Toric Code and Anyons
CERN Summer Program presentation

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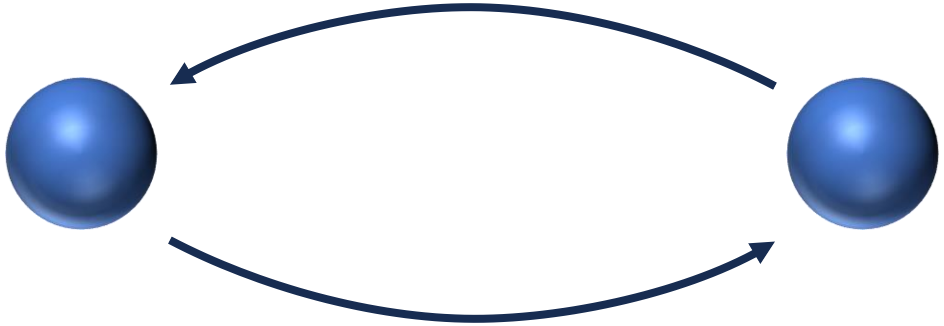
Introduction: anyons

Exchange rules for particles:

$|\psi\rangle' = e^{i\phi}|\psi\rangle,$

where $\phi = \begin{cases} 0, & \text{for bosons,} \\ \pi, & \text{for fermions.} \end{cases}$

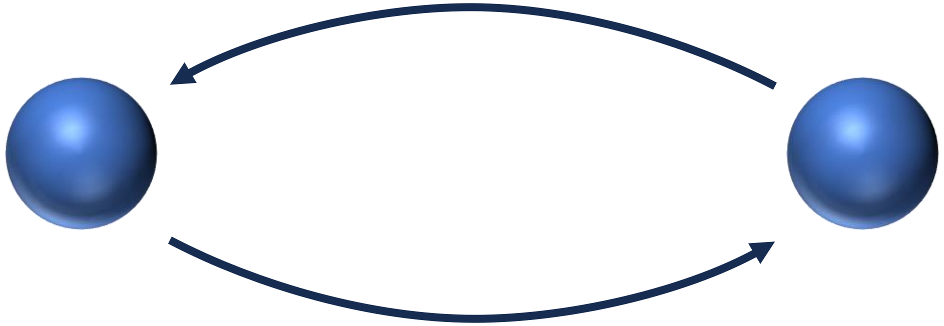


Introduction: anyons

Exchange rules for particles in **3D**:

$|\psi\rangle' = e^{i\phi}|\psi\rangle,$

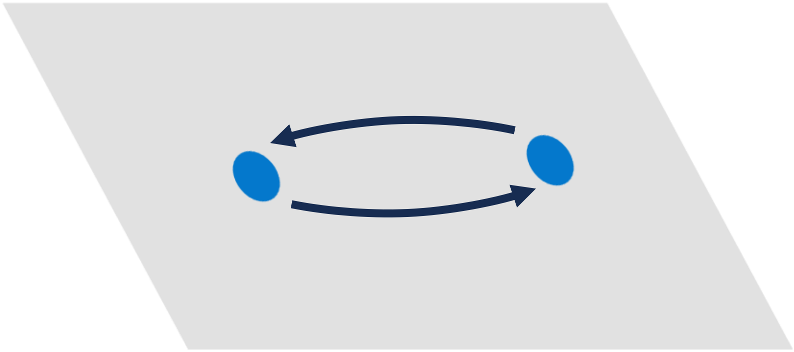
where $\phi = \begin{cases} 0, & \text{for bosons,} \\ \pi, & \text{for fermions.} \end{cases}$



Exchange rules for particles in **2D**:

$|\psi\rangle' = e^{i\phi}|\psi\rangle,$

where $\phi = \begin{cases} 0, & \text{for bosons,} \\ \pi, & \text{for fermions,} \\ \theta, & \text{for anyons.} \end{cases}$



Toric code is a good toy model which shows anyonic behavior of quasiparticles.

Leinaas, J. M. and J. Myrheim (1977). "On the theory of identical particles." Il Nuovo Cimento B (1971-1996) **37(1): 1-23.**

What is Toric Code?

A 2D square lattice with a spin-$\frac{1}{2}$ particle sitting on every edge.

