Special Unitary Groups in Particle Physics

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Special thanks to Adobe photoshop

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Contents

1. Definitions

- Consider a set G and a binary operation $*$. Group is a pair of G and $*$ such that satisfies the following statements.
  1. Closure: $\forall a, b \in G, a * b \in G$
  2. Associativity: $\forall a, b, c \in G, (a * b) * c = a * (b * c)$
  3. Identity: $\forall a \in G, \exists e$ s.t. $a * e = e * a = a$
  4. Inverse: $\forall a \in G, \exists a'$ s.t. $a * a' = a' * a = e$
- In addition, a group which satisfies the following condition is called abelian group.
  5. Commutativity: $\forall a, b \in G, a * b = b * a$

1. Definitions

- A matrix is called to be unitary if its Hermitian conjugate and inverse are identical.
  i.e. U is unitary $\equiv U^{-1} = U^{\dagger}$
- For the eigenvalues of a unitary matrix, their absolute value is 1. The proof is given below.

$$Uv = \lambda v, (Uv)^{\dagger} = v^{\dagger}U^{\dagger} = \bar{\lambda}v^{\dagger}$$
$$v^{\dagger}v = \lambda\bar{\lambda}v^{\dagger}v, |\lambda|^2 = 1, |\lambda| = 1$$

1. Definitions

- The set of $n \times n$ unitary matrices whose determinant is 1 forms a group with general matrix multiplication. (Please check this by yourselves)
- This group is denoted as $\mathrm{SU}(n)$. It is an abbreviation for 'Special Unitary', where special means that the determinant is 1.

1. Definitions

- An exponential of a square matrix A is defined by a Taylor series as follows.

$$\exp A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$

- A logarithm of a square matrix A is the matrix S that satisfies the following.

$$\exp S = A$$

S is denoted as $\ln A$.

2. Properties of SU(2)

- Let us consider $U \in \mathrm{SU}(2)$ and its components. Note that a_{ij} and b_{ij} are real.

$$U = \begin{pmatrix} a_{11} + b_{11}i & a_{12} + b_{12}i \\ a_{21} + b_{21}i & a_{22} + b_{22}i \end{pmatrix}$$

- To be unitary, U satisfies the following.

$$U^{-1} = \frac{1}{\det U} \begin{pmatrix} a_{22} + b_{22}i & -a_{12} - b_{12}i \\ -a_{21} - b_{21}i & a_{11} + b_{11}i \end{pmatrix} = \begin{pmatrix} a_{11} - b_{11}i & a_{21} - b_{21}i \\ a_{12} - b_{12}i & a_{22} - b_{22}i \end{pmatrix} = U^\dagger$$

- Since $\det U = 1$, we obtain the following by comparison.

$$a_{11} = a_{22}, b_{11} = -b_{22}, a_{12} = -a_{21}, b_{12} = -b_{21}$$

2. Properties of SU(2)

- In addition, $\det U = 1$ gives the following.

$$a_{11}{}^2 + b_{11}{}^2 - a_{12}{}^2 - b_{12}{}^2 = 1$$

- To sum up, a general 2×2 complex matrix has a degree of freedom 8. However, it is reduced to 4 due to the unitary condition; the determinant condition reduces one more.
- Therefore, SU(2) can be parametrized with three real parameters.

2. Properties of SU(2)

- Cayley – Klein parametrization is one of the possible parametrizations with three real parameters (ξ, ζ, η) as follows. (Please check that the parametrization generates SU(2))

$$U = \begin{pmatrix} e^{i\xi}\cos\eta & e^{i\zeta}\sin\eta \\ -e^{-i\zeta}\sin\eta & e^{-i\xi}\cos\eta \end{pmatrix}$$

2. Properties of SU(2)

- Every unitary matrix U can be expressed as an exponential of a Hermitian matrix H with a coefficient i as follows.

$$U = \exp(iH)$$

- It is easy to show that $\exp(iH)$ is unitary.
- It is known that a unitary matrix U can be unitary diagonalized as follows: $U = VDV^\dagger$. For all eigenvalues of a unitary matrix, their absolute value is 1. Therefore, D can be expressed as follows.

$$D = \begin{pmatrix} e^{i\theta_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{i\theta_n} \end{pmatrix}$$

2. Properties of SU(2)

- It is enough to set H as follows to satisfy $U = \exp iH$:

$$H = V \begin{pmatrix} \theta_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \theta_n \end{pmatrix} V^\dagger$$

  (Please check that H is Hermitian and produces U by the given exponential.)

