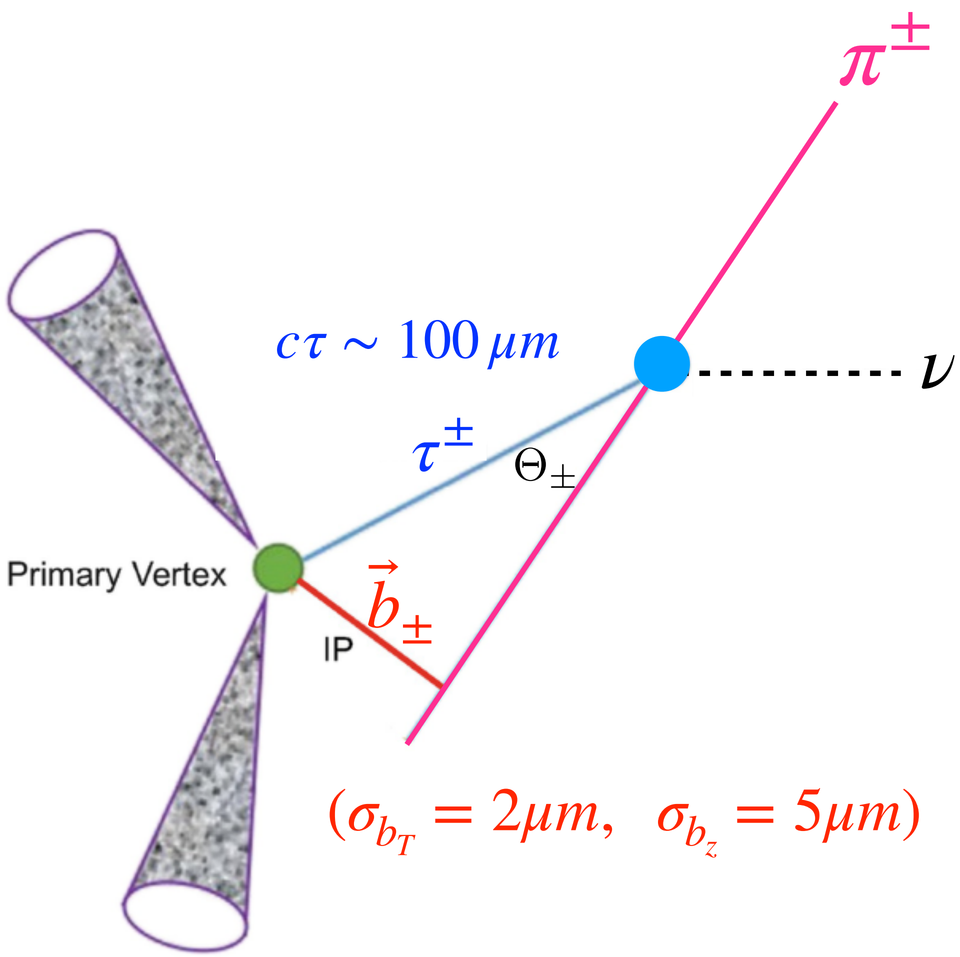

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What we do:

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→ lets us calculate $\vec{b}_{\pm}$ as functions of δ

$$\vec{b}_{\pm}^{\text{reco}}(\mathbf{e}_{\tau^{\pm}}) = |\vec{b}_{\pm}| \cdot \left[\mathbf{e}_{\tau^{\pm}} \cdot \sin^{-1}\Theta_{\pm} - \mathbf{e}_{\pi^{\pm}} \cdot \tan^{-1}\Theta_{\pm}\right]$$

- compare the calculated $\vec{b}_{\pm}^{\text{reco}}(\boldsymbol{\delta})$ and measured $\vec{b}_{\pm}^{\text{obs}}$ and construct the likelihood function

2 fold solutions: $i_s = 1,2$

$$\vec{\Delta}_{b_{\pm}}^{i_s}(\boldsymbol{\delta}) \equiv \vec{b}_{\pm} - \vec{b}_{\pm}^{\text{reco}}\left(\mathbf{e}_{\tau^{+}}^{i_s}(\boldsymbol{\delta})\right)$$

$$L_{\pm}^{i_s}(\boldsymbol{\delta}) = \frac{[\Delta_{b_{\pm}}^{i_s}(\boldsymbol{\delta})]_x^2 + [\Delta_{b_{\pm}}^{i_s}(\boldsymbol{\delta})]_y^2}{\sigma_{b_T}^2} + \frac{[\Delta_{b_{\pm}}^{i_s}(\boldsymbol{\delta})]_z^2}{\sigma_{b_z}^2} + \delta_{\pi^+}^2 + \delta_{\pi^-}^2 + \delta_x^2 + \delta_{\bar{x}}^2 .$$

minimizing $L^{i_s}(\boldsymbol{\delta})$ would give us the correct set of $\boldsymbol{\delta}s$ and solution i_s

Result

2211.10513

	ILC	FCC-ee
C_{ij}	$\begin{pmatrix} 0.830 \pm 0.176 & 0.020 \pm 0.146 & -0.019 \pm 0.159 \\ -0.034 \pm 0.160 & 0.981 \pm 0.1527 & -0.029 \pm 0.156 \\ -0.001 \pm 0.158 & -0.021 \pm 0.155 & -0.729 \pm 0.140 \end{pmatrix}$	$\begin{pmatrix} 0.925 \pm 0.109 & -0.011 \pm 0.110 & 0.038 \pm 0.095 \\ -0.009 \pm 0.110 & 0.929 \pm 0.113 & 0.001 \pm 0.115 \\ -0.026 \pm 0.122 & -0.019 \pm 0.110 & -0.879 \pm 0.098 \end{pmatrix}$
$\mathcal{C}[\rho]$	0.778 ± 0.126	0.871 ± 0.084
$S_{\rm CHSH}/2$	1.103 ± 0.163	1.276 ± 0.094

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$S_{\rm CHSH}/2$	1.103 ± 0.163	1.276 ± 0.094 $\sim 3\sigma$

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Superiority of FCC-ee over ILC is due to a better beam resolution

	ILC	FCC-ee
energy (GeV)	250	240
luminosity (ab^{-1})	3	5
beam resolution e^+ (%)	0.18	$0.83 \cdot 10^{-4}$
beam resolution e^- (%)	0.27	$0.83 \cdot 10^{-4}$

CP measurement

- Under CP, the spin correlation matrix transforms: $C \xrightarrow{CP} C^T$
- This can be used for a *model-independent* test of CP violation. We define:

$$A \equiv (C_{rn} - C_{nr})^2 + (C_{nk} - C_{kn})^2 + (C_{kr} - C_{rk})^2 \;\; \geq 0$$

- Observation of $A \neq 0$ immediately confirms CP violation.

- From our simulation, we observe

$$A = \begin{cases} 0.204 \pm 0.173 & \text{(ILC)} \\ 0.112 \pm 0.085 & \text{(FCC-ee)} \end{cases} \longleftarrow$$

consistent with absence of CPV

- This model independent bounds can be translated to the constraint on the CP-phase δ

$$\mathscr{L}_{\rm int} \propto H \bar{\psi}_\tau (\cos\delta + i\gamma_5 \sin\delta)\,\psi_\tau \;\Rightarrow\; C_{ij} = \begin{pmatrix} \cos 2\delta & \sin 2\delta & 0 \\ -\sin 2\delta & \cos 2\delta & 0 \\ 0 & 0 & -1 \end{pmatrix} \;\Rightarrow\; A(\delta) = 4\sin^2 2\delta$$

CP measurement

- Focusing on the region near $|\delta| = 0$, we find the 1-σ bounds:

$$|\delta| < \begin{cases} 8.9^o & (\text{ILC}) \\ 6.4^o & (\text{FCC-ee}) \end{cases}$$

- Other studies:

$$\Delta\delta \sim 11.5^o \quad (\text{HL-LHC})$$ [Hagiwara, Ma, Mori 2016]

$$\Delta\delta \sim 4.3^o \quad (\text{ILC})$$ [Jeans and G. W. Wilson 2018]

Summary

- High energy tests of entanglement and Bell inequality has recently attracted an attention.
- $\tau^+\tau^-$ pairs from $H \to \tau^+\tau^-$ form the EPR triplet state $|\Psi^{(1,0)}\rangle = \dfrac{|+,-\rangle + |-,+\rangle}{\sqrt{2}}$, and maximally entangled.
- We investigated feasibility of quantum property tests @ ILC and FCC-ee.
- Quantum test requires to a precise reconstruction of the tau rest frames and IP information is crucial to achieve this.
- Spin correlation is sensitive to CP-phase and we can measure the CP-phase as a byproduct of the quantum property measurement.

	Entanglement	Steering	Bell-inquality	CP-phase
ILC	$\sim 5\sigma$	$\sim 3\sigma$		8.9^o
FCC-ee	$\gg 5\sigma$	$\sim 5\sigma$	$\sim 3\sigma$	6.4^o

So far, the literature focuses on **two**-particle entanglement

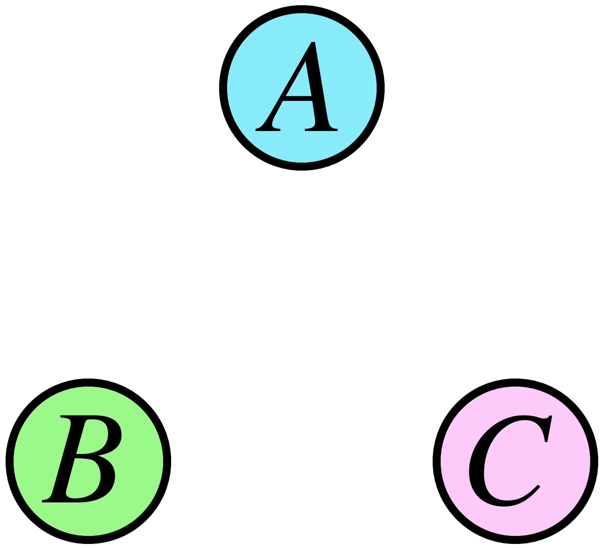
↓

what about **three**-particle entanglement?

✦ KS, M. Spannowsky [2310.01477]

3-Particle Entanglement

3-particle entanglement has a much richer structure that 2-PE !



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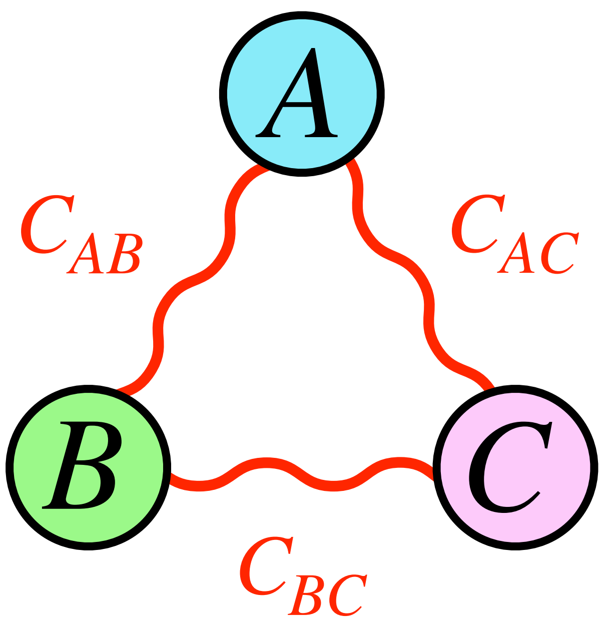

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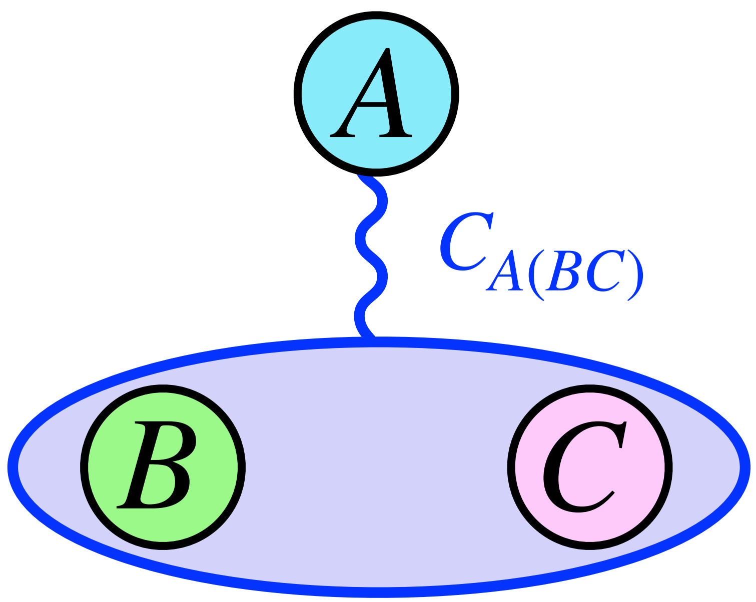

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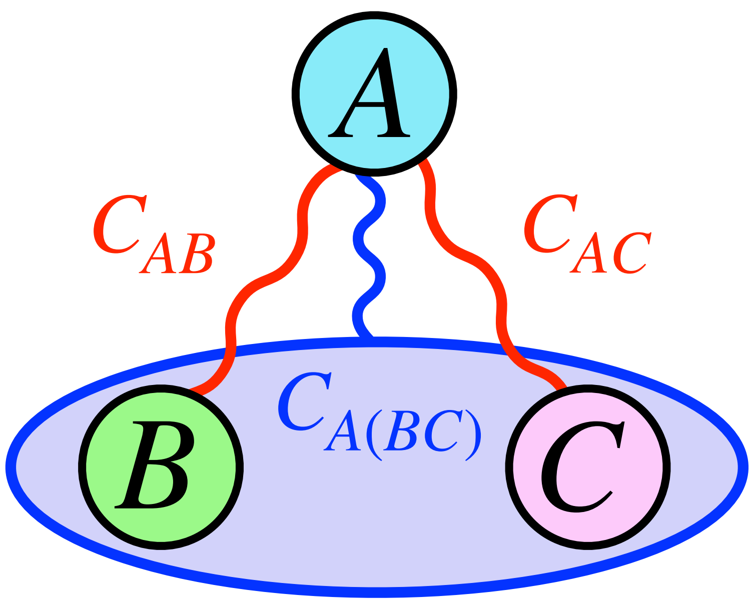

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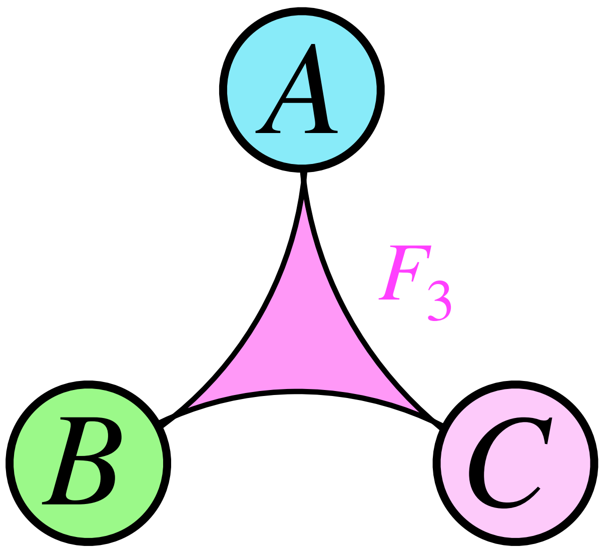

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  (non-separable even partially)

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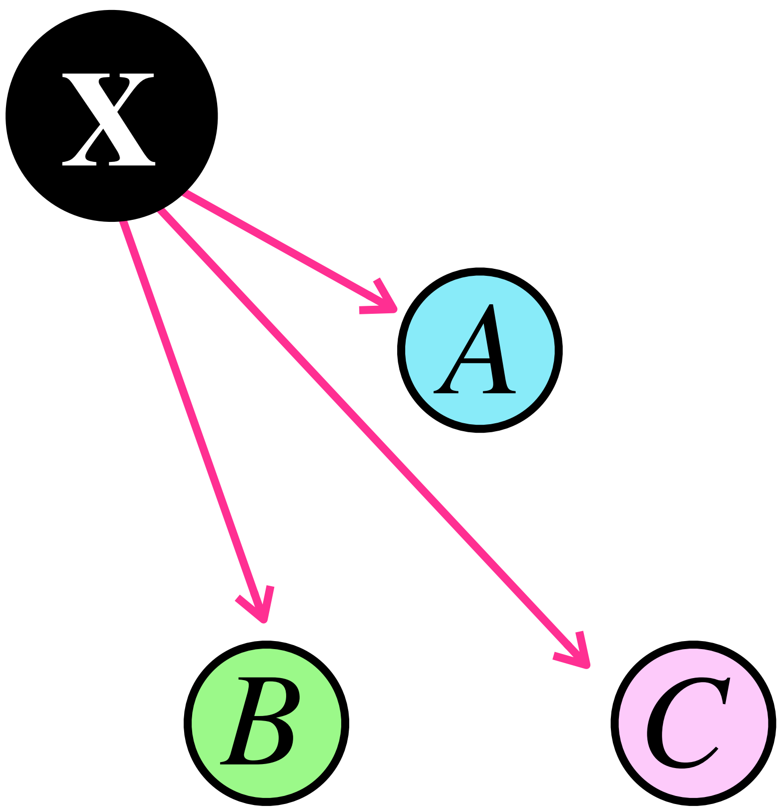

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3-body decay: $X \to ABC$

explore all possible Lorentz invariant interactions

How to quantify entanglement?

Ex.) **Concurrence** [for 2 qubit system]

$$\mathcal{C}[\rho] = \max(0, \eta_1 - \eta_2 - \eta_3 - \eta_4) \in [0, 1]$$

$\eta_1 \geq \eta_2 \geq \eta_3 \geq \eta_4$ are eigenvalues of $\sqrt{\rho\tilde{\rho}}$ with $\tilde{\rho} \equiv (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$.

density matrix $\longrightarrow$ $\rho \equiv \sum_i p_i |\Psi_i\rangle\langle\Psi_i|$

$(p_i \geq 0, \sum_i p_i = 1)$

$$\mathcal{C}[\rho] \begin{cases} = 0 & \longleftarrow \textbf{not}\text{-entangled} \\ > 0 & \longleftarrow \textbf{entangled} \end{cases}$$

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How to compute the entanglement btw. 2-individual qubits?

$$|\Psi\rangle = \sum_{a,b,c} c_{abc} \cdot |a\rangle_A \otimes |b\rangle_B \otimes |c\rangle_C$$

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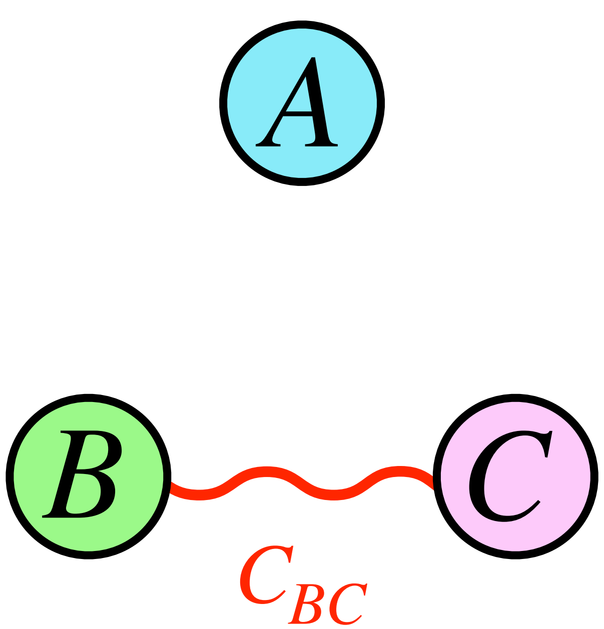

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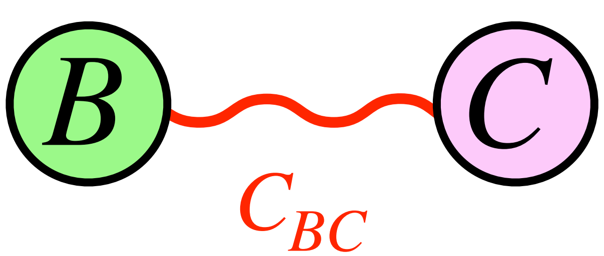
trace out A

$$\Rightarrow \quad \rho_{BC} = \mathrm{Tr}_A |\Psi\rangle\langle\Psi|$$



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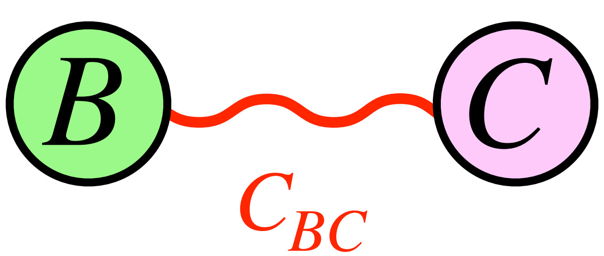

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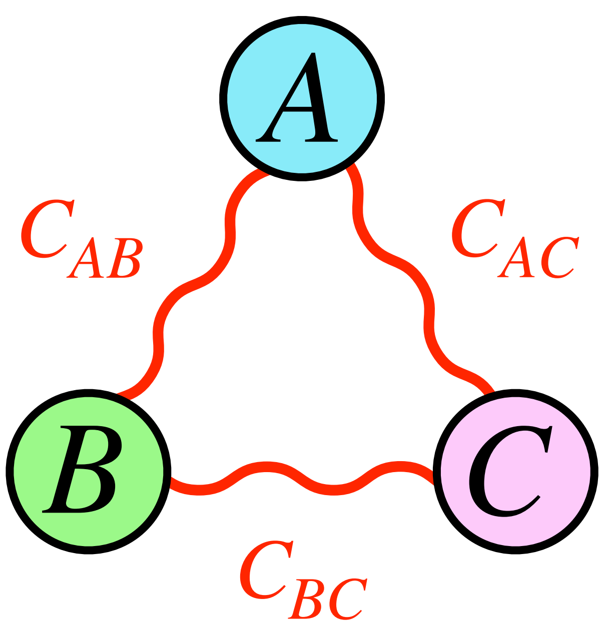
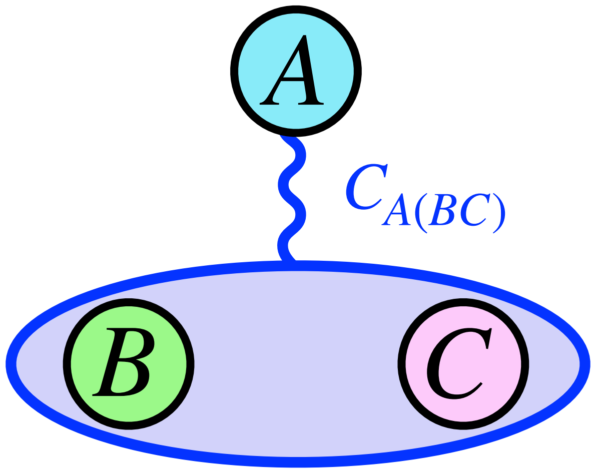


C_{AB}, C_{BC}, C_{AC}

How to calculate entanglement btw A and (BC) ?