Hamiltonian:

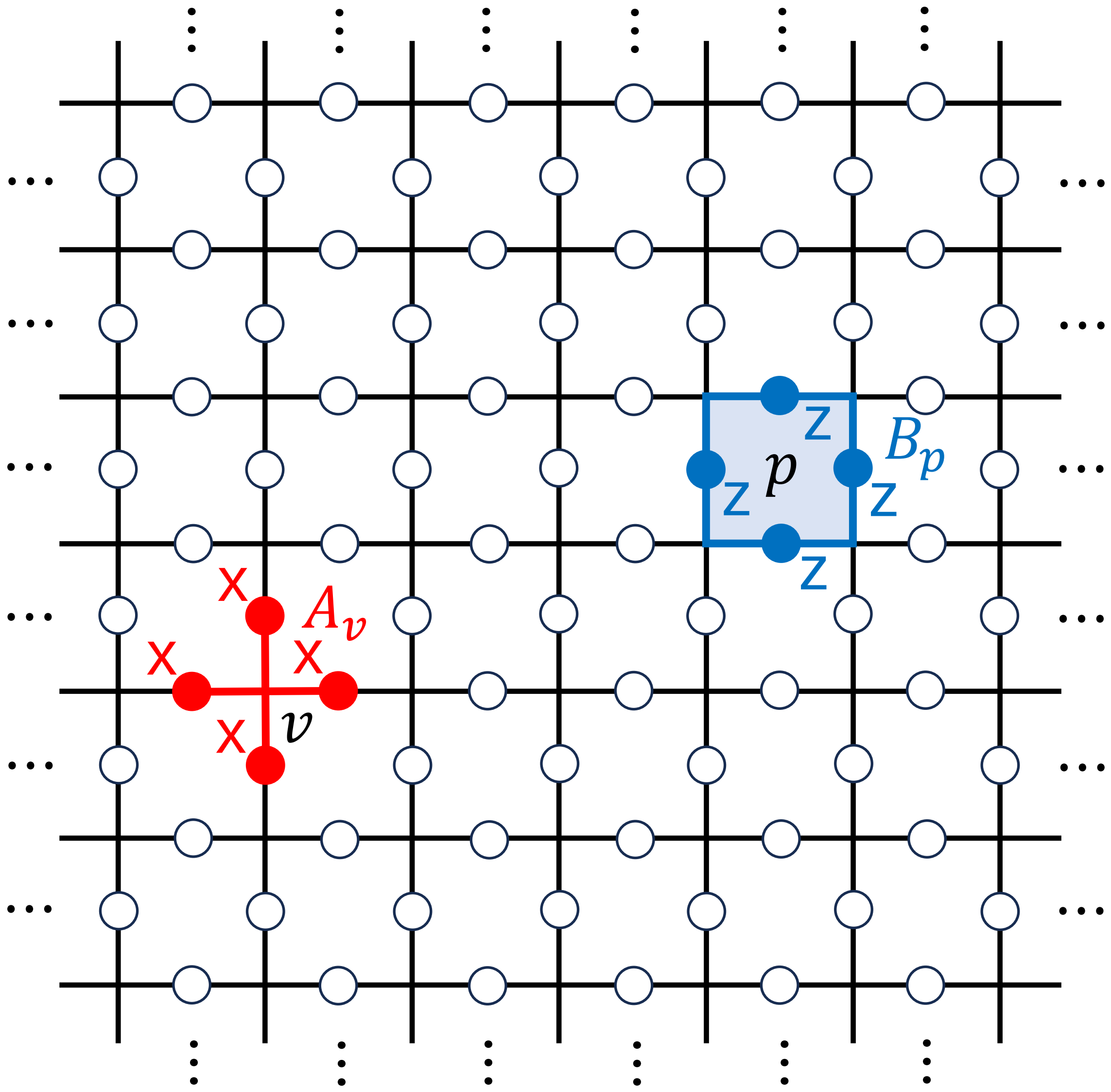
$$H = -\sum_v A_v - \sum_p B_p \,,$$

where v: vertex, p: face.

Stabilizers:

$$A_v = \prod_{j \in *(v)} \sigma_x^j \,, B_p = \prod_{j \in \partial(p)} \sigma_z^j$$

where $*(v)$: sites adjacent to v, $\partial(p)$: sites on the boundary of p.