- Especially, since $\det U = 1$ for $U \in \mathrm{SU}(2)$, $\mathrm{tr}\, H = \mathrm{tr}\,(\ln U) = \ln(\det U) = 0$.

2. Properties of SU(2)

- Consider an one-dimensional space translation as follows. Note that the space translation requires a parameter a which represents how much translation occurs.

$$T(a) = \exp\left(-\frac{i\hat{p}a}{\hbar}\right)$$

- In this structure, there is a multiplication of the parameter a and an Hermitian operator $\hat{p}$ in the exponent. The Hermitian operator (or matrix) which gives a certain symmetry operation with a combination with a parameter is called a generator of the symmetry operation. For instance, $\hat{p}$ is a generator of a space translation.

2. Properties of SU(2)

- Now, let us compute the generator of $\mathrm{SU}(2)$ based on Cayley – Klein parameters. Note that there are three generators (the same with the number of parameters) and they are traceless.
- Let the generators be H_ξ, H_ζ and H_η, which correspond to the Cayley – Klein parameters respectively. Then, $U \in \mathrm{SU}(2)$ will be represented as follows.

$$U = \begin{pmatrix} e^{i\xi}\cos\eta & e^{i\zeta}\sin\eta \\ -e^{-i\zeta}\sin\eta & e^{-i\xi}\cos\eta \end{pmatrix} = \exp\left(i\left(\xi H_\xi + \zeta H_\zeta + \eta H_\eta\right)\right)$$

2. Properties of SU(2)

- To obtain the generators, it is enough to differentiate the Cayley – Klein parametrization with respect to a certain parameters and set the parameters to be zero.

$$H_\xi = -i\left.\frac{\partial U}{\partial \xi}\right|_{\xi,\zeta,\eta=0} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_z$$

$$H_\zeta = -i\left.\frac{\partial U}{\partial \zeta}\right|_{\xi,\zeta,\eta=0} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_x$$

$$H_\eta = -i\left.\frac{\partial U}{\partial \eta}\right|_{\xi,\zeta,\eta=0} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_y$$

2. Properties of SU(2)

- The generators of SU(2) are Pauli matrices!
- SU(2) is well – known mathematical structure that describes the rotation of spin ½ particle. Considering the following rotation, it is natural.

$$R_{\hat{n}}(\theta) = \exp\left(-\frac{i\vec{S}\cdot\hat{n}\theta}{\hbar}\right)$$

2. Properties of SU(2)

- In fact, SU(2) is a Lie group. Roughly, Lie group is a group that a differentiation can be defined well.
- Roughly, in a Lie group, its generators play a role as derivatives or tangent vectors.
- The generators forms a new algebraic structure, which is known as a Lie algebra. Every Lie group has corresponding Lie algebra.

2. Properties of SU(2)

- For SU(2), its Lie algebra is given by the following. Note that the generators T_i are a half of the Pauli matrices and ε_{ijk} is a Levi – Civita symbol. (Please check the following relations)

$$[T_i, T_j] = i\varepsilon_{ijk}T_k$$

- In general, the Lie algebra of a certain Lie group can be written as below, with general coefficients f_{ijk} which are called structure constants.

$$[T_i, T_j] = if_{ijk}T_k$$

Note: $i = \sqrt{-1}$

2. Properties of SU(2)

- Lie algebra determines a unique structure of the corresponding Lie group. Therefore, analyzing the Lie algebra for a certain Lie group is critical.
- In SU(2), its Lie algebra shows the commutation relation of a spin angular momentum. The commutation relation is, in fact, a foundation to construct the elegant mathematical structure of the spin angular momentum.

3. SU(2) in the Particle Physics

- Yang – Mills theory say that, the electroweak interaction shows $SU(2) \times U(1)$ gauge symmetry.
- The generators of the gauge symmetry play a role as a gauge charge, which is a time – invariant quantum number.
- In the view of Quantum Field Theory, each gauge charge must be described as a field. The field gives a gauge boson.
- The number of generators in the gauge symmetry is equal to the number of gauge bosons.
  e.g. 1 (photon) for electromagnetic force / 3 ($W^{\pm}$, Z^0) for weak force / 8 (8 gluons, explained in the next section) for strong force

3. SU(2) in the Particle Physics

- In electroweak interaction, a weak isospin (denoted as T_i) and a weak hypercharge (denoted as Y_W) are the generators of SU(2) and U(1), respectively.
- Weak isospin is analogous to the spin angular momentum of a non – relativistic electron. Therefore, weak isospin can be described by T^2 and T_3, as the non – relativistic spin angular momentum is described by S^2 and S_z.