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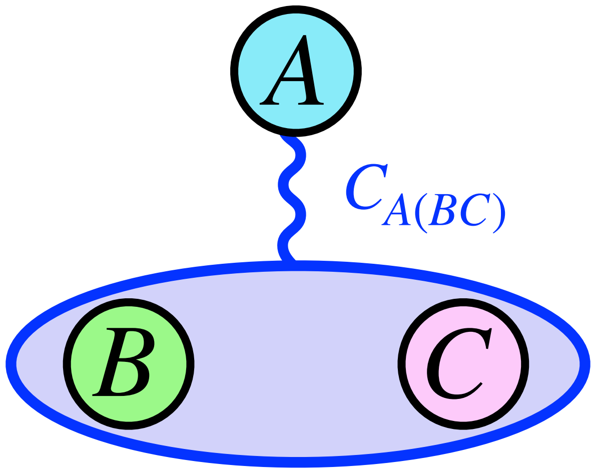

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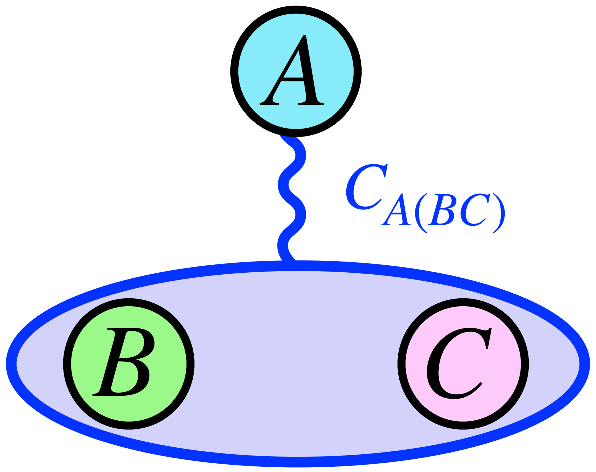

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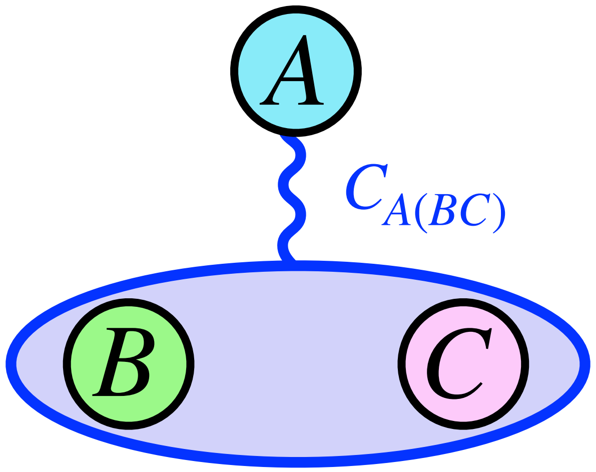

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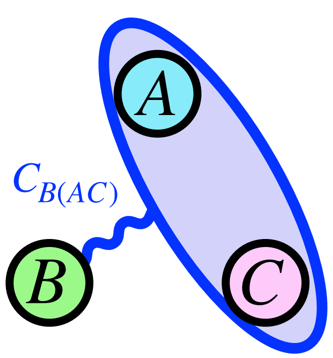
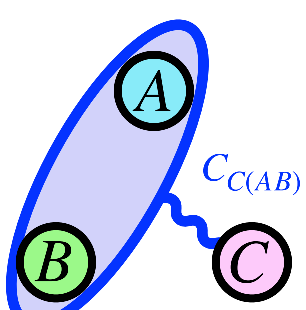


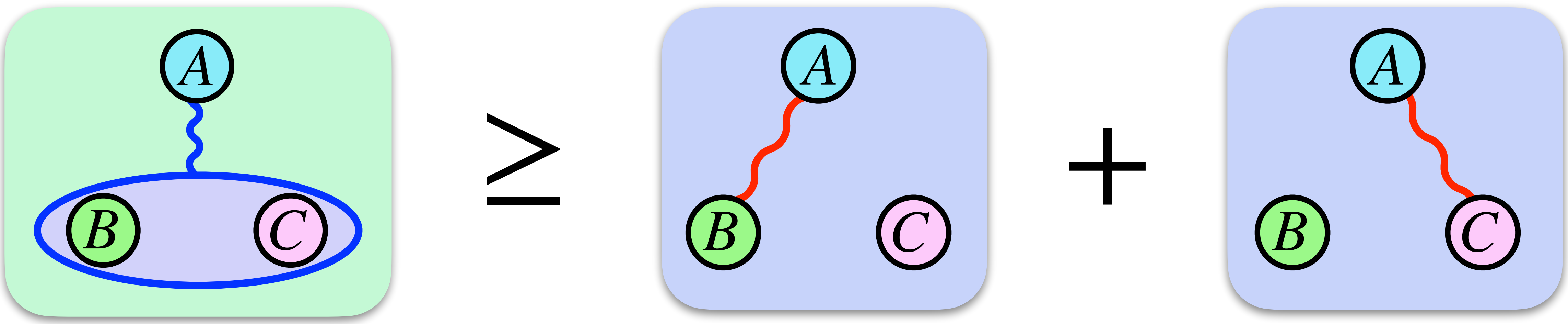
$$\mathcal{C}_{B(AC)} \equiv \mathcal{C}[|\Psi\rangle] = \sqrt{2(1 - \mathrm{Tr}\rho^2_{AC})} \qquad \rho_{AC} \equiv \mathrm{Tr}_B |\Psi\rangle\langle\Psi|$$



$$\mathcal{C}_{C(AB)} \equiv \mathcal{C}[|\Psi\rangle] = \sqrt{2(1 - \mathrm{Tr}\rho^2_{AB})} \qquad \rho_{AB} \equiv \mathrm{Tr}_C |\Psi\rangle\langle\Psi|$$

Monogamy

- **A**-(**BC**) entanglement limits **A**-**B** and **A**-**C** entanglements



[Coffman, Kundu, Wootters '99]

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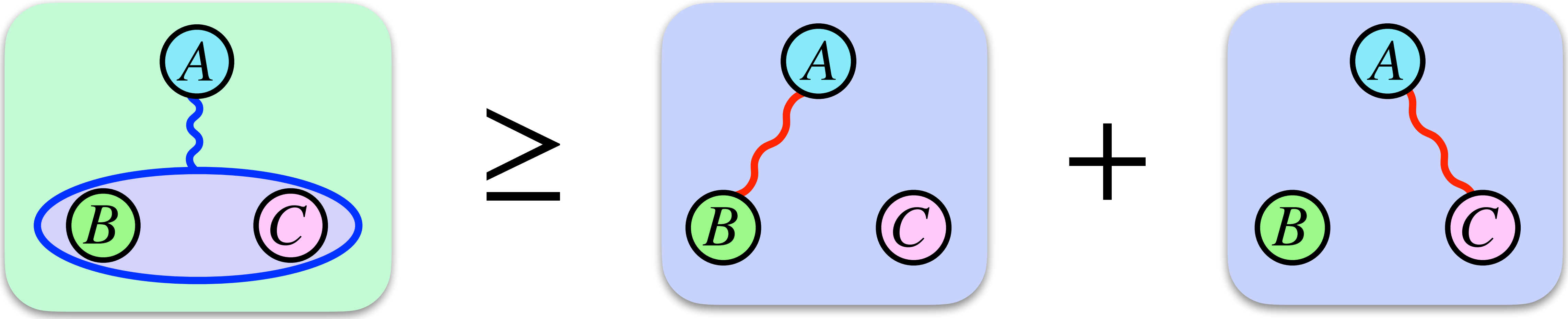

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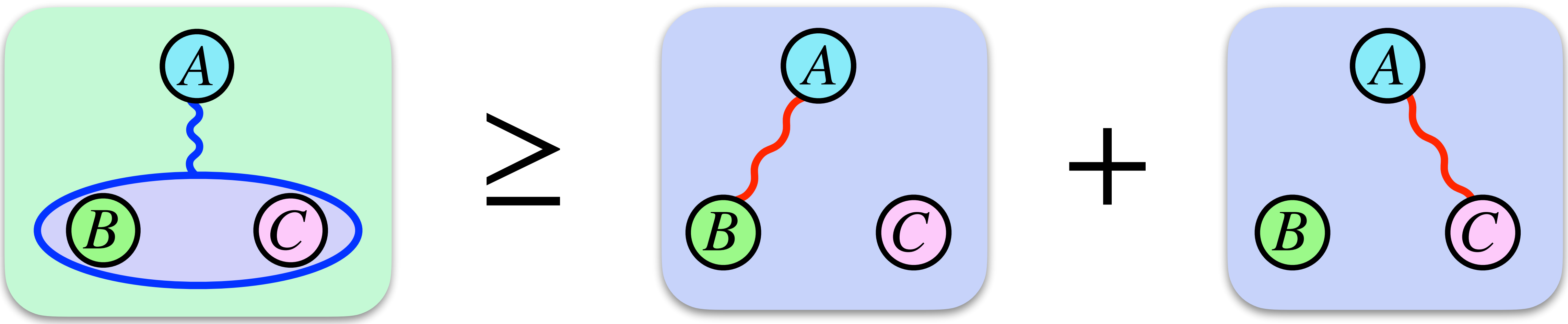

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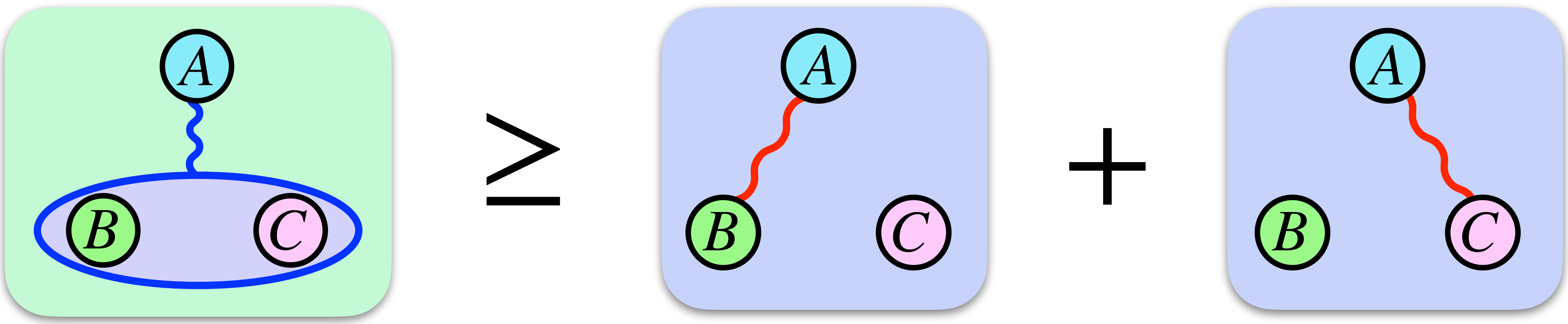

- Coffman-Kundu-Wootters (CKW) monogamy inequality [Coffman, Kundu, Wootters '99]

$$C^2_{A(BC)} \geq C^2_{AB} + C^2_{AC}$$

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Monogamy

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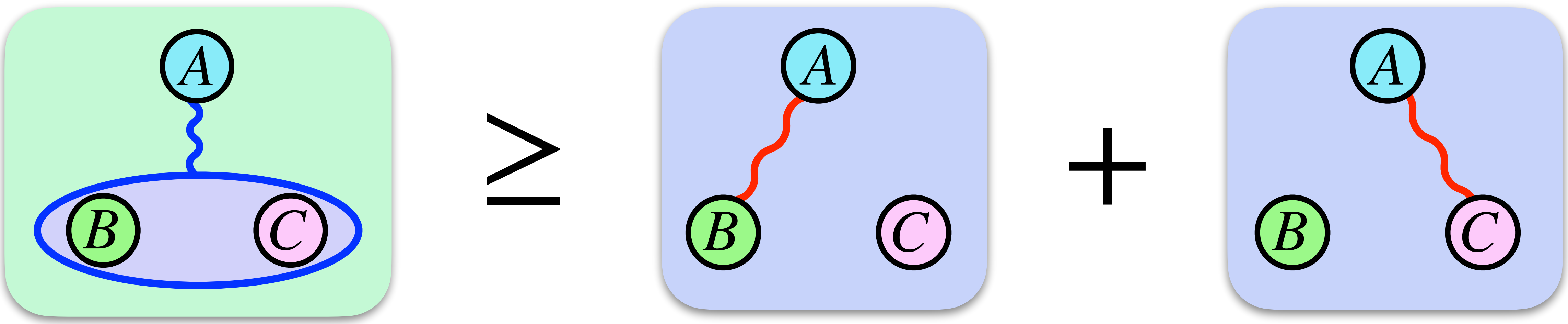

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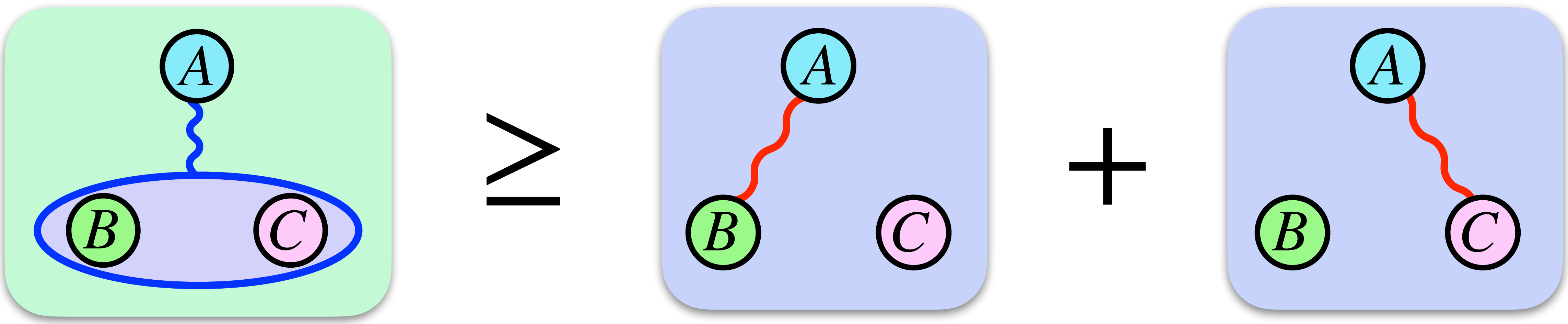

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$$\Downarrow$$

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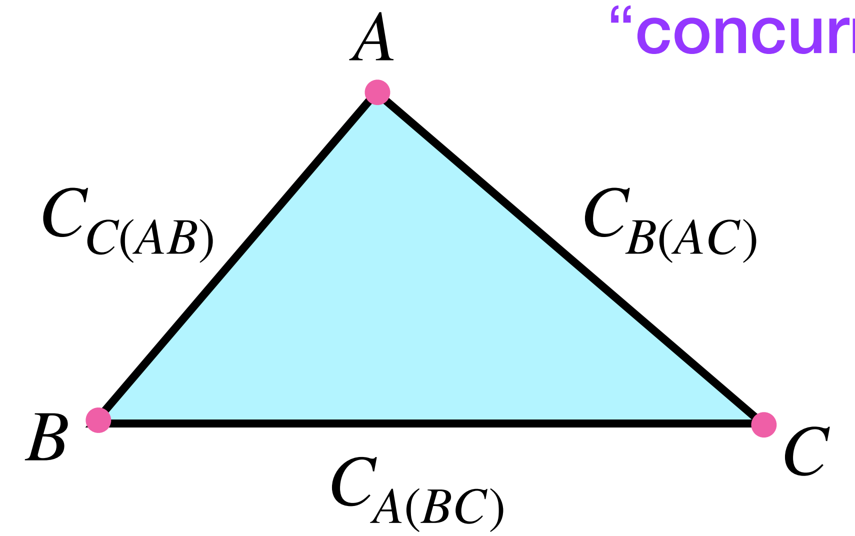
$\Rightarrow$



"concurrence triangle"

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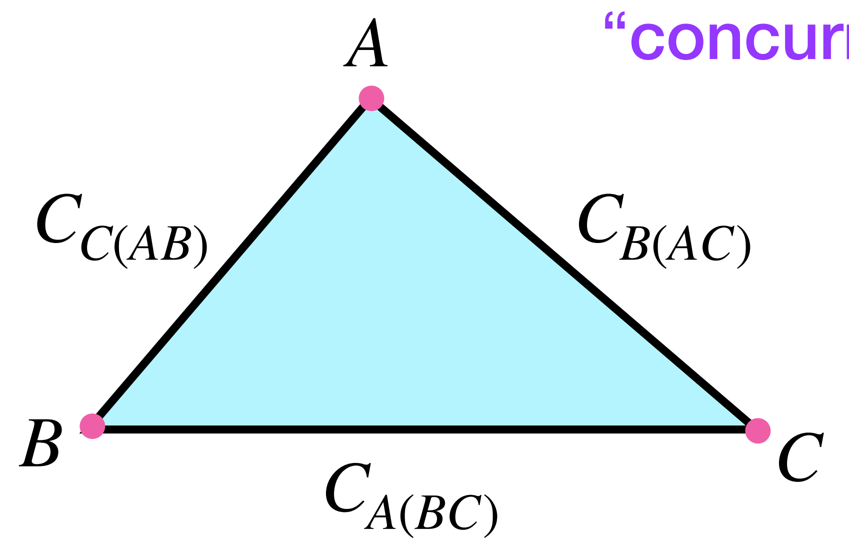
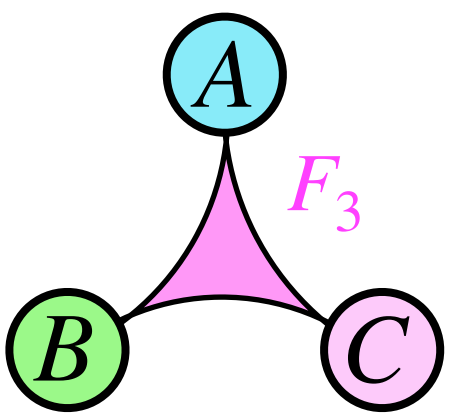
$$\boxed{C_{A(BC)} + C_{B(AC)} \geq C_{C(AB)}} \quad \Rightarrow$$



Genuine Multi-particle Entanglement (GME) measure: [Dur, Vidal, Cirac '00, Ma, Chen, Chen, Spengler, Gabriel, Huber '11, Xie, Eberly '21]



GME should satisfy the following properties:

(1) vanish for all product and biseparable states $\Rightarrow$ unseparable even partially

(2) positive for all non-biseparable states

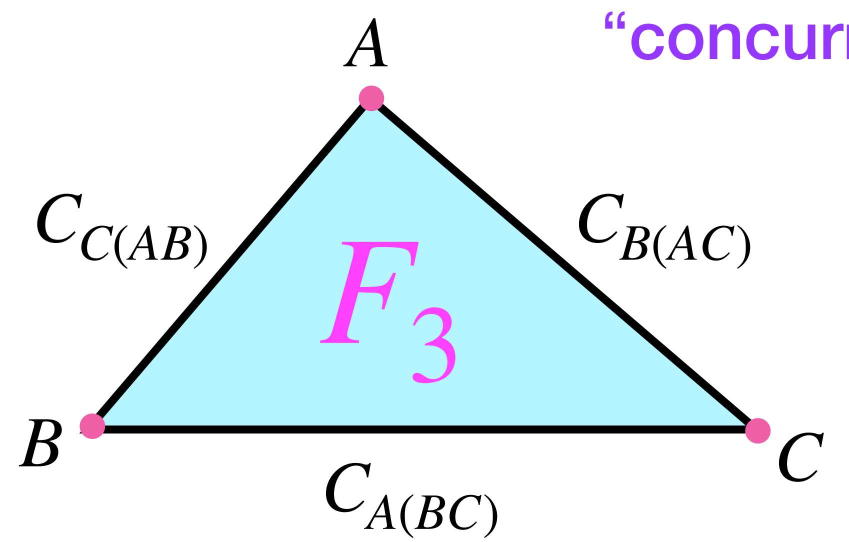
(3) not increase under LOCC

$$|\psi\rangle_A \otimes (|00\rangle_{BC} + |11\rangle_{BC})$$

$$\Rightarrow F_3 = 0$$

"concurrence triangle"

$$C_{\mathbf{A}(\mathbf{BC})} + C_{\mathbf{B}(\mathbf{AC})} \geq C_{\mathbf{C}(\mathbf{AB})} \quad \Rightarrow$$



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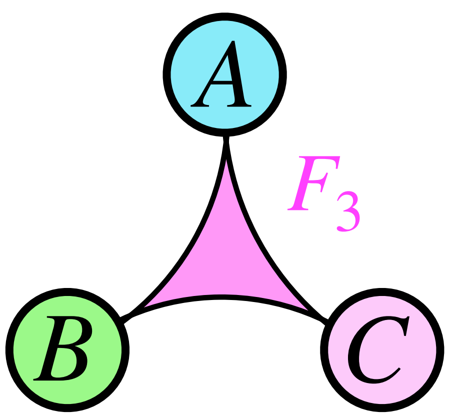

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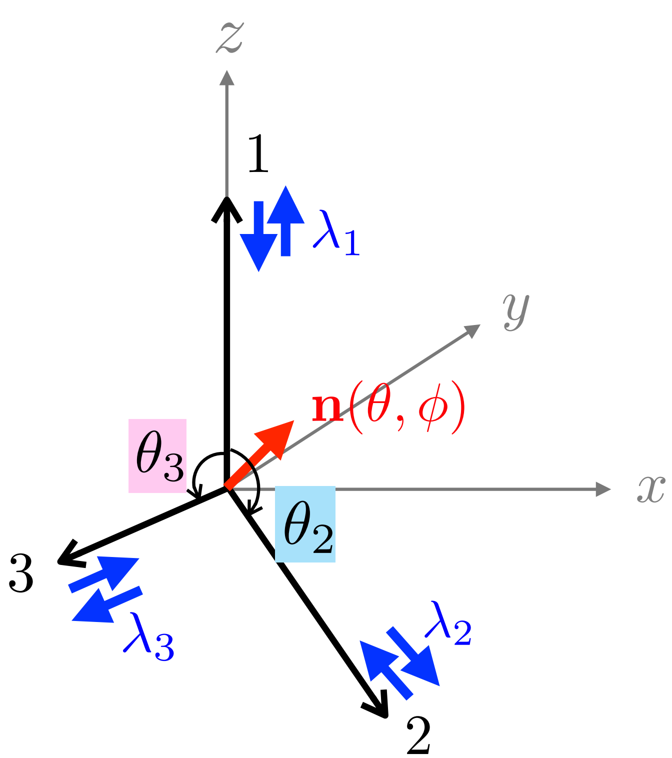
The **area** of the "**concurrence triangle**" satisfies (1), (2), (3) ! [Jin, Tao, Gui, Fei, Li-Jost, Qiao (2023)]

$$F_3 \equiv \left[\tfrac{16}{3} Q (Q - \mathcal{C}_{A(BC)})(Q - \mathcal{C}_{B(AC)})(Q - \mathcal{C}_{C(AB)})\right]^{\frac{1}{2}} \in [0, 1]$$

$$Q \equiv \frac{1}{2}\left[\mathcal{C}_{A(BC)} + \mathcal{C}_{B(AC)} + \mathcal{C}_{C(AB)}\right]$$

3-body decay: $\psi_0 \to \psi_1 \bar{\psi}_2 \psi_3$

[KS, M.Spannowsky 2310.01477]



Assumptions:

- all particles have spin 1/2
- all final particles 1,2,3 are massless

Kinematics:

- rest frame of the initial particle 0
- p_1 is in the z-axis
- decay is in the x-z plane

$$p_1^\mu = p_1(1, 0, 0, 1)$$
$$p_2^\mu = p_2(1, \sin\theta_2, 0, \cos\theta_2)$$
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$\mathbf{n}(\theta, \phi)$: polarisation of initial spin

$\lambda_1, \lambda_2, \lambda_3 \in (+, -)$: helicities of 1,2,3

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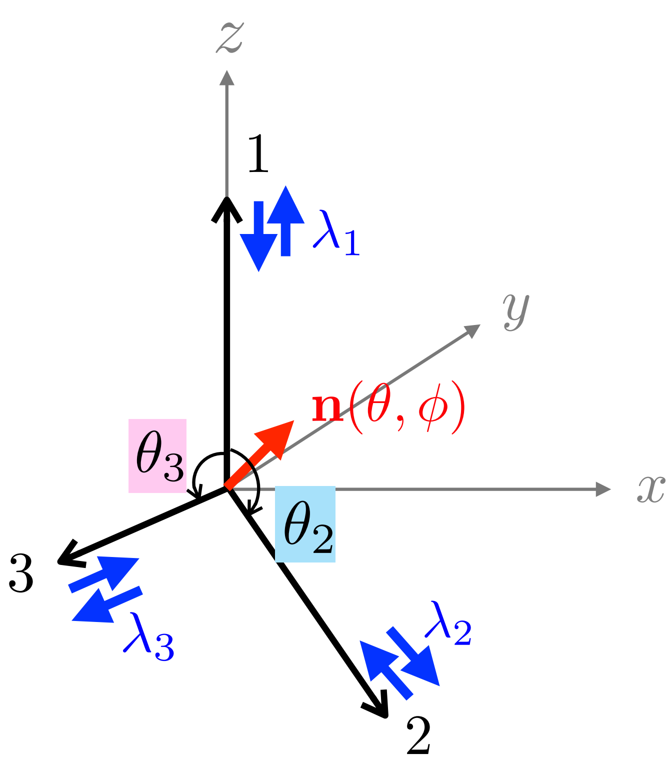

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initial state

$$|\mathbf{n}(\theta, \phi)\rangle$$

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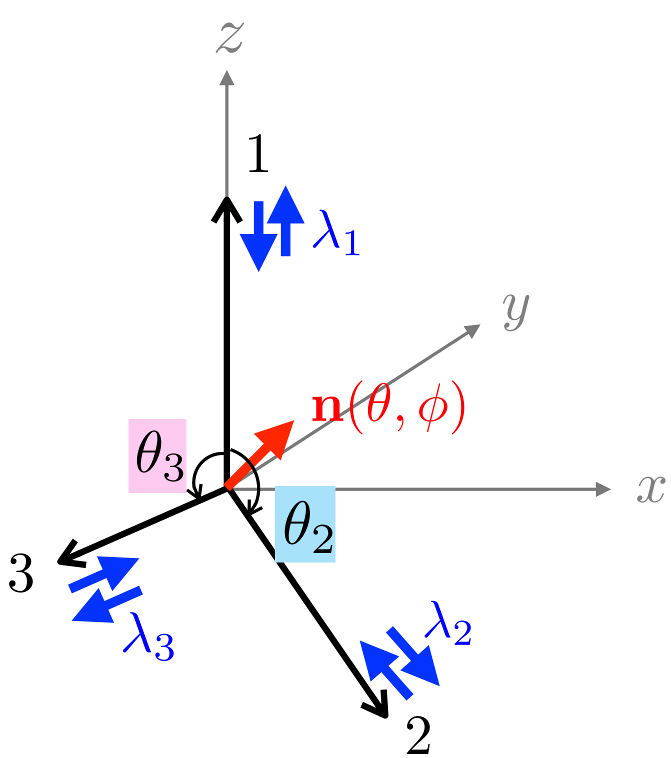
[KS, M.Spannowsky 2310.01477]



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amplitude

$$\hat{1} = \sum_{\lambda_1,\lambda_2,\lambda_3} |\lambda_1, \lambda_2, \lambda_3\rangle\langle\lambda_1, \lambda_2, \lambda_3| \qquad \mathcal{M}^{\mathbf{n}}_{\lambda_1,\lambda_2,\lambda_3} = \langle\lambda_1, \lambda_2, \lambda_3|\mathbf{n}(\theta, \phi)\rangle$$

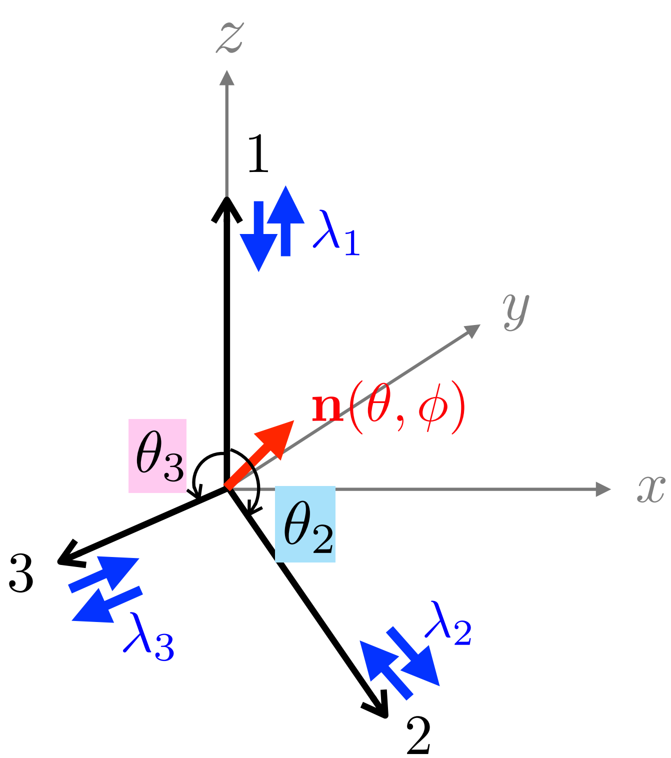
initial state

$$|\mathbf{n}(\theta, \phi)\rangle = \sum_{\lambda_1,\lambda_2,\lambda_3} \mathcal{M}^{\mathbf{n}}_{\lambda_1,\lambda_2,\lambda_3} |\lambda_1, \lambda_2, \lambda_3\rangle + \cdots$$

final state

3-body decay: $\psi_0 \to \psi_1 \bar{\psi}_2 \psi_3$

[KS, M.Spannowsky 2310.01477]



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$$|\mathbf{n}(\theta,\phi)\rangle = \boxed{\sum_{\lambda_1,\lambda_2,\lambda_3} \mathcal{M}^{\mathbf{n}}_{\lambda_1,\lambda_2,\lambda_3}|\lambda_1,\lambda_2,\lambda_3\rangle} = |\Psi\rangle \leftarrow$$

pure (entangled) **3-spin state**

Interaction

- Consider **most general** Lorentz invariant 4-fermion interactions

$$\mathcal{L}_{\rm int} = (\bar{\psi}_1 \Gamma_A \psi_0)(\bar{\psi}_3 \Gamma_B \psi_2)$$

$$\Gamma_{A/B} \in \{1, \gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, \sigma^{\mu\nu}\}$$

$$\psi_0 \to \psi_1 \bar{\psi}_2 \psi_3$$

❖ Scalar-type

$$[\bar{\psi}_1(c_S + i c_A \gamma_5)\psi_0][\bar{\psi}_3(d_S + i d_A \gamma_5)\psi_2]$$

$$c \equiv c_S + i c_A = e^{i\delta_1}$$
$$d \equiv d_S + i d_A = e^{i\delta_2}$$

❖ Vector-type

$$[\bar{\psi}_1 \gamma_\mu (c_L P_L + c_R P_R)\psi_0][\bar{\psi}_3 \gamma^\mu (d_L P_L + d_R P_R)\psi_2]$$

$$P_{R/L} = \frac{1 \pm \gamma^5}{2}$$
$$c_L, c_R, d_L, d_R \in \mathbb{R}$$

❖ Tensor-type

$$[\bar{\psi}_1(c_M + i c_E \gamma_5)\sigma^{\mu\nu}\psi_0][\bar{\psi}_3(d_M + i d_E \gamma_5)\sigma_{\mu\nu}\psi_2]$$

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[KS, M.Spannowsky 2310.01477]

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[KS, M.Spannowsky 2310.01477]

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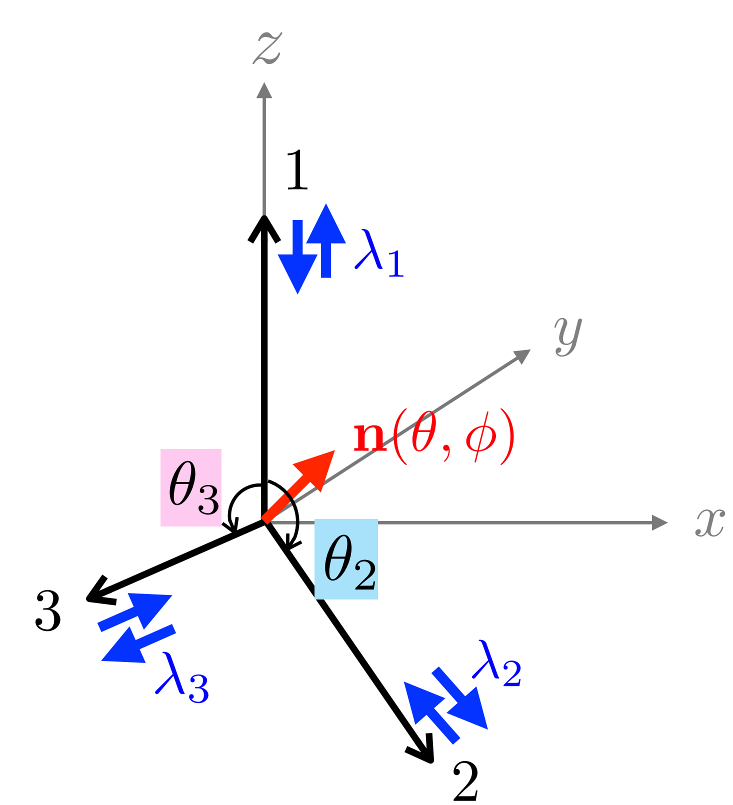
independent of final state momenta θ_2, θ_3



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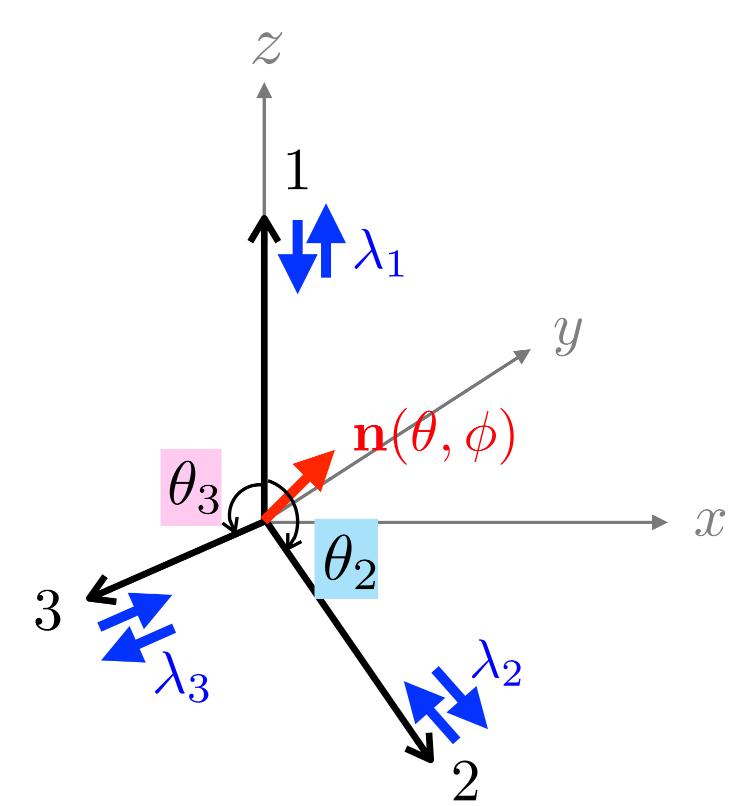
bi-separable



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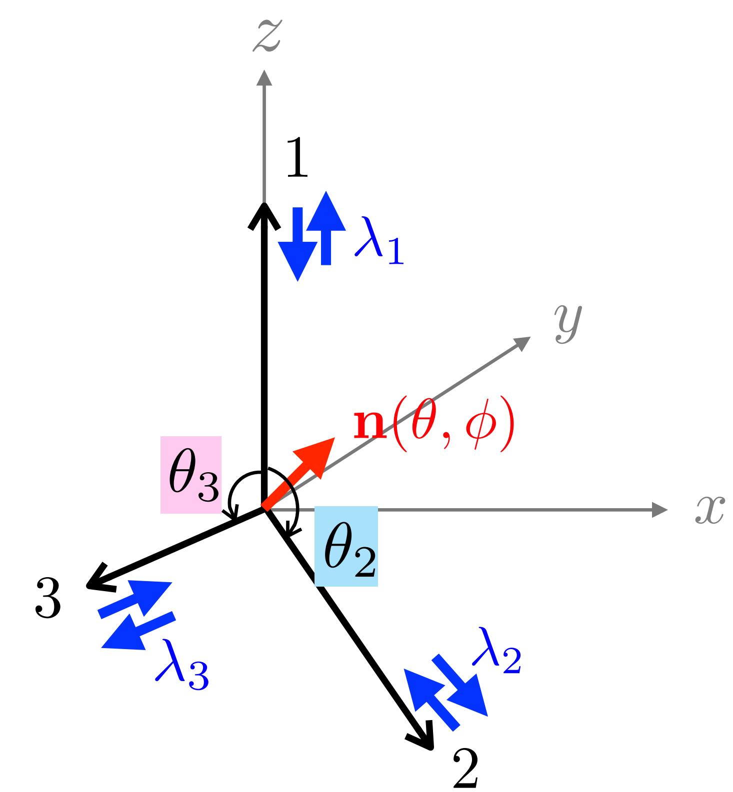

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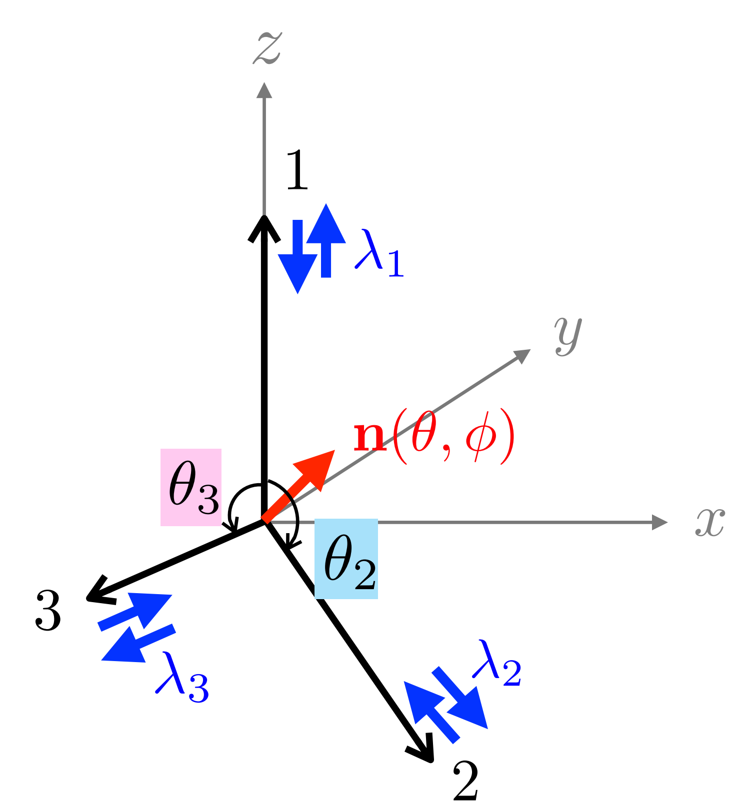
$$\mathcal{C}_{12} = \mathcal{C}_{13} = \mathcal{C}_{1(23)} = 0$$



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independent of final state momenta θ_2, θ_3

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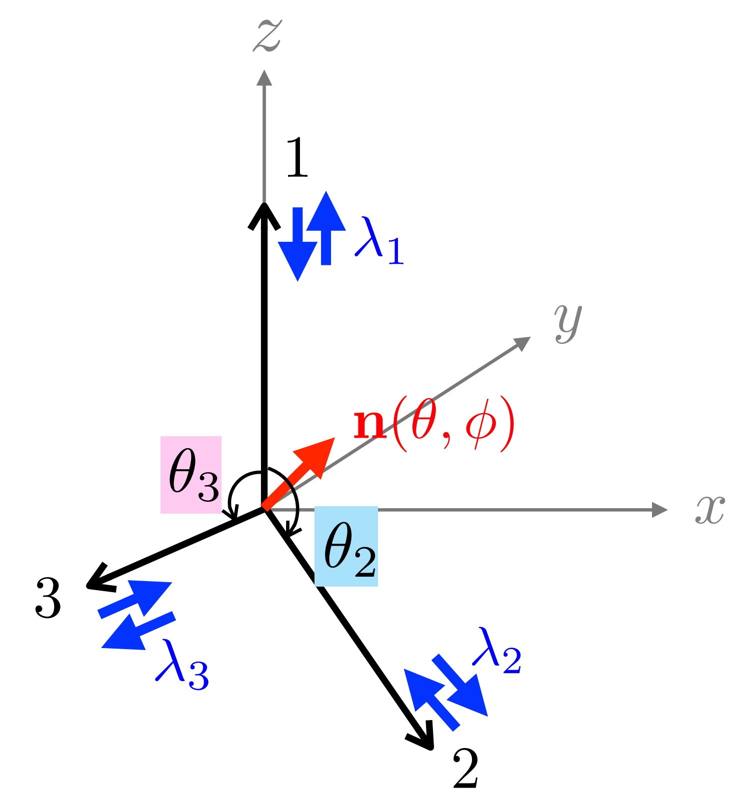

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✤ Due to **monogamy**, **2** and **3** are **maximally entangled** with the rest

$$\mathcal{C}_{2(13)} = \mathcal{C}_{3(12)} = 1$$

Monogamy

$$\mathcal{C}^2_{2(13)} \geq \underbrace{\mathcal{C}^2_{12}}_{=0} + \underbrace{\mathcal{C}^2_{23}}_{=1}$$

$$\mathcal{C}^2_{3(12)} \geq \underbrace{\mathcal{C}^2_{13}}_{=0} + \underbrace{\mathcal{C}^2_{23}}_{=1}$$

[KS, M.Spannowsky
2310.01477]

Vector

$$P_{R/L} = \frac{1 \pm \gamma^5}{2}$$

$$\mathcal{L}_{\text{int}} = [\bar{\psi}_1 \gamma_\mu (c_L P_L + c_R P_R) \psi_0][\bar{\psi}_3 \gamma^\mu (d_L P_L + d_R P_R) \psi_2] \qquad c_L, c_R, d_L, d_R \in \mathbb{R}$$

[KS, M.Spannowsky 2310.01477]

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[KS, M.Spannowsky 2310.01477]

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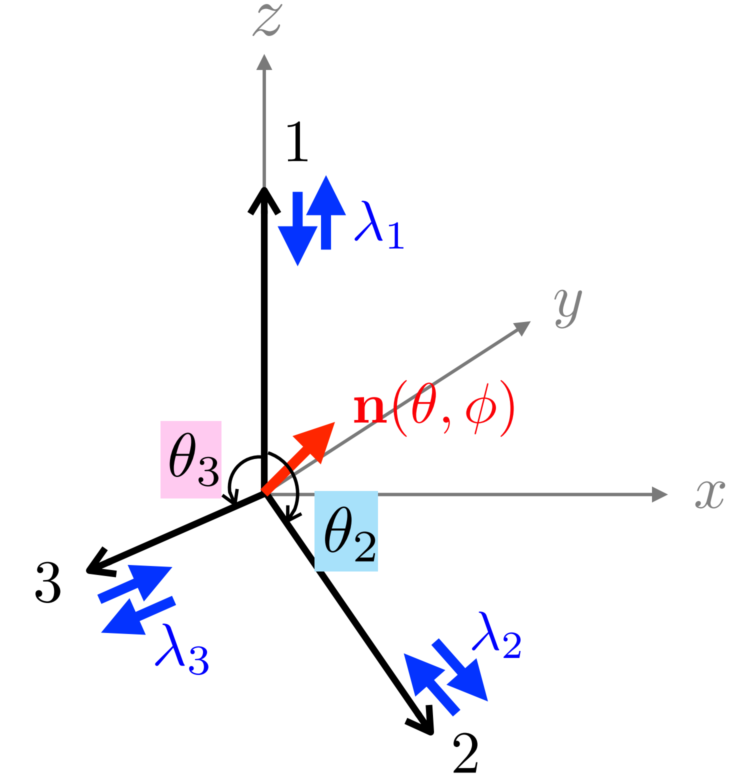
$$+ \; c_R d_L s_{\frac{\theta_2}{2}} \left[c_{\frac{\theta}{2}} s_{\frac{\theta_3}{2}} - e^{i\phi} s_{\frac{\theta}{2}} c_{\frac{\theta_3}{2}}\right] |++-\rangle \; + \; c_R d_R s_{\frac{\theta_3}{2}} \left[c_{\frac{\theta}{2}} s_{\frac{\theta_2}{2}} - e^{i\phi} s_{\frac{\theta}{2}} c_{\frac{\theta_2}{2}}\right] |+-+\rangle$$



[KS, M.Spannowsky 2310.01477]

Vector

$$P_{R/L} = \frac{1 \pm \gamma^5}{2}$$

$$\mathcal{L}_{\rm int} = [\bar{\psi}_1 \gamma_\mu (c_L P_L + c_R P_R) \psi_0][\bar{\psi}_3 \gamma^\mu (d_L P_L + d_R P_R) \psi_2] \qquad c_L, c_R, d_L, d_R \in \mathbb{R}$$

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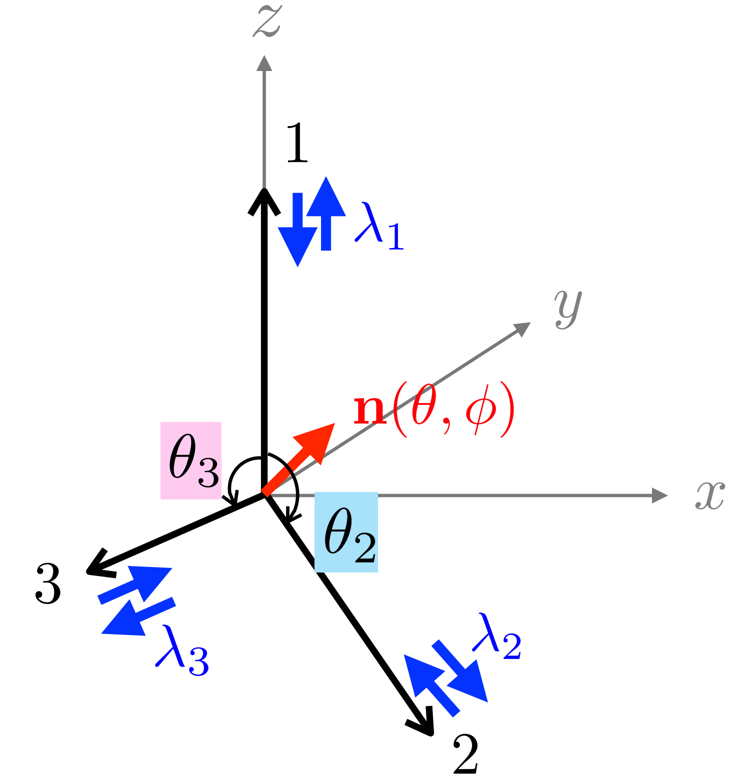
$$\mathcal{C}_{12} = \mathcal{C}_{13} = 0\,, \quad \mathcal{C}_{23} = 2|M_{LL} M_{LR}^* + M_{RL} M_{RR}^*|$$