3. SU(2) in the Particle Physics

- For the spin angular momentum, S_z is a z – directional component. However, T_3 of weak isospin does not have spatial meaning. Instead, T_3 is defined by a component that gives a charge.
- Explicitly, the charge Q can be calculated as follows.

$$Q = T_3 + \frac{1}{2} Y_W$$

This formula is called Gell-mann – Nishijima formula.

3. SU(2) in the Particle Physics

- SU(2) is used not only in Yang – Mills theory but also the theory of strong interactions.
- Since the mass of up quark and down quark are similar, it is possible to construct a spin – like quantity. This quantity is called isospin which is denoted as I.
- Isospin is described by I^2 and I_3 like the weak isospin and the spin angular momentum.

3. SU(2) in the Particle Physics

- Isospin has an approximate SU(2) symmetry. It means the hadron composed with up and down (anti)quarks can be analyzed by spin ½ particle combination. (As we learned in the undergraduate QM course)
- For instance, let us consider mesons (quark + antiquark) that are composed with up or down quarks. If we combine two spin ½ particles, we obtain a triplet state and a singlet state.
- The triplet state corresponds to three pions (π^+, π^0, π^-) and the singlet state corresponds to one eta (η).

3. SU(2) in the Particle Physics

- In the same way, baryon with up and down quarks can be considered (quark + quark + quark). In this case, a quadruplet state and two doublet states are obtained.
- The quadruplet state corresponds to delta (Δ^{++}, Δ^{+}, Δ^{0}, Δ^{-}) and the doublet states correspond to proton and neutron.

4. SU(3) in Particle Physics

- Like SU(2), we can consider SU(3), which is known as the gauge symmetry for the strong interaction.
- 3×3 complex matrix has 18 real parameters. The unitary condition reduces them to 9; the determinant condition reduces one more.
- Therefore, SU(3) can be parametrized with 8 real parameters, and thus has 8 generators.

4. SU(3) in Particle Physics

- The generators of SU(3) are Gell – Mann matrices.

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

4. SU(3) in Particle Physics

- The strong interaction has SU(3) gauge symmetry. In this case, the corresponding gauge charge is a color charge.
- There are three color charges (r, g, b) and three anti – color charges ($\bar{r}, \bar{g}, \bar{b}$). Gluon has one color charge and one anti – color charge, so nine combinations are possible.
- However, one of a possible combination as below is impossible.

$$\frac{1}{\sqrt{3}}\left(r\bar{r} + b\bar{b} + g\bar{g}\right)$$

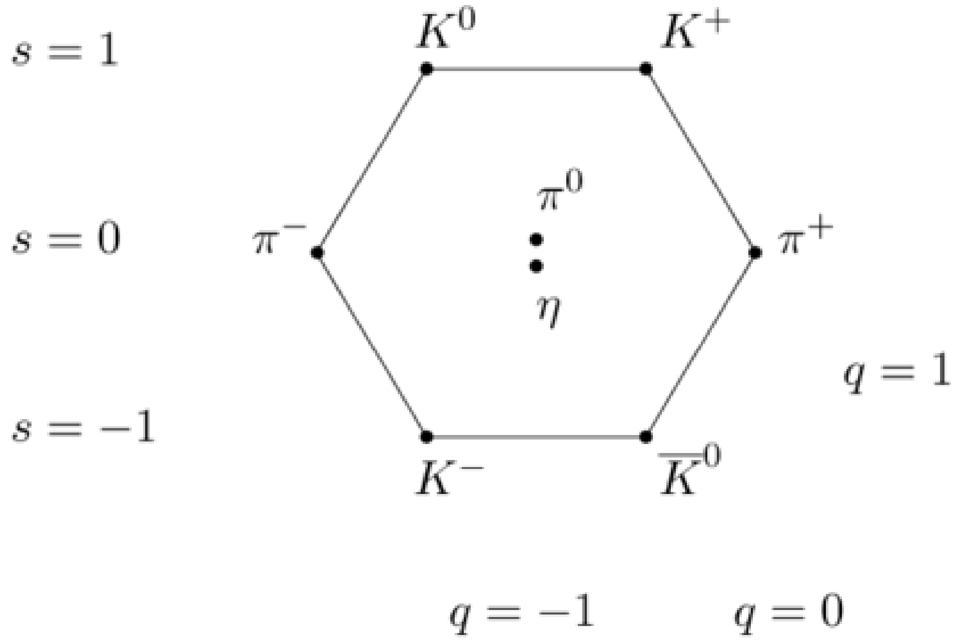
Thus, only 8 independent gluons exists and they correspond to the Gell – Mann matrices.