[KS, M.Spannowsky 2310.01477]

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$$\mathcal{C}_{2(13)} = \mathcal{C}_{3(12)} = 2\sqrt{(|M_{LL}|^2 + |M_{RL}|^2)\,(|M_{LR}|^2 + |M_{RR}|^2)}$$

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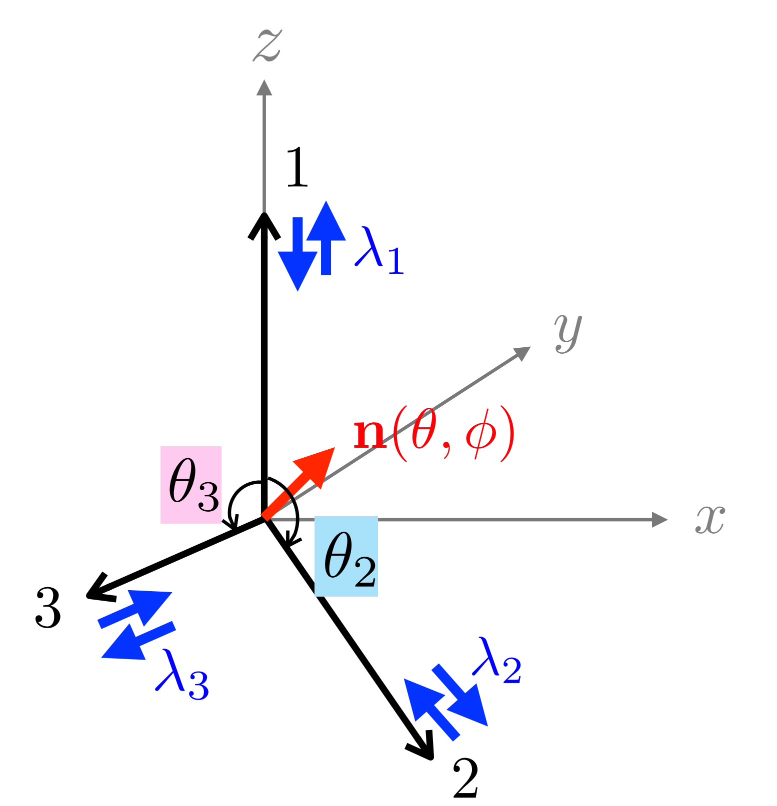

[KS, M.Spannowsky 2310.01477]

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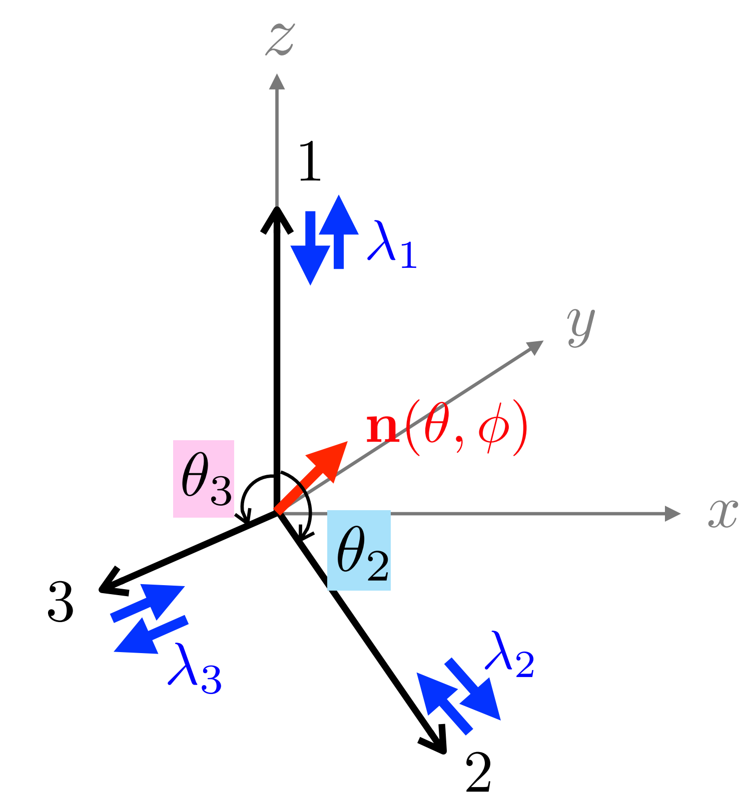

✤ **Monogamy**

$$M_i \equiv \mathcal{C}^2_{i(jk)} - [\mathcal{C}^2_{ij} + \mathcal{C}^2_{ik}] \; \Rightarrow \; M_1 = M_2 = M_3 = \mathcal{C}^2_{1(23)} \geq 0$$

[KS, M.Spannowsky 2310.01477]

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- Individual 2-party entanglement:

$$\mathcal{C}_{12} = \mathcal{C}_{13} = 0\,, \quad \mathcal{C}_{23} = 2|M_{LL}M^*_{LR} + M_{RL}M^*_{RR}| \quad \longleftarrow \text{vanish if } d_L d_R = 0$$

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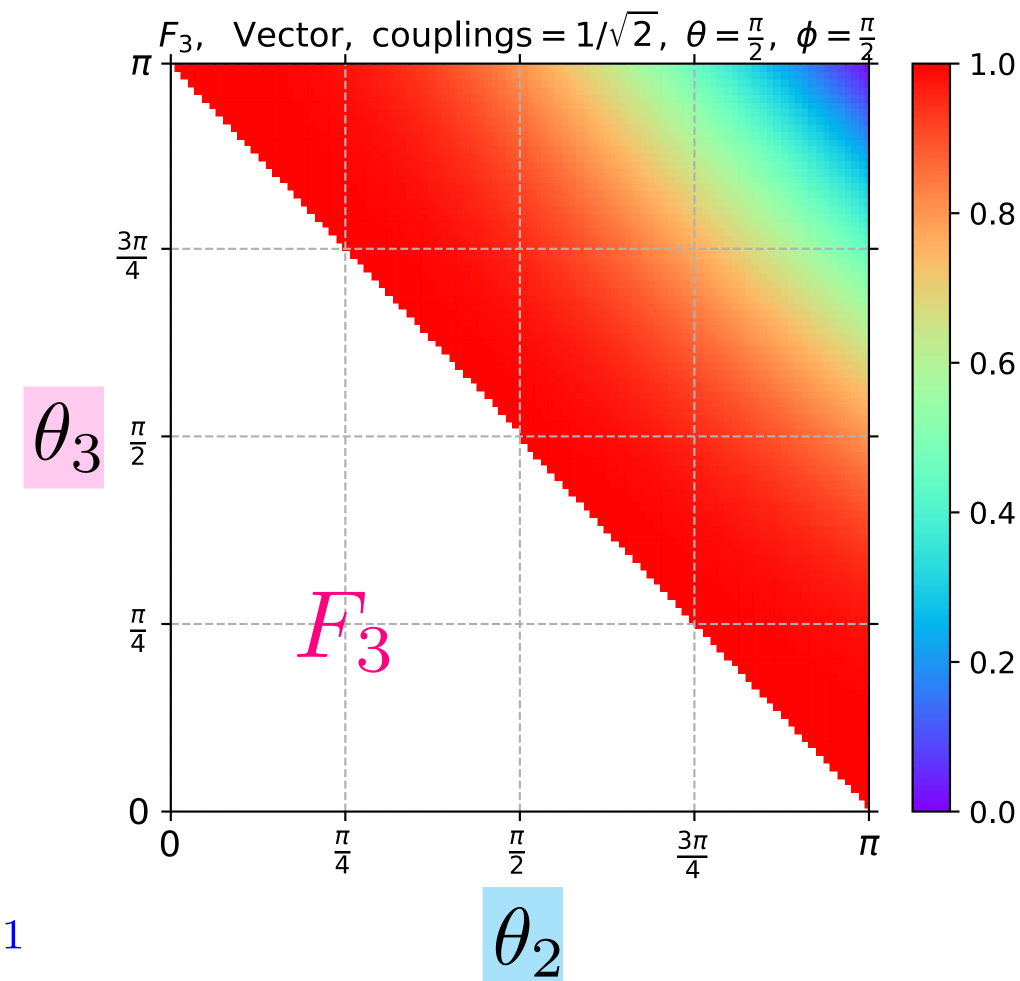
All entanglements vanish for weak decays

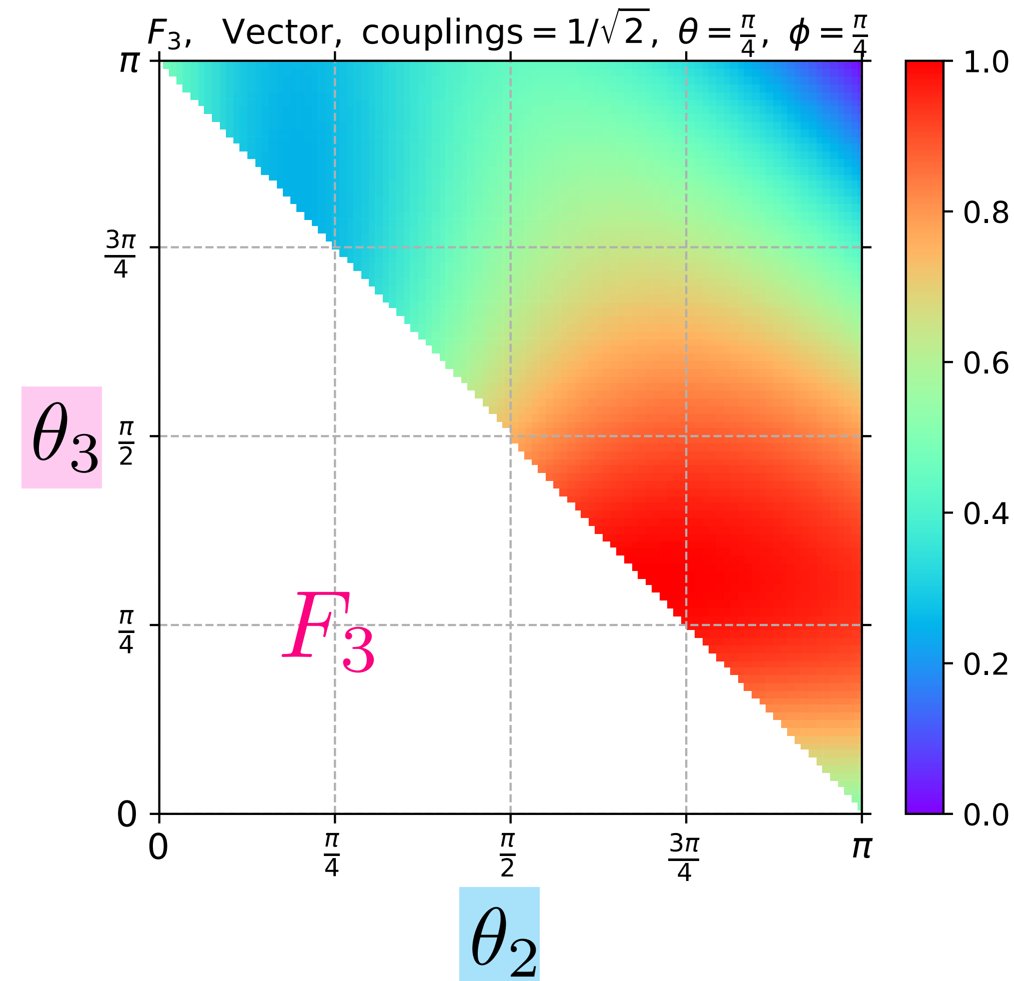
$$c_R = d_R = 0$$

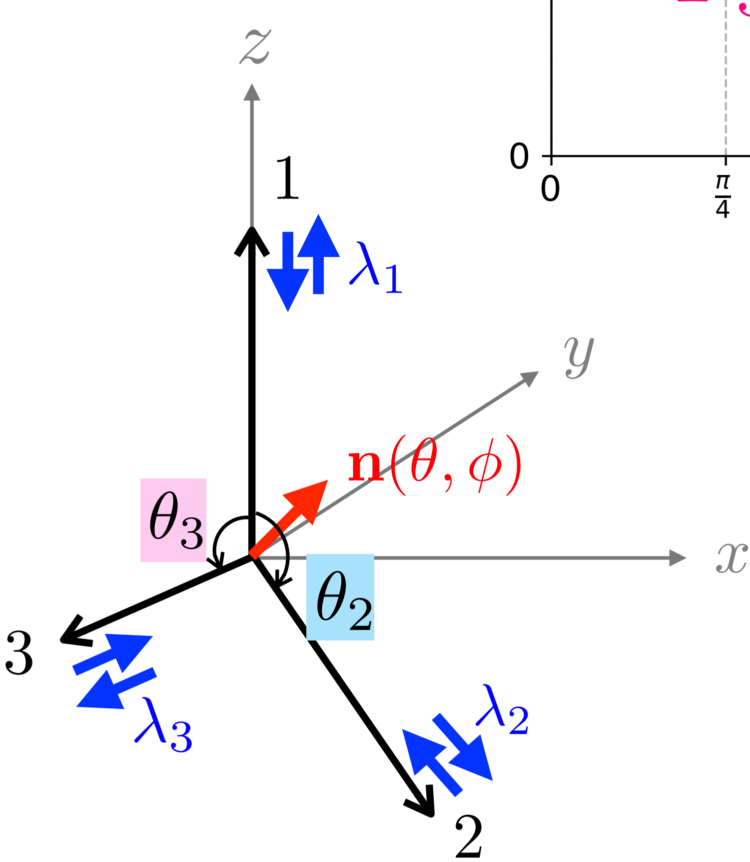
F_3 for Vector

$$\mathbf{n} = \mathbf{e}_y$$

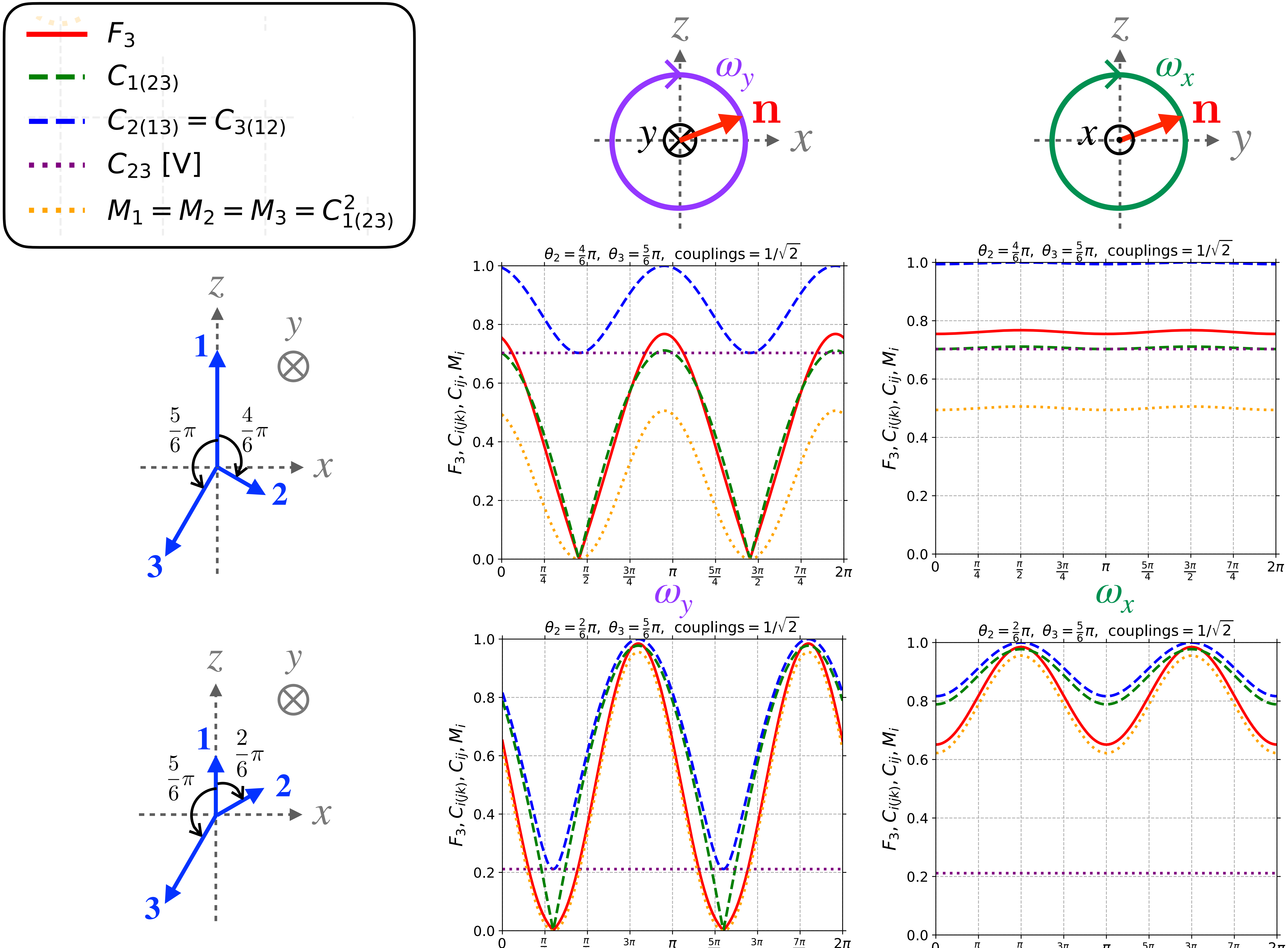
$$\mathbf{n} = \frac{1}{\sqrt{3}}(\mathbf{e}_x + \mathbf{e}_y + \mathbf{e}_z)$$







[KS, M.Spannowsky 2310.01477]



[KS, M.Spannowsky 2310.01477]

[KS, M.Spannowsky
2310.01477]

Tensor

$$\mathcal{L}_{\text{int}} = [\bar{\psi}_1(c_M + ic_E\gamma_5)\sigma^{\mu\nu}\psi_0][\bar{\psi}_3(d_M + id_E\gamma_5)\sigma_{\mu\nu}\psi_2]$$

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[KS, M.Spannowsky 2310.01477]

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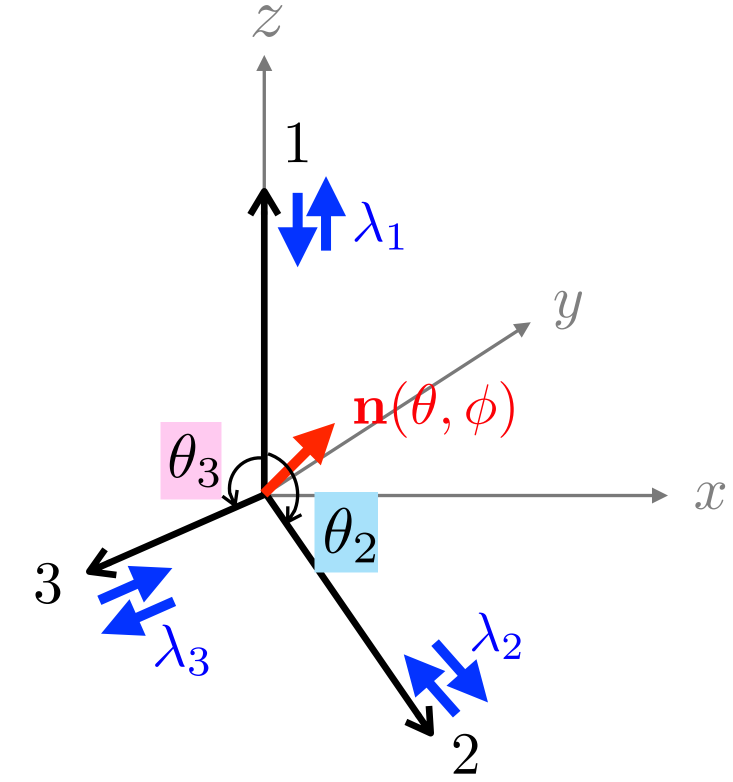

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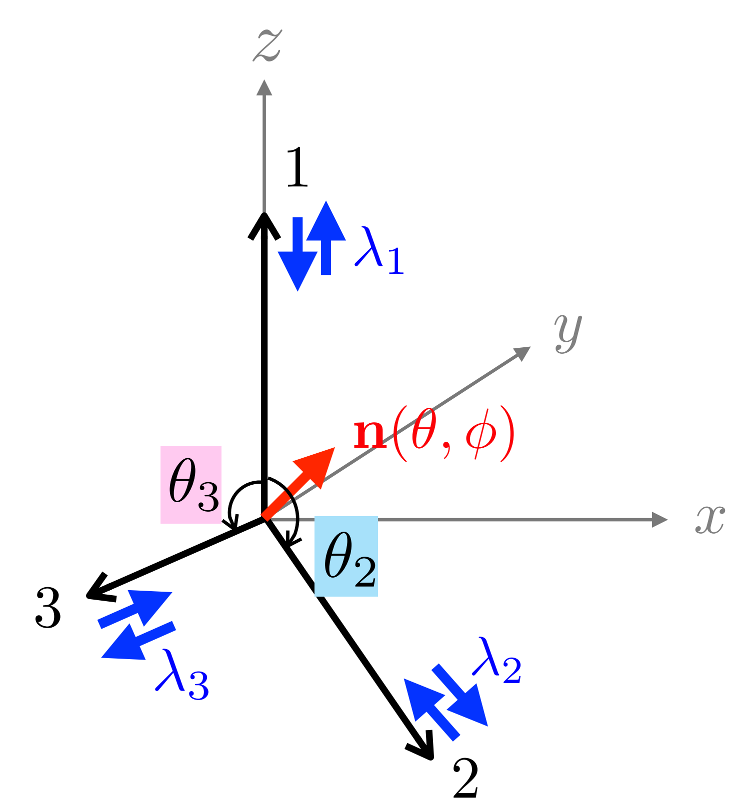

[KS, M.Spannowsky 2310.01477]

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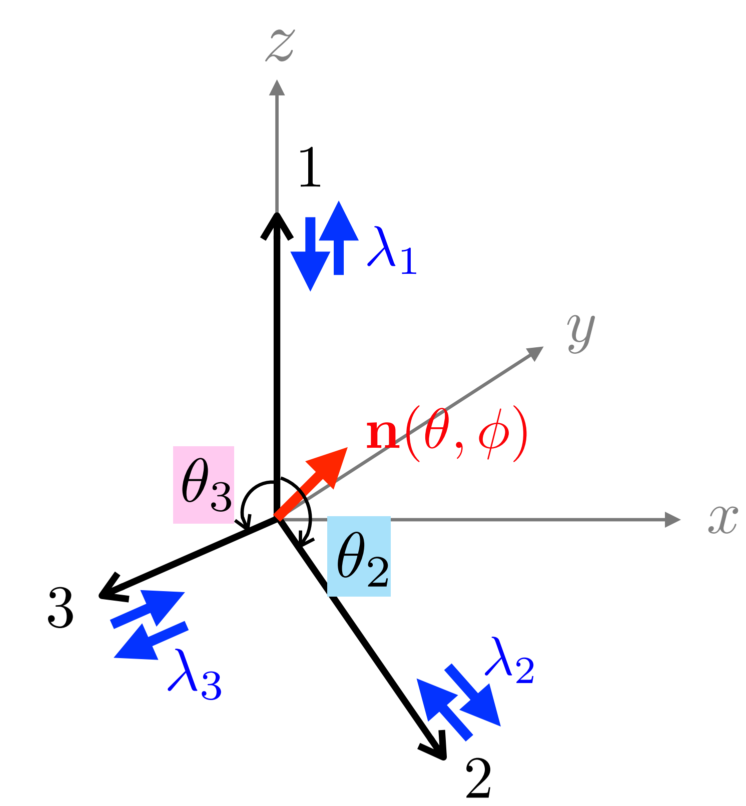

[KS, M.Spannowsky 2310.01477]

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[KS, M.Spannowsky 2310.01477]

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✤ one-to-other entanglements are **universal**:

$$\mathcal{C}_{1(23)} = \mathcal{C}_{2(13)} = \mathcal{C}_{3(12)} = 2|M_R M_L|$$

$$F_3 = 4|M_R M_L|^2$$

[KS, M.Spannowsky 2310.01477]

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independent of the coupling structure (CP phases)

$$\omega_1, \omega_2$$

F_3 for Tensor

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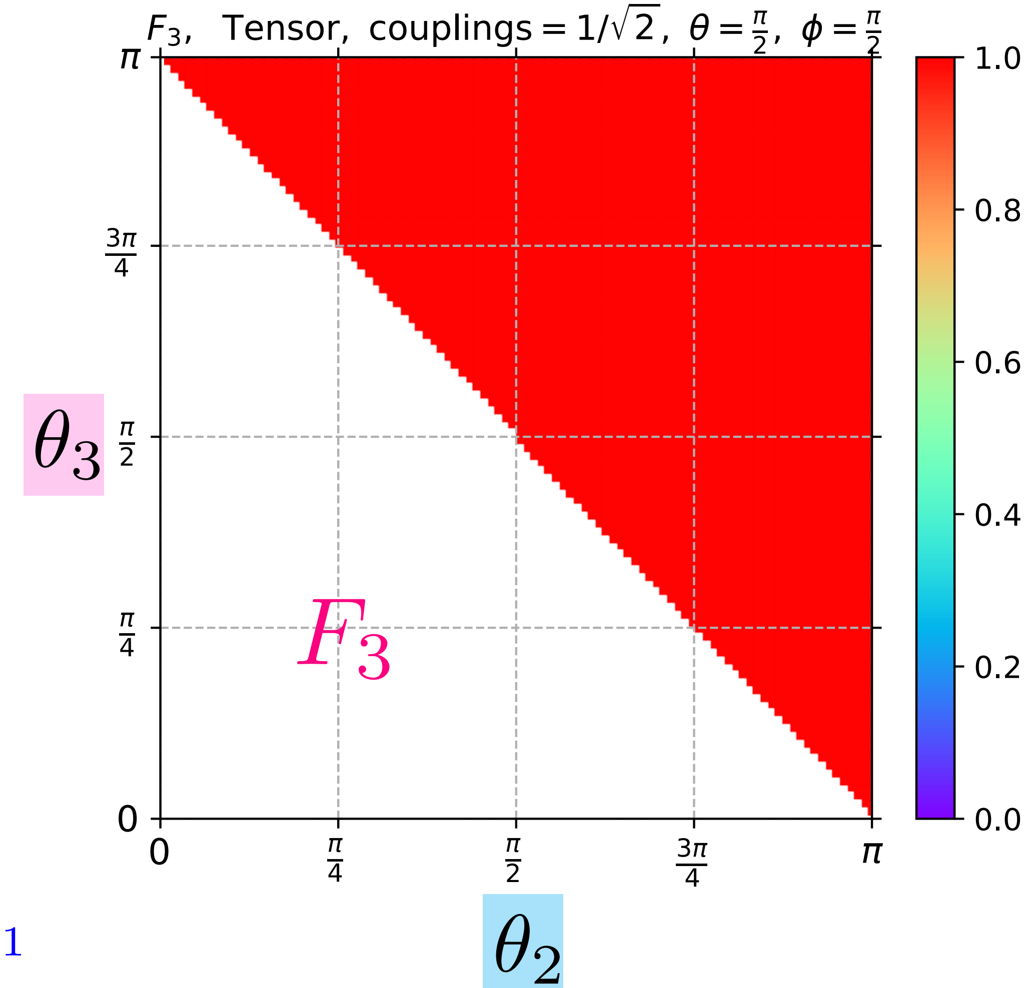

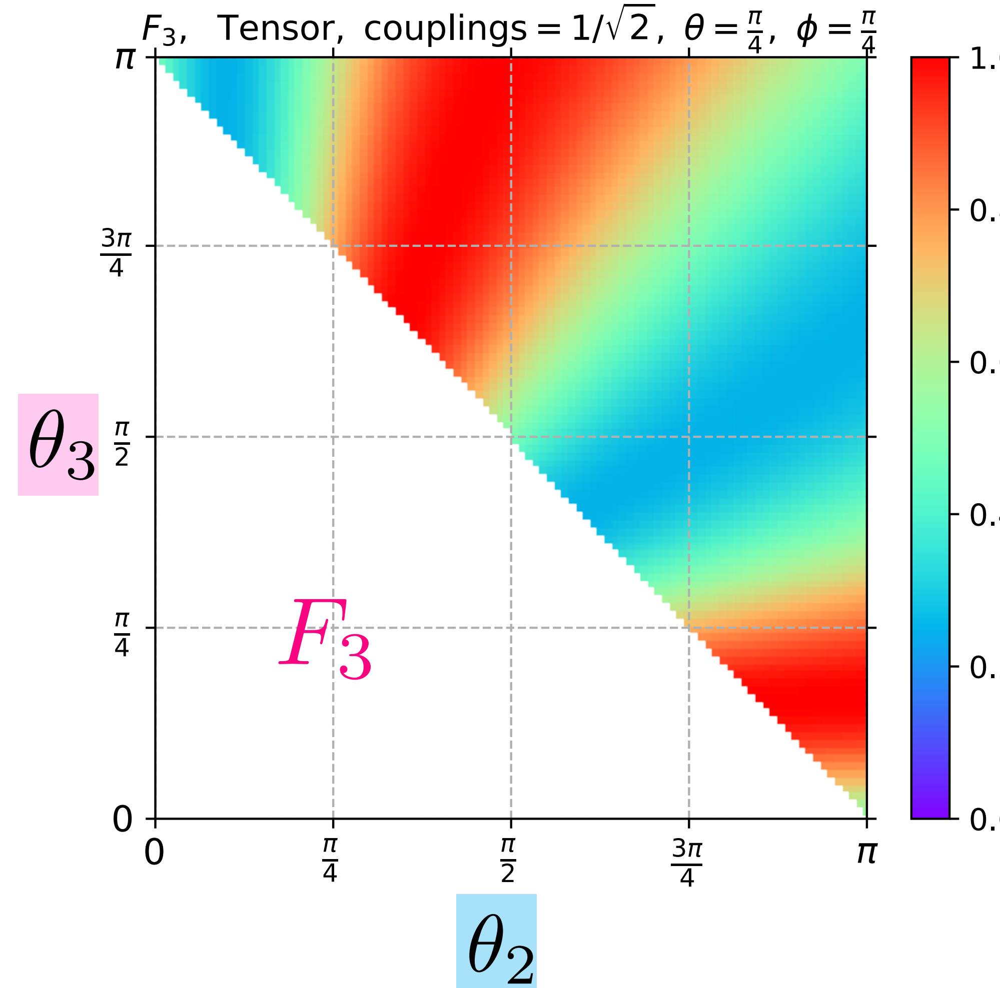

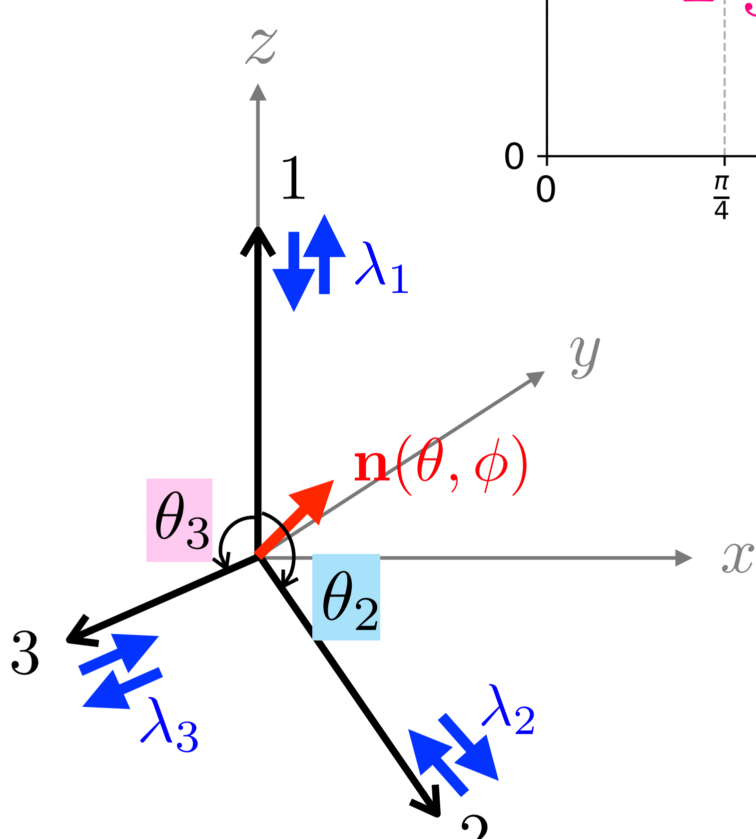

[KS, M.Spannowsky 2310.01477]

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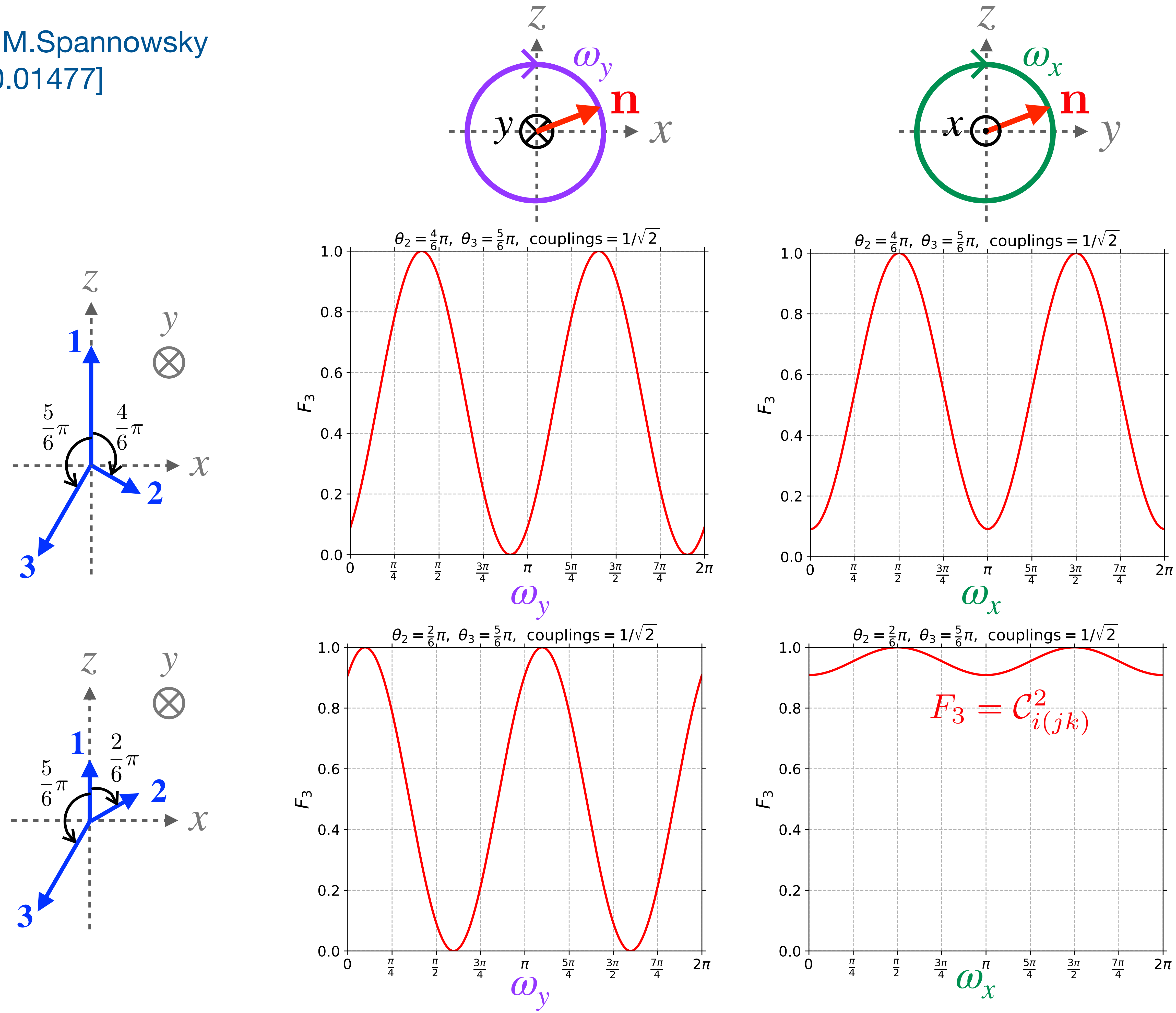
2310.01477]



Discussion

What to do with it?

✤ **measure**/study 3-body entanglements **experimentally** *e.g*. in **hadron decays**

✤ look for **theories** to **maximise/minimise** the **entanglement**

Carena, Low, Wagner, Xiao [2307.08112]

minimising entanglement
in $\Phi_a \Phi_b \to \Phi_c \Phi_d$

general 2HD → maximum SO(8) sym. → SM-like Higgs

Future directions:

✤ Effect of **masses** in the final particles

✤ More **spin structures**: $SFFV, VVFF, SFVF_{3/2}, SVVT \cdots$

✤ 3-body **non-locality** [Mermin '90, Svetlichny '87]

Mermin ineq: $\langle \mathcal{B}_{\rm M} \rangle_{\rm LR} \leq 2 \quad \langle \mathcal{B}_{\rm M} \rangle_{\rm QM} \leq 4 \qquad \mathcal{B}_{\rm M} = abc' + ab'c + a'bc - a'b'c'$

Horodecki, KS, Spannowsky, *in progress*

Thank you for listening!

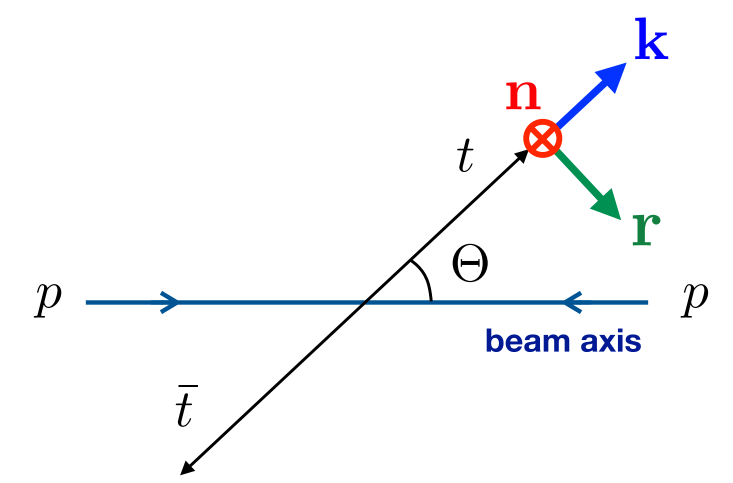
The research leading to the results presented in this talk has received funding from the Norwegian Financial Mechanism for years 2014-2021, grant nr 2019/34/H/ST2/00707

Understanding the Early Universe:
interplay of theory and collider experiments

Joint research project between the University of Warsaw & University of Bergen

[Afik, Nova (2021, 2022)]

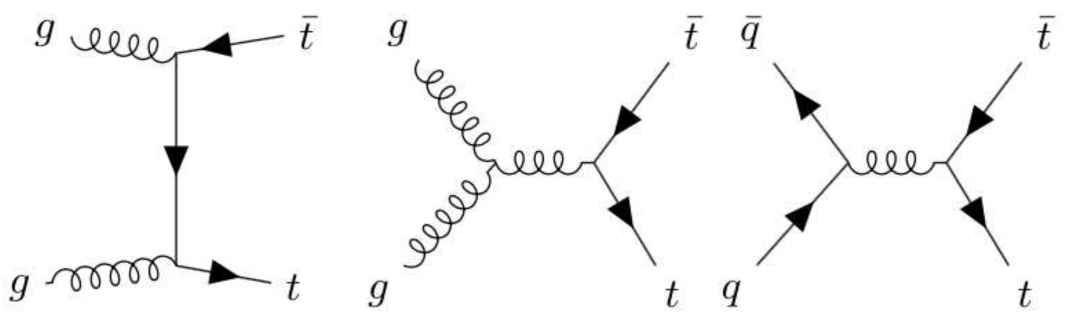
$pp \to t\bar{t}$ @ LHC



- At the rest frame of $t\bar{t}$, the kinematics is determined by:

  Θ : the angle between t and the beam line ($0 \le \Theta \le \pi/2$)

  $M_{t\bar{t}}$: the inv. mass of $t\bar{t}$



- gg and $q\bar{q}$ initial states contribute stochastically
  $\Rightarrow$ the $t\bar{t}$ spin state is necessarily **mixed**

$$\rho(M_{t\bar{t}}, \Theta) = \sum_{I=gg,q\bar{q}} w_I(M_{t\bar{t}}, \Theta) \cdot \rho^I(M_{t\bar{t}}, \Theta)$$

$$w_I(M_{t\bar{t}}, \Theta) = \frac{L_I(M_{t\bar{t}})\tilde{A}^I(M_{t\bar{t}}, \Theta)}{\sum_J L_J(M_{t\bar{t}})\tilde{A}^J(M_{t\bar{t}}, \Theta)}$$

$\tilde{A}^I(M_{t\bar{t}}, \Theta)$: partonic differential x-section

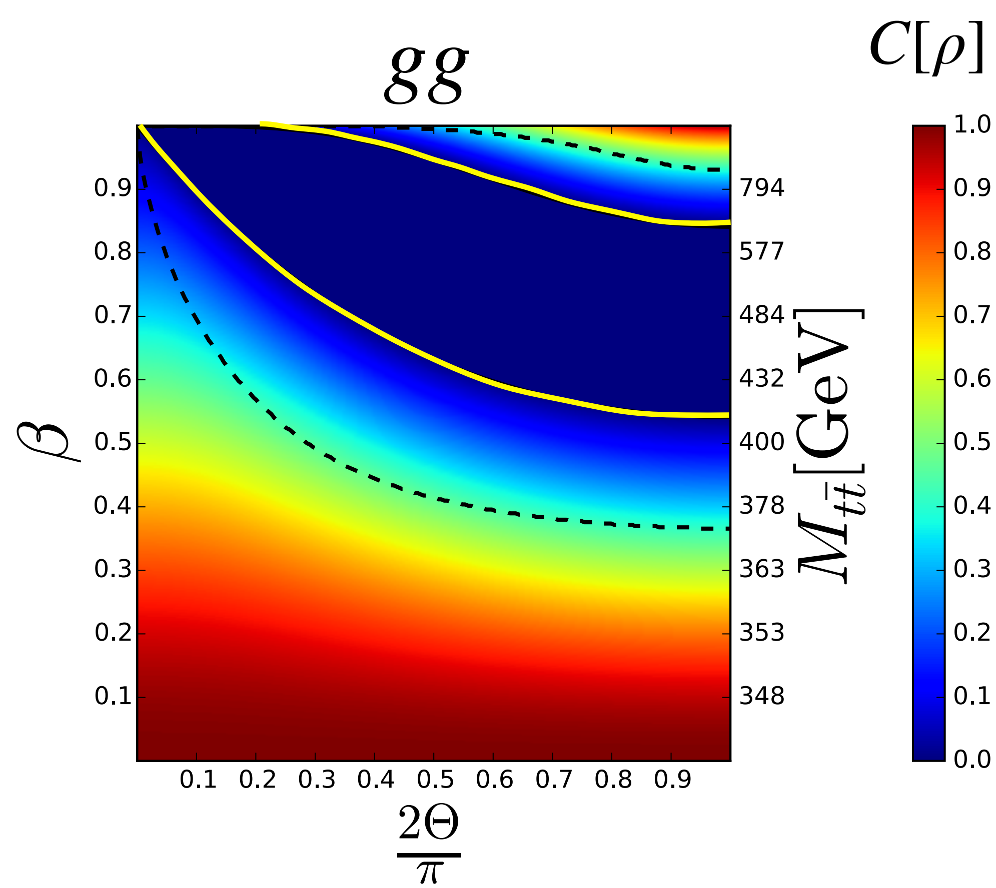
$L_I(M_{t\bar{t}})$: luminosity function