4. SU(3) in Particle Physics

- It is possible to construct a new isospin that includes not only up and down quarks but also a strange quark.
- In this case, the new isospin has an approximate SU(3) symmetry. By using the same technique in the previous section, it is possible to classify mesons and baryons which are including up, down and strange quarks.
- The schematic of the classification is known as the ‘Eightfold way’, which is named by Murray Gell – Mann.

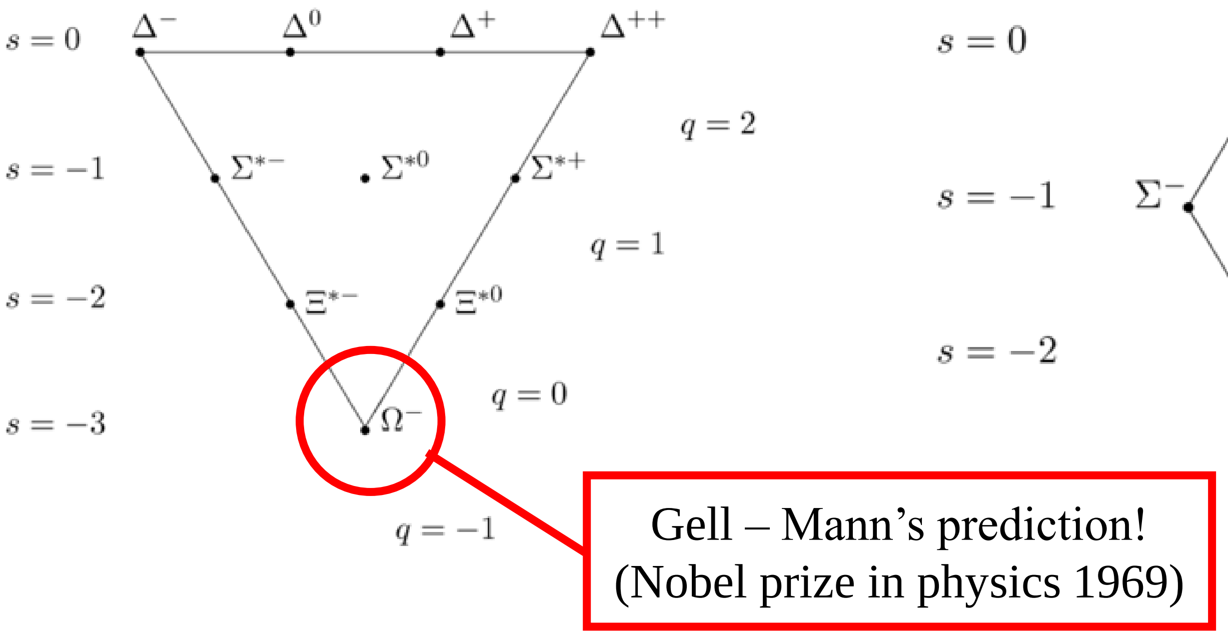
4. SU(3) in Particle Physics

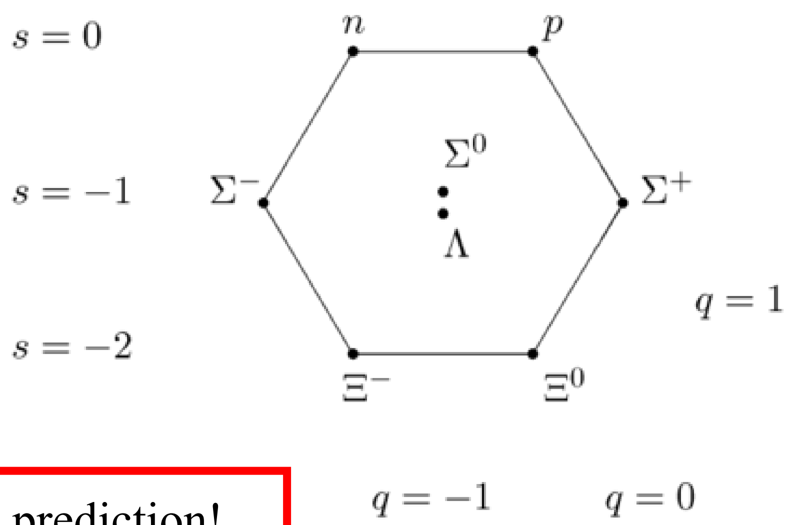
- The 'eightfold way' diagram for mesons is shown below.



4. SU(3) in Particle Physics

- The 'eightfold way' diagrams for baryons are shown below.





5. Q & A