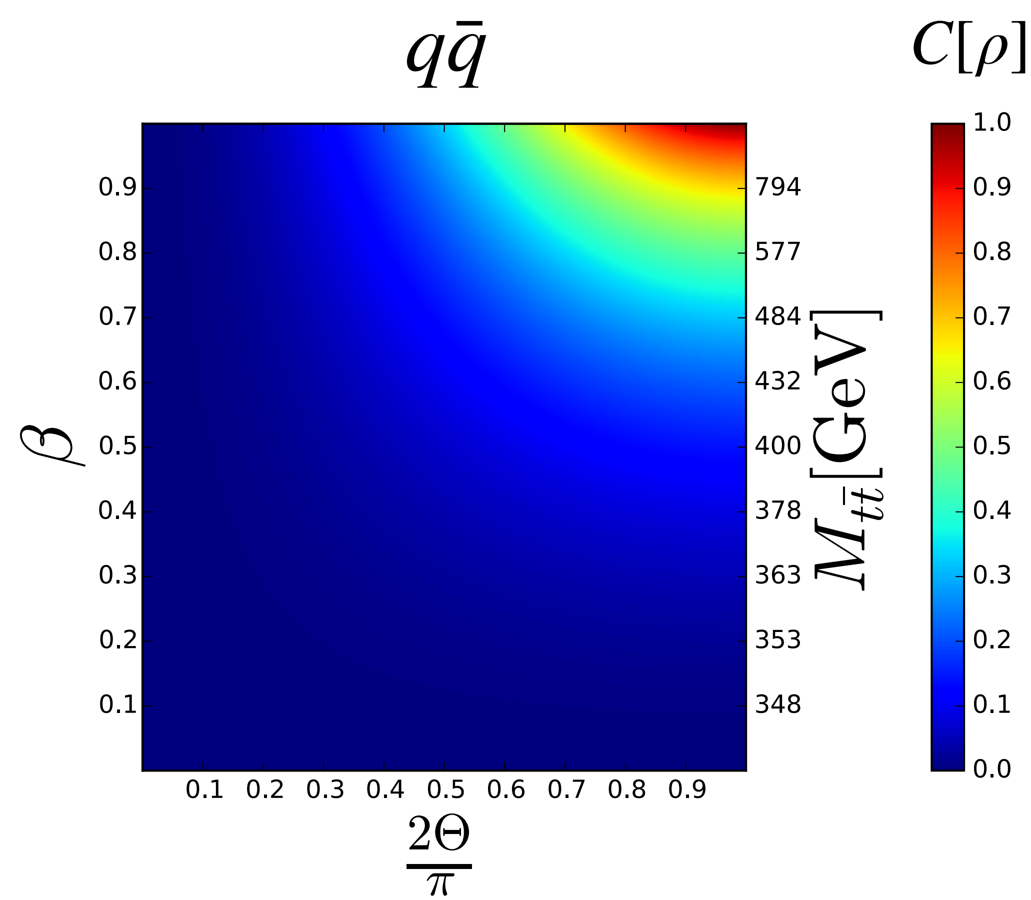
Concurrence

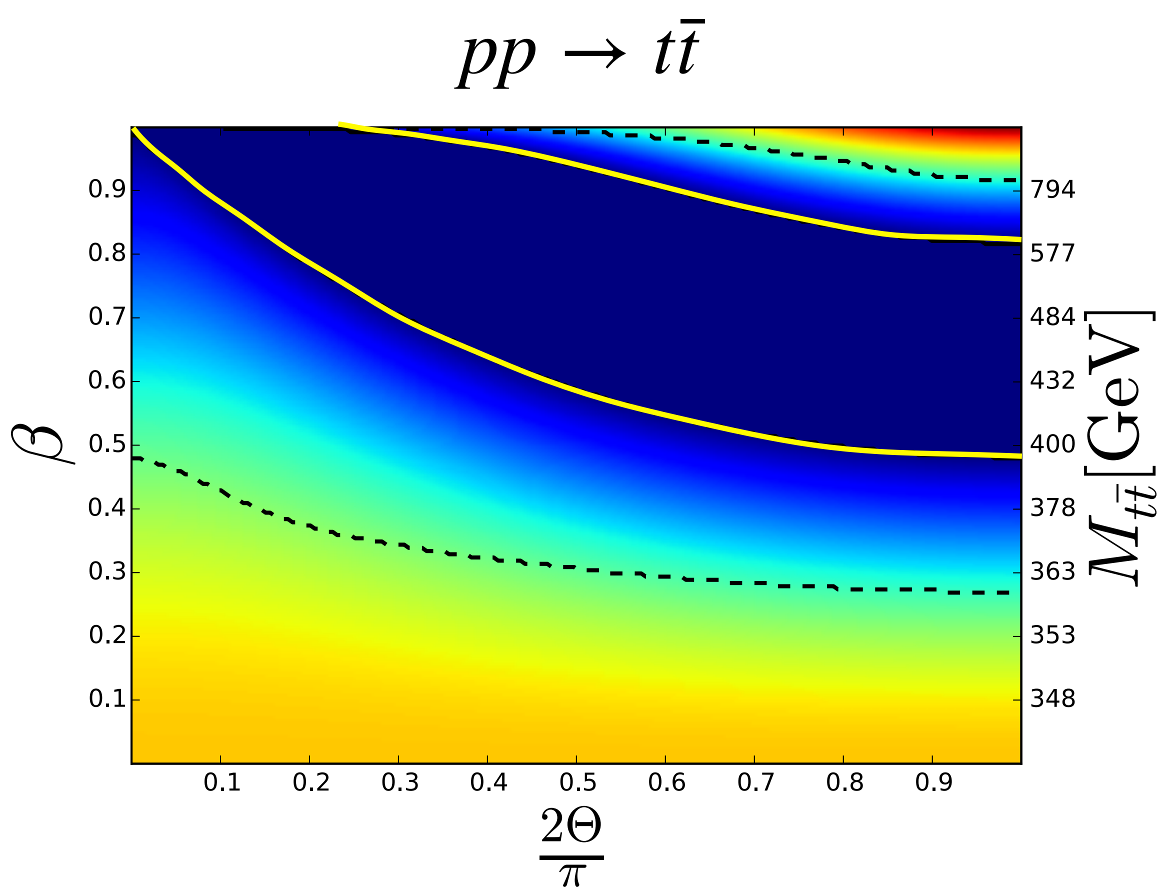
Yellow solid: $C[\rho] \geq 0$

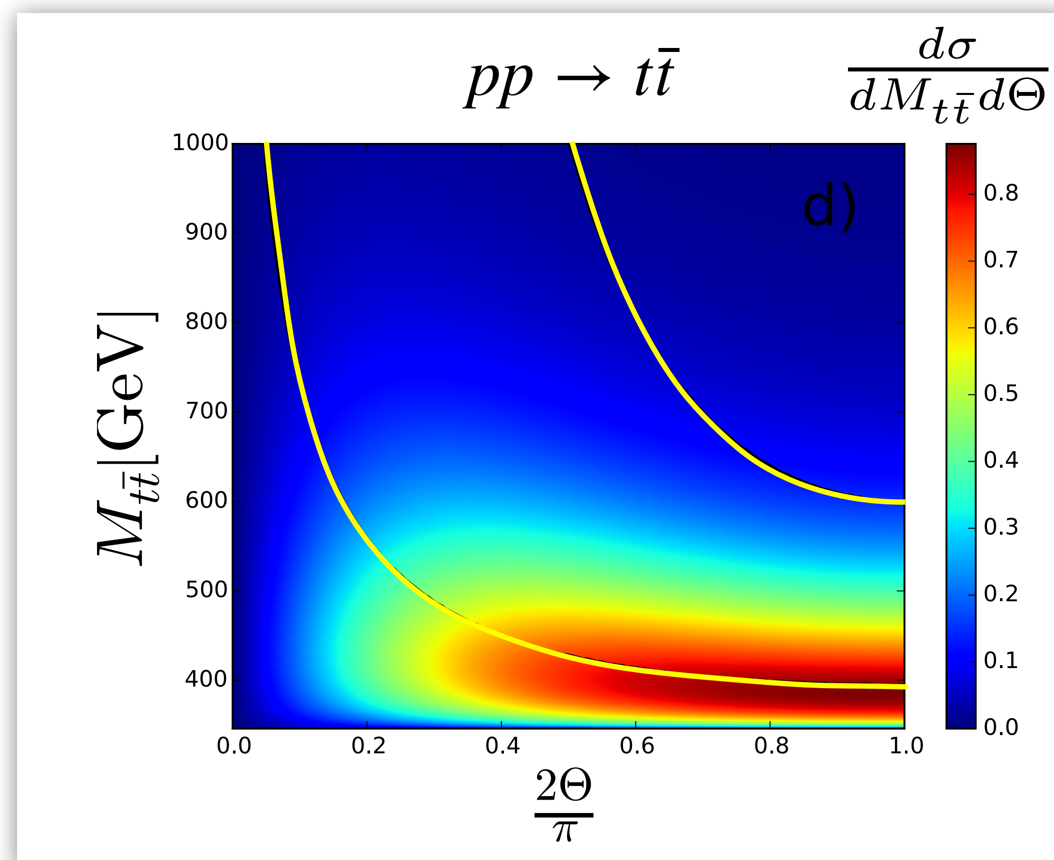
Black dashed: $S_{\mathrm{CHSH}} \geq 2$

[Afik, Nova (2021, 2022)]

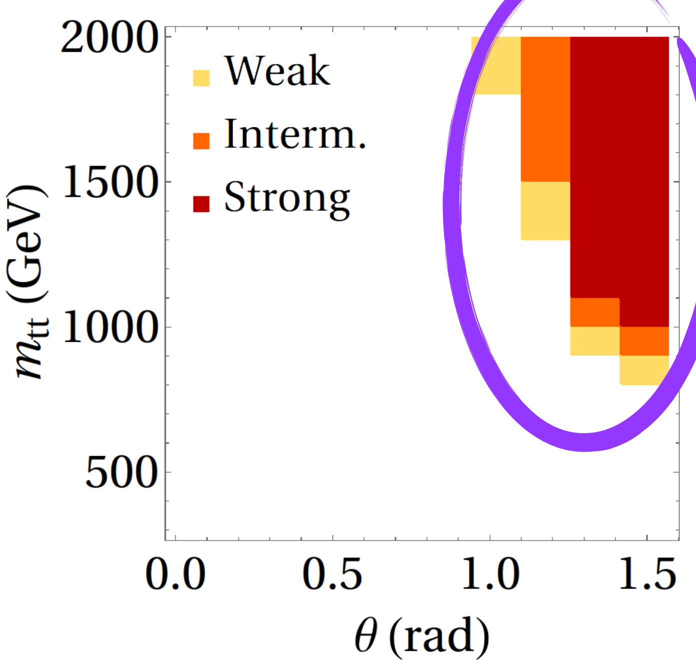








MC-sim: di-leptonic decay, $pp \to t\bar{t} \to (b\ell^+\nu)(\bar{b}\ell^-\bar{\nu})$



selecting events here

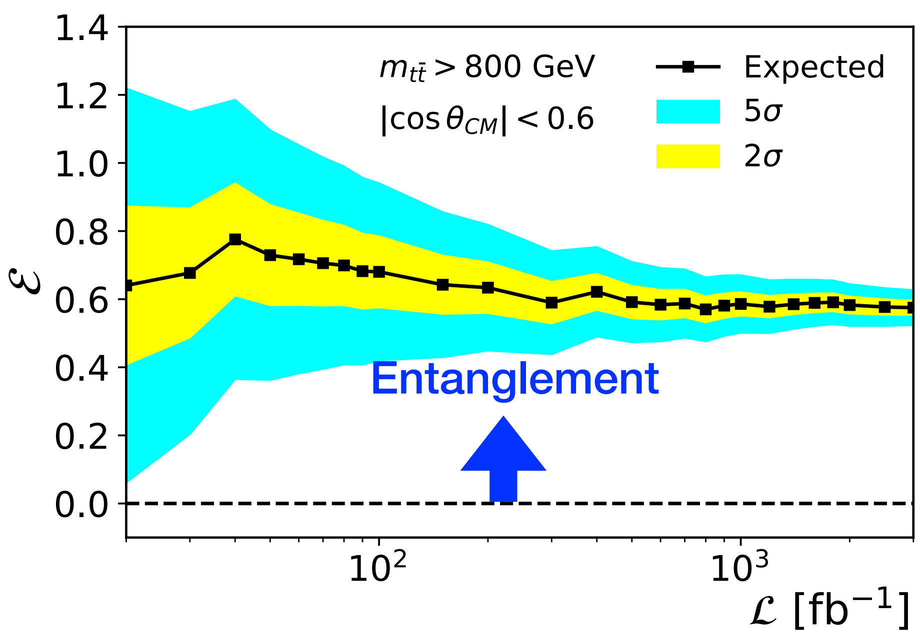
HL-LHC $(L = 3\ \mathrm{ab}^{-1})$ [Severi, Boschi, Maltoni, Sioli (2022)]

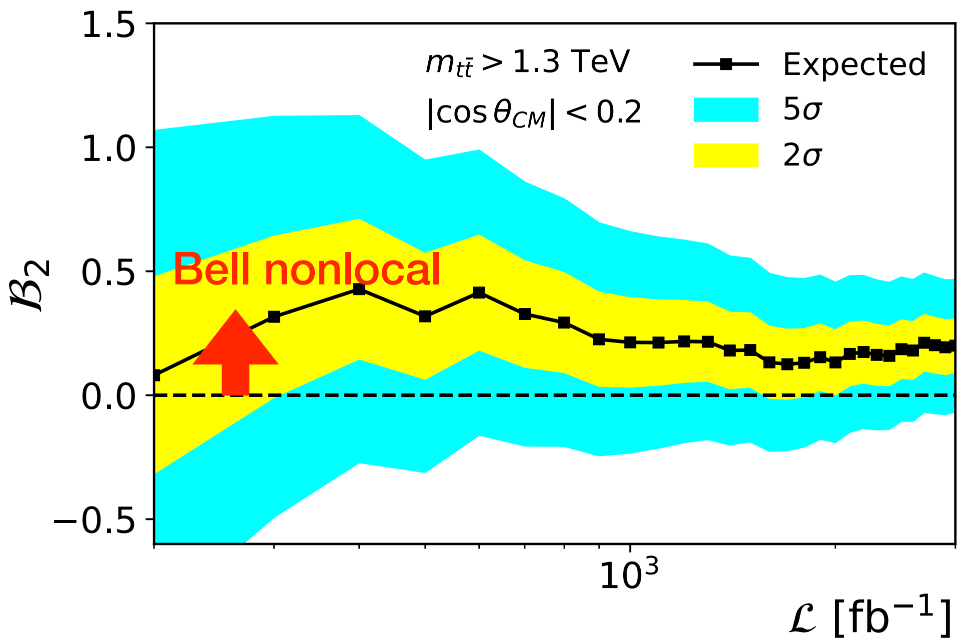
$$|C_{kk} + C_{rr}| - C_{nn} = 1.36 \pm 0.07 \; > 1 \;\Rightarrow \text{Entanglement} \gg 5\sigma$$

$$\sqrt{2}S_{\mathrm{CHSH}}/2 = 2.20 \pm 0.1 \; > 2 \;\Rightarrow \text{Bell nonlocality} \sim 1.8\sigma$$

MC-sim: semi-leptonic decay, $pp \to t\bar{t} \to (b\ell\nu)(\underbrace{bjj}_{\text{boosted top-tagging}})$

[Dong, Goncalves, Kong, Navarro (2023)]





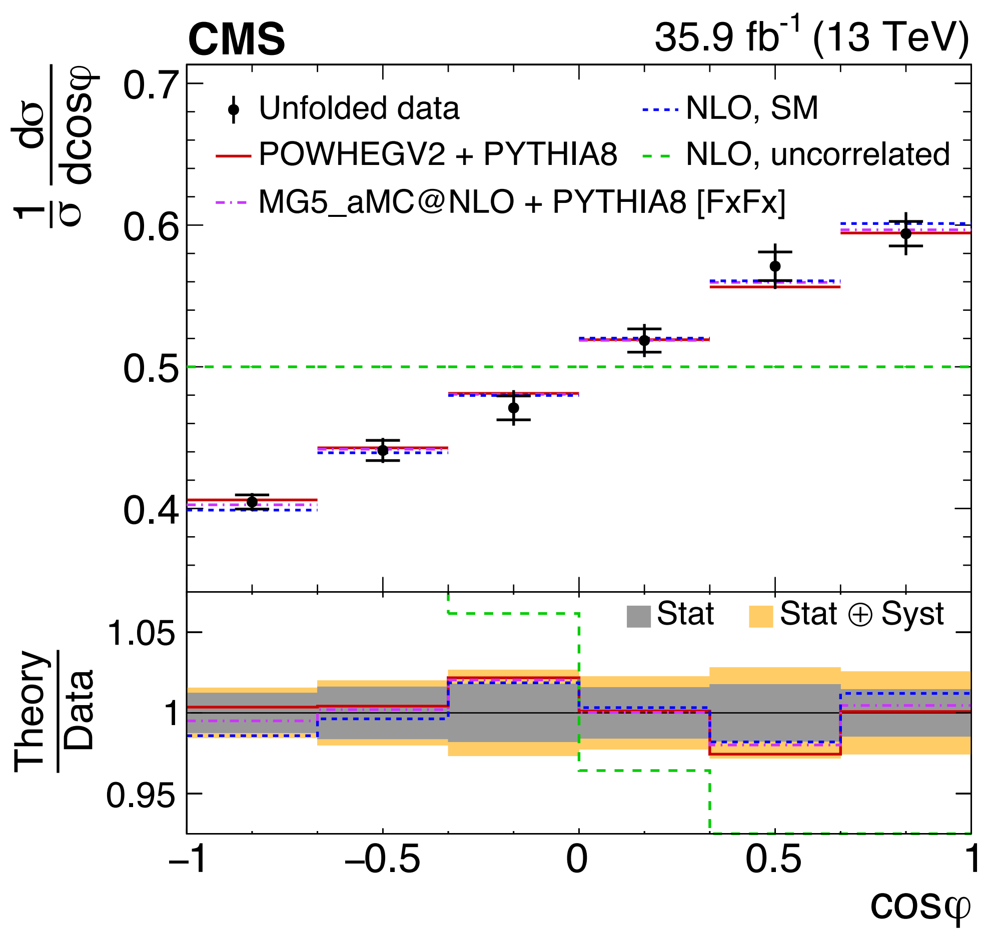
Entanglement in CMS

[Phys. Rev. D 100, 072002]

Recently, D has been measure by CMS in di-leptonic $pp \to t\bar{t}$

$$D \equiv \frac{\mathrm{Tr}[C]}{3} < -\frac{1}{3}$$ entanglement

$$\frac{1}{\sigma}\frac{d\sigma}{d\cos\varphi} = \frac{1 - D\cos\varphi}{2}$$



CMS result:

$$D = -0.237 \pm 0.011 > -\frac{1}{3}$$ entanglement is not detected

To see the entanglement, selecting certain kinematical regions is crucial.

A dedicated analysis is needed.

$H \to WW^*, ZZ^*$

[Barr (2022)]
[Aguilar-Saavedra ,Bernal, Casas, Moreno (2022)]
[Aguilar-Saavedra (2023)]
[Fabbrichesi, Floreanini, Gabrielli, Marzola (2023)]

- Conceptually less clear since one particle is off-shell.

  ⇒ virtual particle with mass shifted: $m_{V*} = f \cdot m_V \quad (0 < f < 1)$

- two qutrits (rather than qubits)

- the final state is pure:

$$\left.\begin{array}{l} |\Psi_{VV*}\rangle \simeq |+-\rangle - \beta|00\rangle + |-+\rangle \\[2em] \beta = 1 + \dfrac{m_H^2 - (1+f)^2 m_V^2}{2 f m_V^2} \sim 1 \end{array}\right\} \Rightarrow \text{(almost) maximally entangled}$$

CGLMP Qutrit inequality

CGLMP function [Collins Gisin Linden Massar Popescu (2002)]

$$I_3 \equiv P(A_1 = B_1) + P(B_1 = A_2 + 1) + P(A_2 = B_2) + P(B_2 = A_1)$$
$$- P(A_1 = B_1 - 1) - P(B_1 = A_2) - P(A_2 = B_2 - 1) - P(B_2 = A_1 - 1)$$

*) $P(A_i = B_j + k)$ is the probability that A_i and B_j are differ by k mod 3

$$I_3 \le \begin{cases} 2 & \text{Local theories} \\ 1 + \sqrt{11/3} \simeq 2.9149 & \text{Quantum Mechanics} \end{cases}$$

Quantum state tomography

- It is convenient to reconstruct the density matrix from the kinematics, then analysis entanglement and nonlocality

- density matrix is 9 x 9 Hermitian matrix with unit trace. It can be expanded by two sets of Gell-Mann matrices and $8+8+64=80$ real parameters (9^2-1)

$$\rho = \frac{1}{9}(\mathbf{1}\otimes\mathbf{1}) + \frac{1}{3}\sum_{i=1}^{8} a_i(\lambda_i\otimes\mathbf{1}) + \frac{1}{3}\sum_{j=1}^{8} b_j(\mathbf{1}\otimes\lambda_j) + \sum_{i,j=1}^{8} c_{ij}(\lambda_i\otimes\lambda_j)$$

- real parameters a_i, b_j, c_{ij} can be reconstructed from the directions of two charged leptons, $\mathbf{n}_1$ and $\mathbf{n}_2$, using the eight Wigner P functions, Φ_i^P

$$a_i = \frac{1}{2}\left\langle \Phi_i^P \mathbf{n_1} \right\rangle_{\rm av} \qquad b_i = \frac{1}{2}\left\langle \Phi_i^P \mathbf{n_2} \right\rangle_{\rm av} \qquad c_{ij} = \frac{1}{4}\left\langle (\Phi_i^P \mathbf{n_1})(\Phi_j^P \mathbf{n_2}) \right\rangle_{\rm av}$$

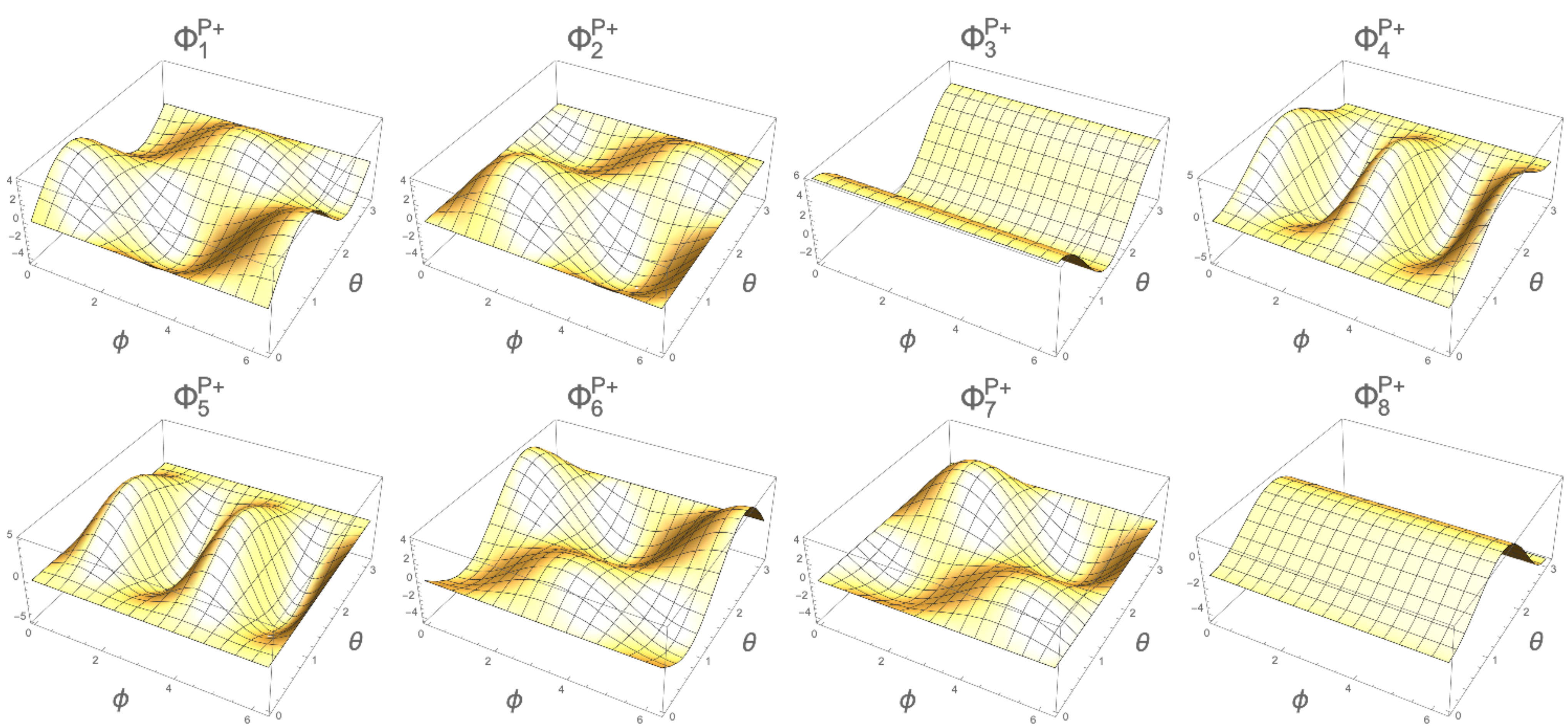
[Ashby-Pickering, Barr, Wierzchucka (2022)]

Quantum state tomography

- Wigner functions for $W^{\pm} \to \ell^{\pm}\nu$

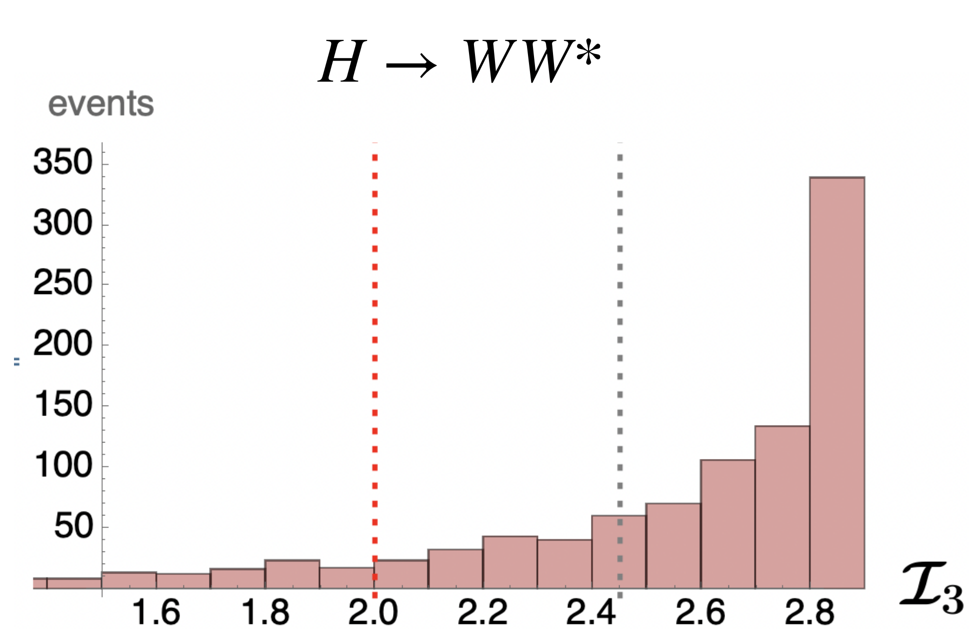
[Ashby-Pickering, Barr, Wierzchucka (2022)]

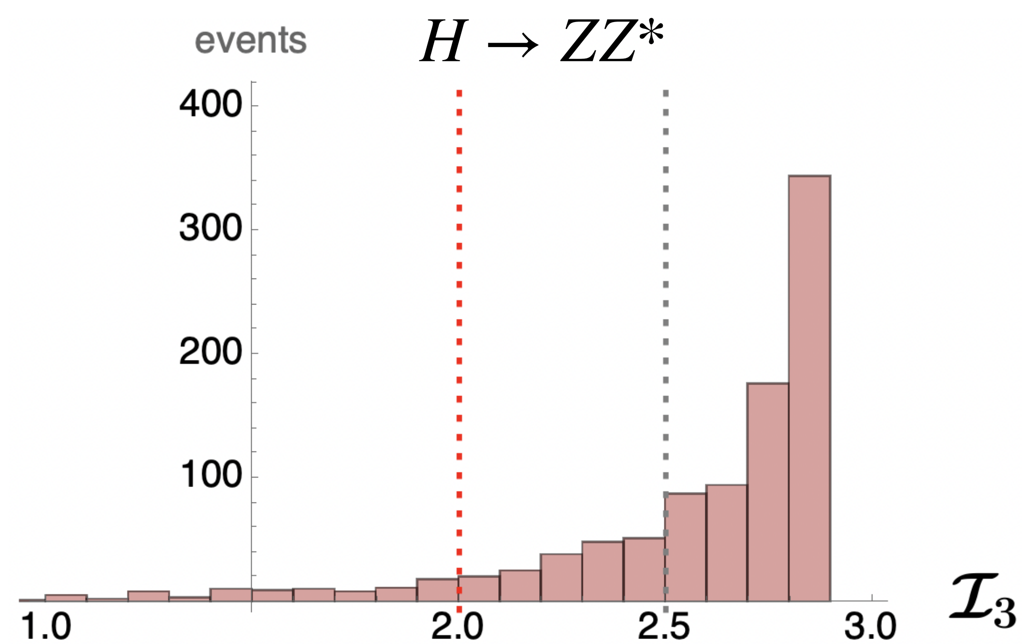
$$\Phi_1^{P\pm} = \sqrt{2}(5\cos\theta \pm 1)\sin\theta\cos\phi$$
$$\Phi_2^{P\pm} = \sqrt{2}(5\cos\theta \pm 1)\sin\theta\sin\phi$$
$$\Phi_3^{P\pm} = \frac{1}{4}(\pm 4\cos\theta + 15\cos 2\theta + 5)$$
$$\Phi_4^{P\pm} = 5\sin^2\theta\cos 2\phi$$
$$\Phi_5^{P\pm} = 5\sin^2\theta\sin 2\phi$$
$$\Phi_6^{P\pm} = \sqrt{2}(\pm 1 - 5\cos\theta)\sin\theta\cos\phi$$
$$\Phi_7^{P\pm} = \sqrt{2}(\pm 1 - 5\cos\theta)\sin\theta\sin\phi$$
$$\Phi_8^{P\pm} = \frac{1}{4\sqrt{3}}(\pm 12\cos\theta - 15\cos 2\theta - 5)$$

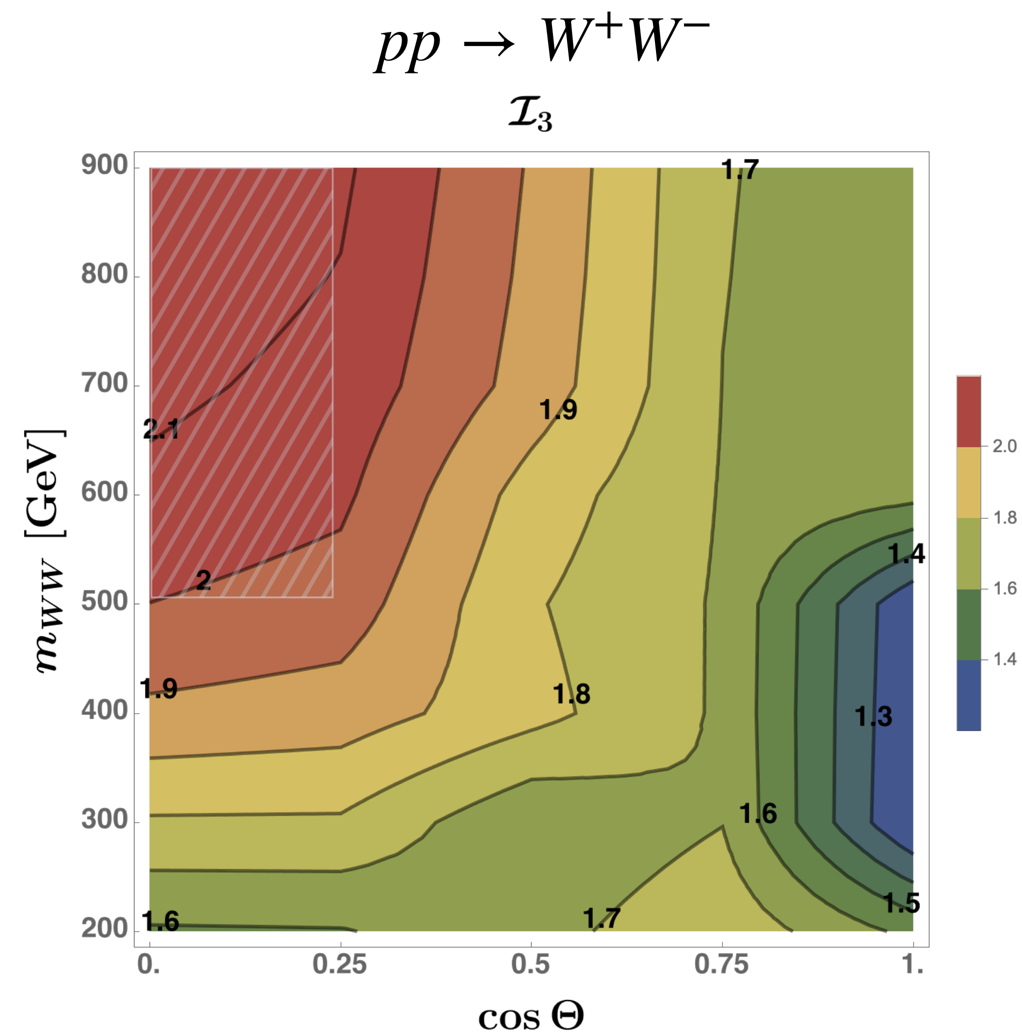


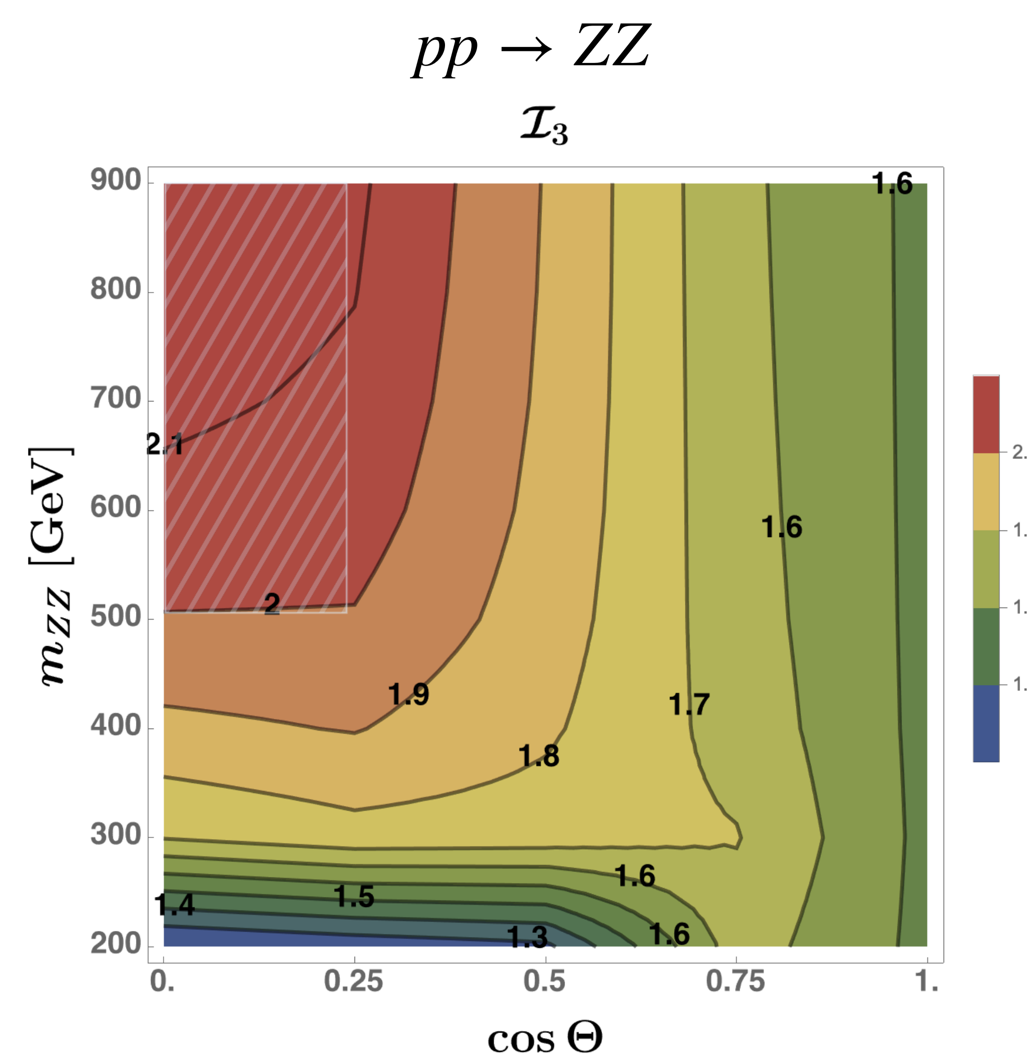
CGLMP function I_3 in optimal measurement axes

[Fabbrichesi, Floreanini, Gabrielli, Marzola (2023)]





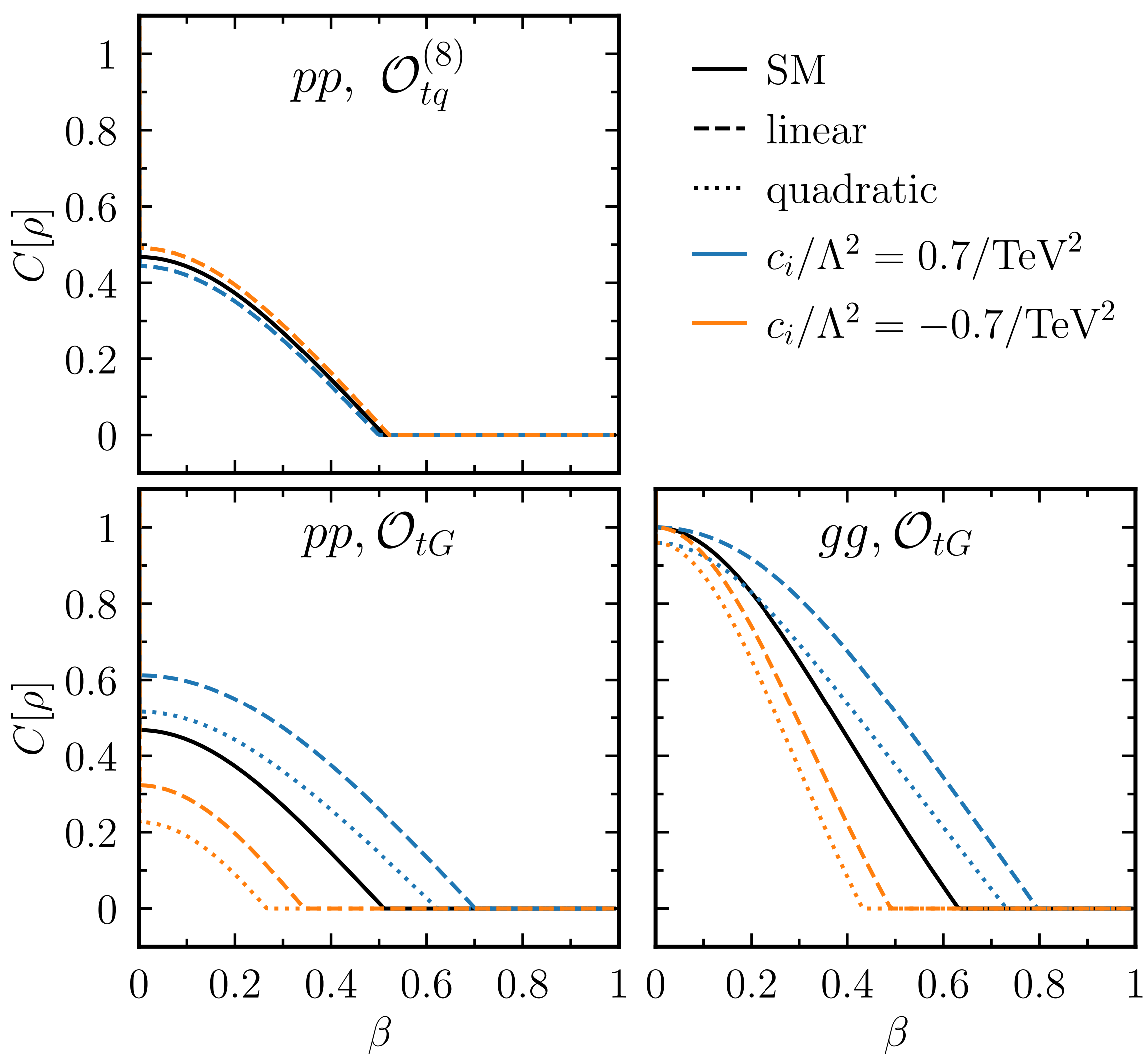




Effect of BSM $pp \to t\bar{t}$

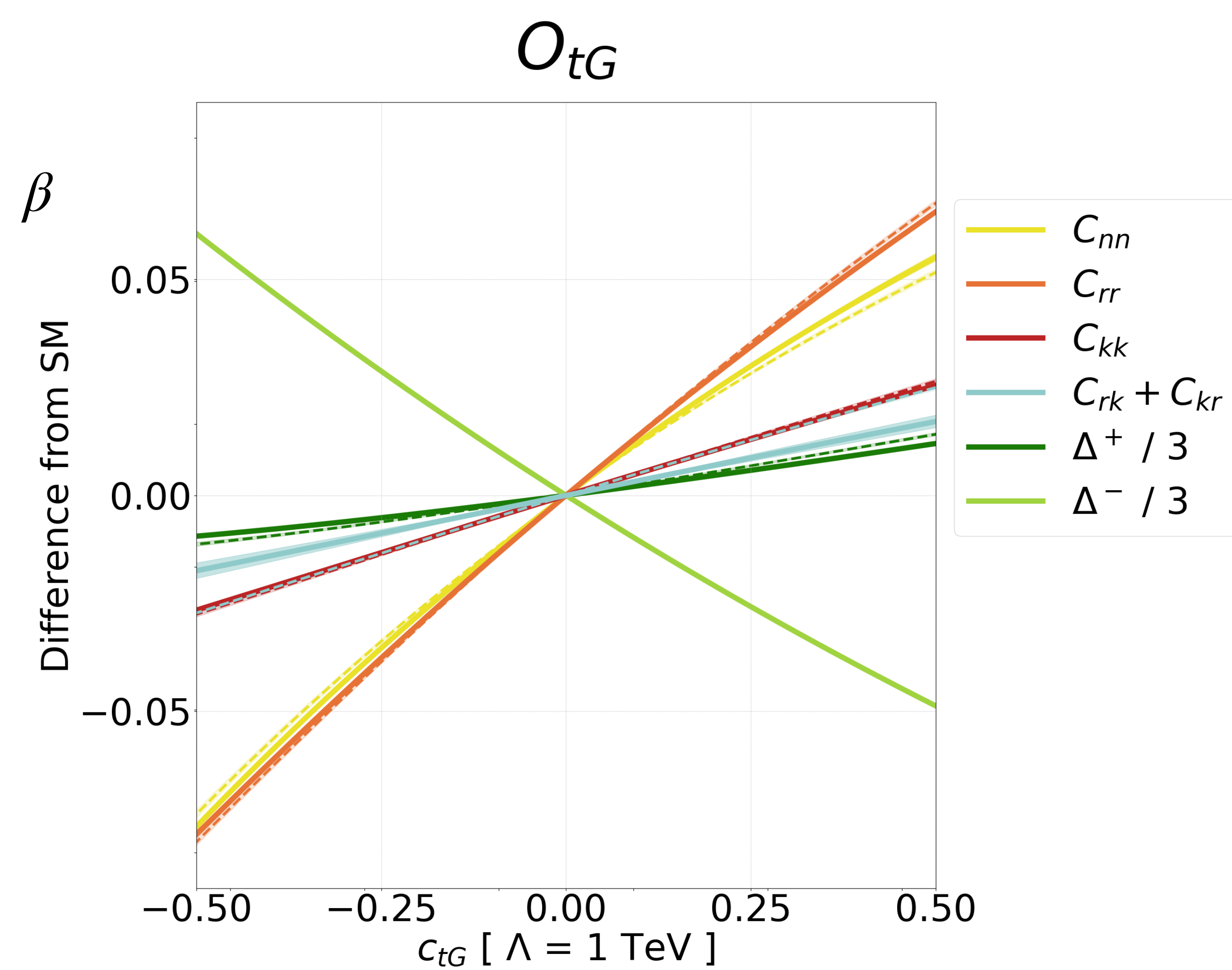
$$\mathcal{O}_{tG} = g_S\,\overline{Q}T_A\tilde{\varphi}\sigma^{\mu\nu}t\,G^A_{\mu\nu} \qquad \mathcal{O}^8_{tq} = \textstyle\sum_{\mathrm{f}=1}^{2}(\bar{q}_\mathrm{f}\gamma_\mu T_A q_\mathrm{f})(\bar{t}\gamma^\mu T^A t)$$

[Aoude Madge Maltoni Mantani (2022)]



β: top velocity in the $t\bar{t}$ rest frame

[Severi Vryonidou (2023)]



$$\Delta^\pm = \pm(C_{kk} + C_{rr}) - C_{nn} - 1$$

Local **R**eal Hidden Variable theories:

Mermin ineq:

$$P(abc|XYZ) = \sum_{\lambda} q_\lambda P_\lambda(a|X) P_\lambda(b|Y) P_\lambda(c|Z) \longrightarrow \langle \mathcal{B}_\mathrm{M} \rangle_\mathrm{LR} \leq 2 \quad \langle \mathcal{B}_\mathrm{M} \rangle_\mathrm{QM} \leq 4$$

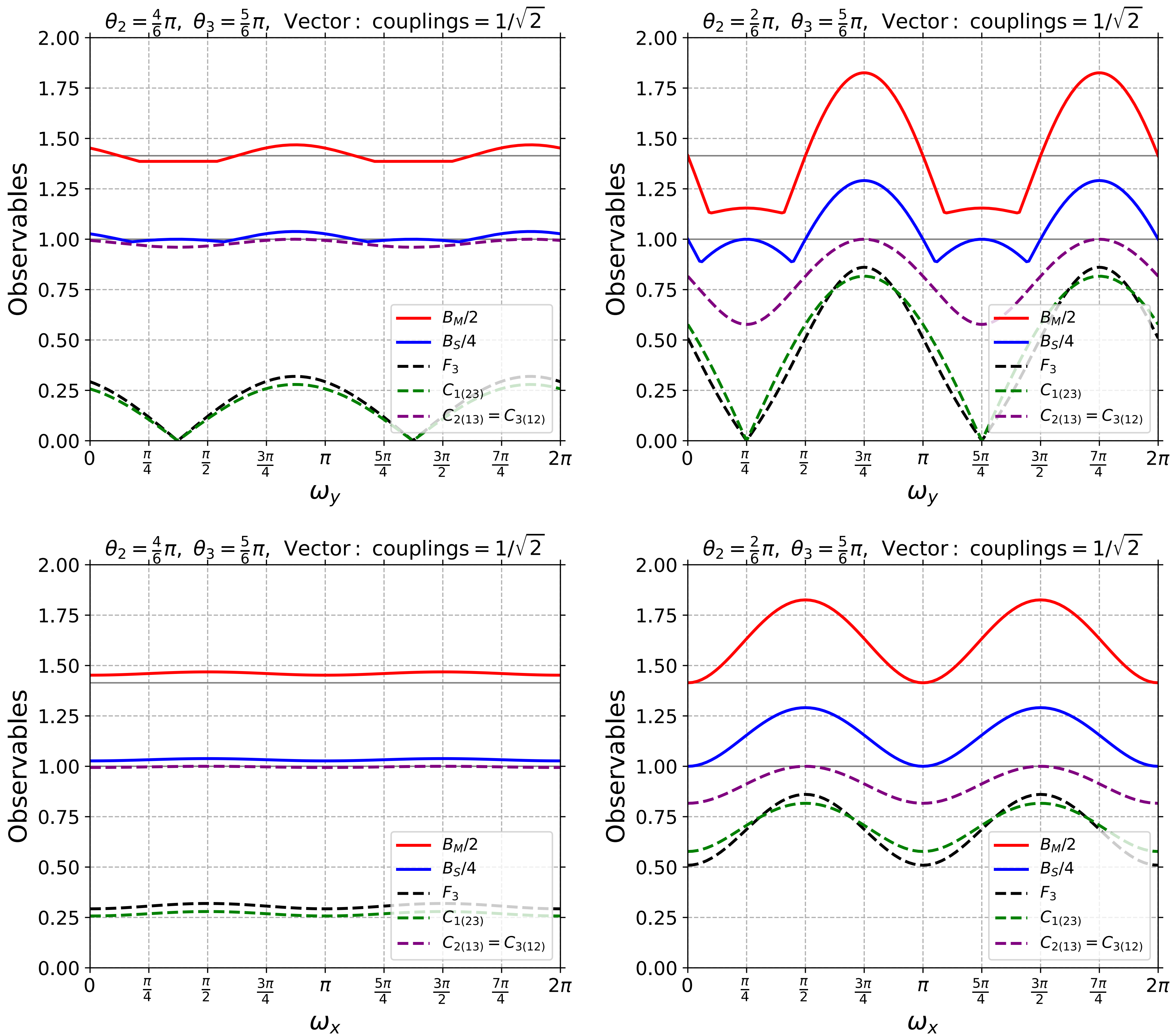
Hybrid (**L**ocal-Nonlocal) **R**eal theories:

$$P(abc|XYZ) = \sum_\lambda q_\lambda P_\lambda(ab|XY) P_\lambda(c|Z) + \sum_\mu q_\mu P_\mu(ac|XZ) P_\mu(b|Y) + \sum_v q_v P_v(bc|YZ) P_v(a|X)$$

$$\longrightarrow \langle \mathcal{B}_\mathrm{S} \rangle_\mathrm{HLR} \leq 4 \quad \langle \mathcal{B}_\mathrm{S} \rangle_\mathrm{QM} \leq 4\sqrt{2}$$

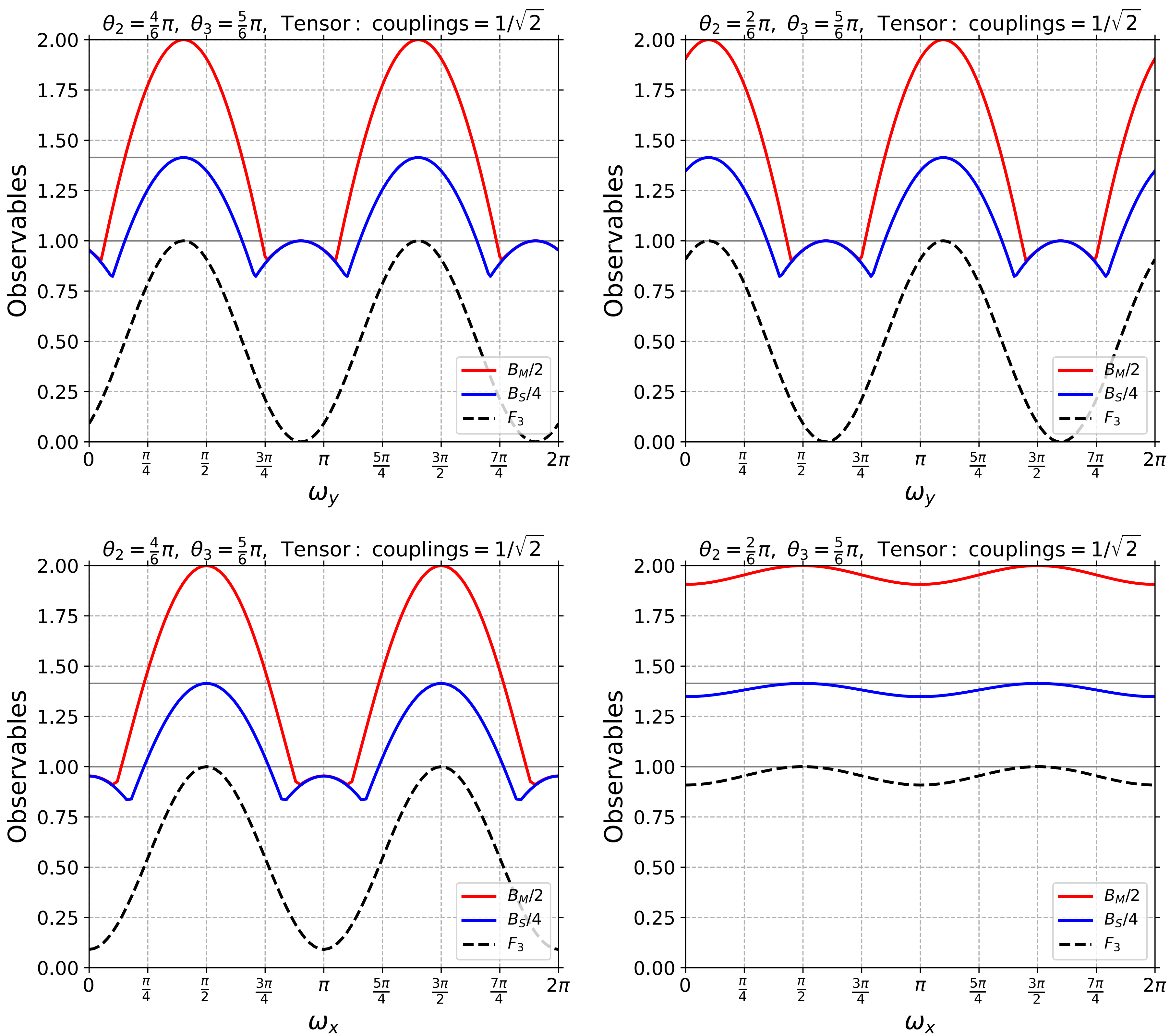
Svetlichny ineq

Nonlocality for Vector



[KS, Spannowsky, Horodecki, *in progress*]

Nonlocality for Tensor

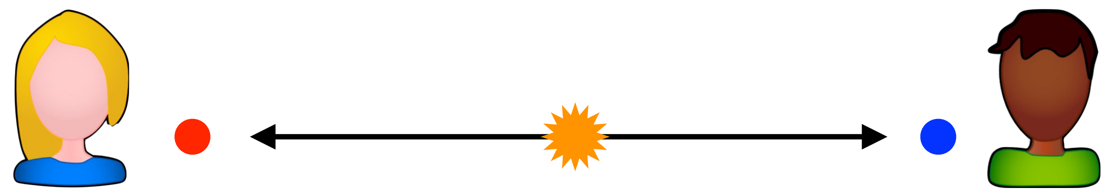


[KS, Spannowsky, Horodecki, *in progress*]

measures axis: $x \in \{\vec{n}\}$

outcome: $a \in \{-1, +1\}$

probability for a: $p_A(a\,|\,x)$



$y \in \{\vec{n}\}$

$b \in \{-1, +1\}$

$p_B(b\,|\,y)$

Any data is expressed by the *joint distribution* $p(a, b\,|\,x, y)$

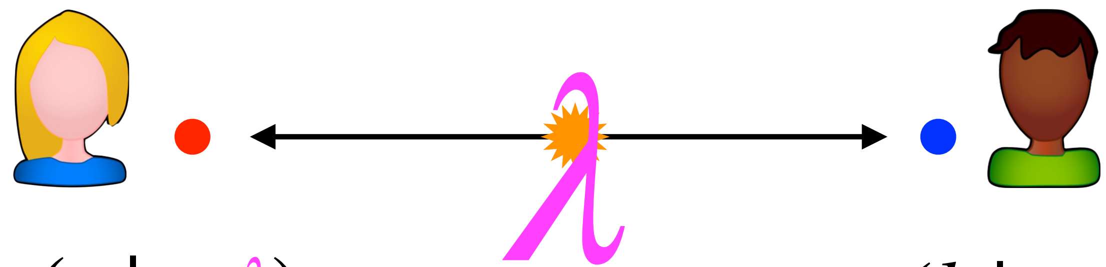
⇒ Models describing the experiment can be classified by possible forms of $p(a, b\,|\,x, y)$

measures axis: $x \in \{\vec{n}\}$

outcome: $a \in \{-1, +1\}$

probability for a: $p_A(a\,|\,x)$

q_λ



$p_A(a\,|\,x, \lambda)$ $\qquad$ $p_B(b\,|\,y, \lambda)$

$y \in \{\vec{n}\}$

$b \in \{-1, +1\}$

$p_B(b\,|\,y)$

Any data is expressed by the *joint distribution* $\quad p(a, b\,|\,x, y)$

⇒ Models describing the experiment can be classified by possible forms of $p(a, b\,|\,x, y)$

probability for λ

Local theories

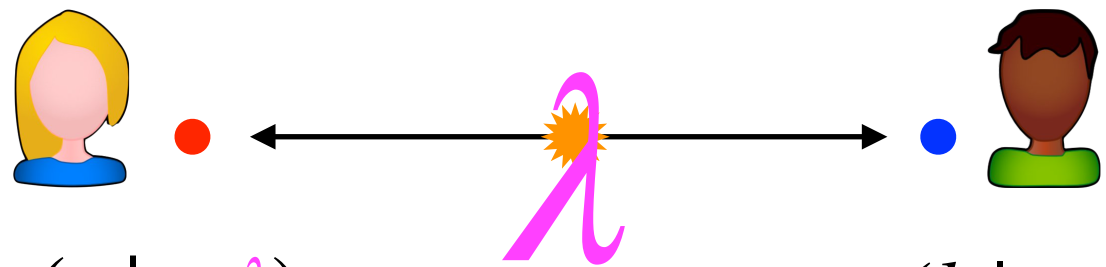
$$p_L(a, b\,|\,x, y) = \sum_\lambda q_\lambda \cdot p_A(a\,|\,x, \lambda) \cdot p_B(b\,|\,y, \lambda)$$

measures axis: $x \in \{\vec{n}\}$

outcome: $a \in \{-1, +1\}$

probability for a: $p_A(a \mid x)$

q_λ



$y \in \{\vec{n}\}$

$b \in \{-1, +1\}$

$p_B(b \mid y)$

$p_A(a \mid x, \lambda)$ $p_B(b \mid y, \lambda)$

Any data is expressed by the *joint distribution* $p(a, b \mid x, y)$

⇒ Models describing the experiment can be classified by possible forms of $p(a, b \mid x, y)$

probability for λ

Local theories $$p_L(a, b \mid x, y) = \sum_\lambda q_\lambda \cdot p_A(a \mid x, \lambda) \cdot p_B(b \mid y, \lambda)$$

density operator

Quantum Mechanics $$p_Q(a, b \mid x, y) = \mathrm{Tr}\left[\rho_{AB}\left(M_{a|x} \otimes M_{b|y}\right)\right]$$

For projective measurement:

$$M_{a|x} = |a_x\rangle\langle a_x|$$

$$\hat{s}_x |a_x\rangle = a_x |a_x\rangle$$

measures axis: $x \in \{\vec{n}\}$

outcome: $a \in \{-1, +1\}$

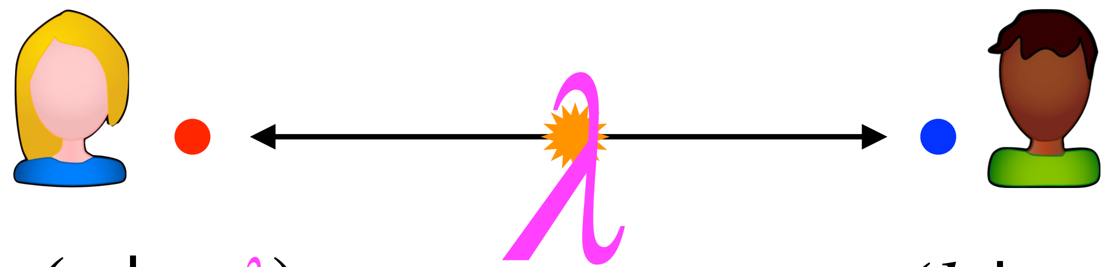
probability for a: $p_A(a\,|\,x)$

q_λ

$y \in \{\vec{n}\}$

$b \in \{-1, +1\}$

$p_B(b\,|\,y)$



$p_A(a\,|\,x, \lambda)$ $\quad$ $p_B(b\,|\,y, \lambda)$

Any data is expressed by the ***joint distribution*** $\quad p(a, b\,|\,x, y)$

⇒ Models describing the experiment can be classified by possible forms of $p(a, b\,|\,x, y)$

probability for λ

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For projective measurement:

$$M_{a|x} = |a_x\rangle\langle a_x|$$

$$\hat{s}_x\,|a_x\rangle = a_x\,|a_x\rangle$$

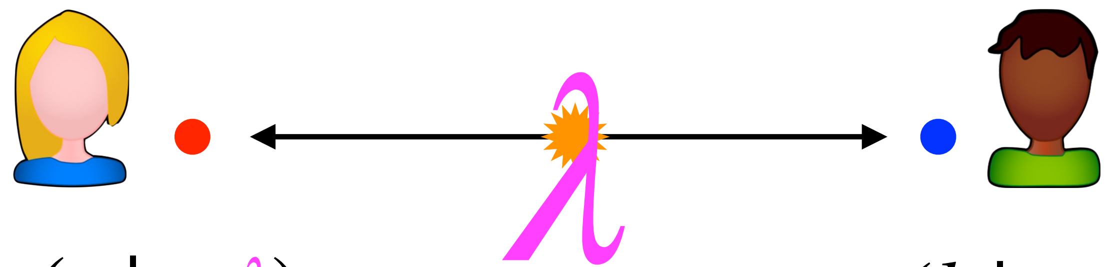
For separable quantum states:

$$\rho_{AB} = \sum_\lambda q_\lambda \cdot \rho_A^\lambda \otimes \rho_B^\lambda \Longrightarrow p_{Q_{\mathrm{sep}}}(a, b\,|\,x, y) = \sum_\lambda q_\lambda \cdot \mathrm{Tr}\left[\rho_A^\lambda M_{a|x}\right] \cdot \mathrm{Tr}\left[\rho_B^\lambda M_{a|x}\right]$$

measures axis: $x \in \{\vec{n}\}$ $\qquad q_\lambda \qquad$ $y \in \{\vec{n}\}$

outcome: $a \in \{-1, +1\}$ $\qquad$ $b \in \{-1, +1\}$

probability for a: $p_A(a|x)$ $\qquad$ $p_B(b|y)$



$p_A(a|x,\lambda)$ $\qquad\qquad$ $p_B(b|y,\lambda)$

Any data is expressed by the ***joint distribution*** $\quad p(a,b|x,y)$

⇒ Models describing the experiment can be classified by possible forms of $p(a,b|x,y)$

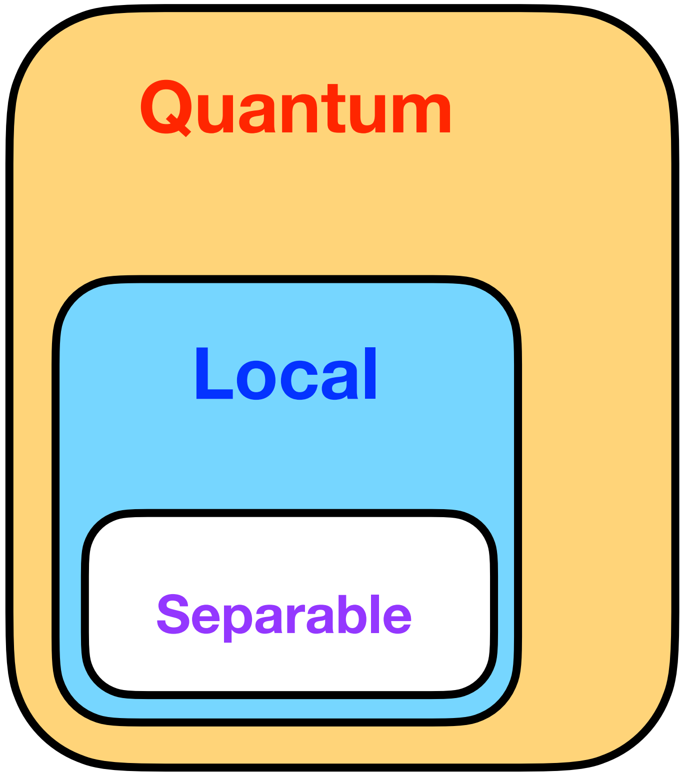
Local theories $\quad p_L(a,b|x,y) = \boxed{\sum_\lambda q_\lambda \cdot p_A(a|x,\lambda) \cdot p_B(b|y,\lambda)}$

density operator

Quantum Mechanics $\quad p_Q(a,b|x,y) = \mathrm{Tr}\left[\rho_{AB}\left(M_{a|x} \otimes M_{b|y}\right)\right]$

For separable quantum states: $\qquad$ **Local**

$$\rho_{AB} = \sum_\lambda q_\lambda \cdot \rho_A^\lambda \otimes \rho_B^\lambda \implies p_{Q_{\mathrm{sep}}}(a,b|x,y) = \boxed{\sum_\lambda q_\lambda \cdot \mathrm{Tr}\left[\rho_A^\lambda M_{a|x}\right] \cdot \mathrm{Tr}\left[\rho_B^\lambda M_{a|x}\right]}$$



Quantum $\supset$ Local $\supset$ Separable

Local theories

$$p_L(a, b\,|\,x, y) = \boxed{\sum_{\lambda} q_{\lambda} \cdot p_A(a\,|\,x, \lambda) \cdot p_B(b\,|\,y, \lambda)}$$

density operator

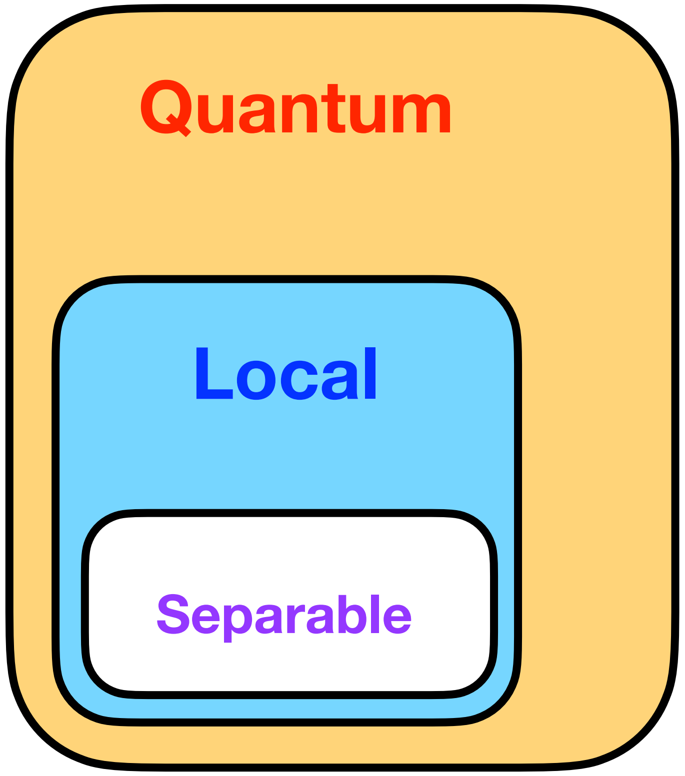
Quantum Mechanics

$$p_Q(a, b\,|\,x, y) = \mathrm{Tr}\left[\rho_{AB}\left(M_{a|x} \otimes M_{b|y}\right)\right]$$

For separable quantum states:

Local

$$\rho_{AB} = \sum_{\lambda} q_{\lambda} \cdot \rho_A^{\lambda} \otimes \rho_B^{\lambda} \implies p_{Q_{\mathrm{sep}}}(a, b\,|\,x, y) = \boxed{\sum_{\lambda} q_{\lambda} \cdot \mathrm{Tr}\left[\rho_A^{\lambda} M_{a|x}\right] \cdot \mathrm{Tr}\left[\rho_B^{\lambda} M_{a|x}\right]}$$



$\text{Quantum} \supset \text{Local} \supset \text{Separable}$

$\text{Nonlocal} \subset \text{Entanglement}$

Local theories

$$p_L(a, b \,|\, x, y) = \boxed{\sum_{\lambda} q_{\lambda} \cdot p_A(a \,|\, x, \lambda) \cdot p_B(b \,|\, y, \lambda)}$$

density operator

Quantum Mechanics

$$p_Q(a, b \,|\, x, y) = \mathrm{Tr}\left[\rho_{AB}\left(M_{a|x} \otimes M_{b|y}\right)\right]$$

For separable quantum states:

Local

$$\rho_{AB} = \sum_{\lambda} q_{\lambda} \cdot \rho_A^{\lambda} \otimes \rho_B^{\lambda} \implies p_{Q_{\mathrm{sep}}}(a, b \,|\, x, y) = \boxed{\sum_{\lambda} q_{\lambda} \cdot \mathrm{Tr}\left[\rho_A^{\lambda} M_{a|x}\right] \cdot \mathrm{Tr}\left[\rho_B^{\lambda} M_{a|x}\right]}$$

- Nonlocal states in QM does not violate causality

$$p(a\,|\,x,y) \equiv \sum_b p(a,b\,|\,x,y)$$

Condition for no causality violation: No-Signalling [Cirel'son(1980), Popescu, Rohrlich(1994)]

$$^\forall a,b,x,x',y,y' \quad \begin{cases} p(a\,|\,x,y) = p(a\,|\,x,y') \\ p(b\,|\,x,y) = p(b\,|\,x',y) \end{cases}$$

Alice's dist. is indep. of Bob's choice for meas. axis

Bob's dist. is indep. of Alice's choice for meas. axis

No-signalling ⊃ Quantum ⊃ Local ⊃ Separable

Bell Inequalities

- Bell-type inequalities (in general) are the inequalities that separate different types of distributions (No-signalling, Quantum, Local).

- Define the correlator $C_{xy} = \langle A_x B_y \rangle \equiv \sum_{a,b} abp(a, b \,|\, x, y)$

- CHSH inequality [Clauser-Horne-Shimony-Holt(1969)]

For $a, b \in \{\pm 1\},\ x \in \{\mathbf{n}_1, \mathbf{n}_2\},\ y \in \{\mathbf{e}_1, \mathbf{e}_2\}$

$$S_{\mathrm{CHSH}} \equiv C_{\mathbf{n}_1,\mathbf{e}_1} + C_{\mathbf{n}_1,\mathbf{e}_2} + C_{\mathbf{n}_2,\mathbf{e}_1} - C_{\mathbf{n}_2,\mathbf{e}_2}$$

$$S_{\mathrm{CHSH}} \leq \begin{cases} 2 & \text{Local theories [CHSH(1969)]} \\ 2\sqrt{2} & \text{Quantum Mechanics [Tsirelson(1987)]} \\ 4 & \text{No-signalling [Popescu, Rohrlich(1994)]} \end{cases}$$