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Properties of stabilizers

$$H = -\sum_v A_v - \sum_p B_p$$

$$A_v = \prod_{j \in *(v)} \sigma_x^j, B_p = \prod_{j \in \partial(p)} \sigma_z^j$$

Recall the Pauli matrices:

$$\begin{cases} [\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k, \\ \{\sigma_i, \sigma_j\} = 2\delta_{ij}1. \end{cases}$$

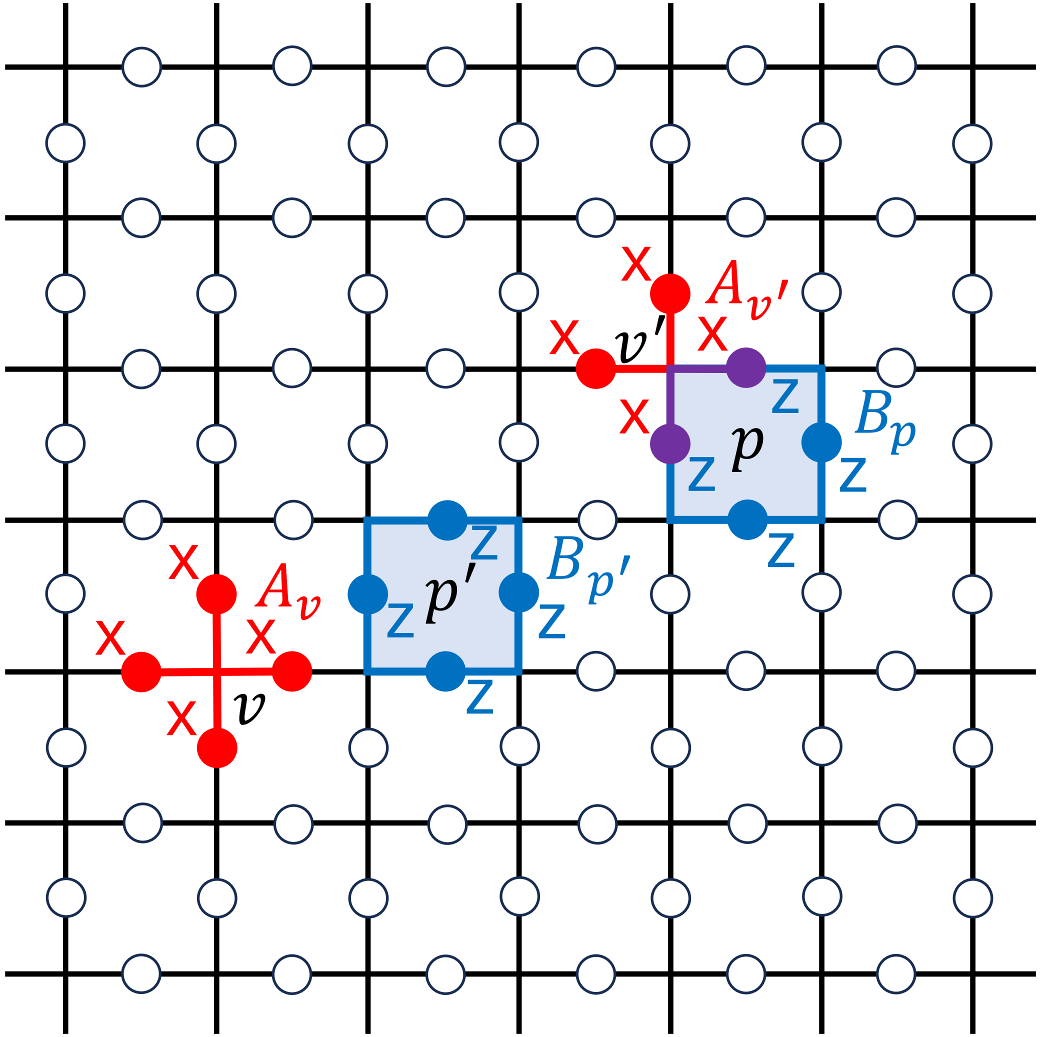
$$\Longrightarrow \sigma_i \sigma_j = \begin{cases} +\sigma_j\sigma_i, \text{if } i = j, \\ -\sigma_j\sigma_i, \text{if } i \neq j. \end{cases}$$

Commutation relations of stabilizers:

$$[A_v, A_{v'}] = [B_p, B_{p'}] = [A_v, B_p] = 0,$$

for any vertices v, v' and faces p, p'.

Eigenvalues: ± 1 for all A_v, B_p.



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Properties of stabilizers

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Under periodic boundary conditions:

# of spin-$\frac{1}{2}$ particles: $2L^2$

# of stabilizers: $\mathrm{N_A} = \mathrm{N_B} = L^2$

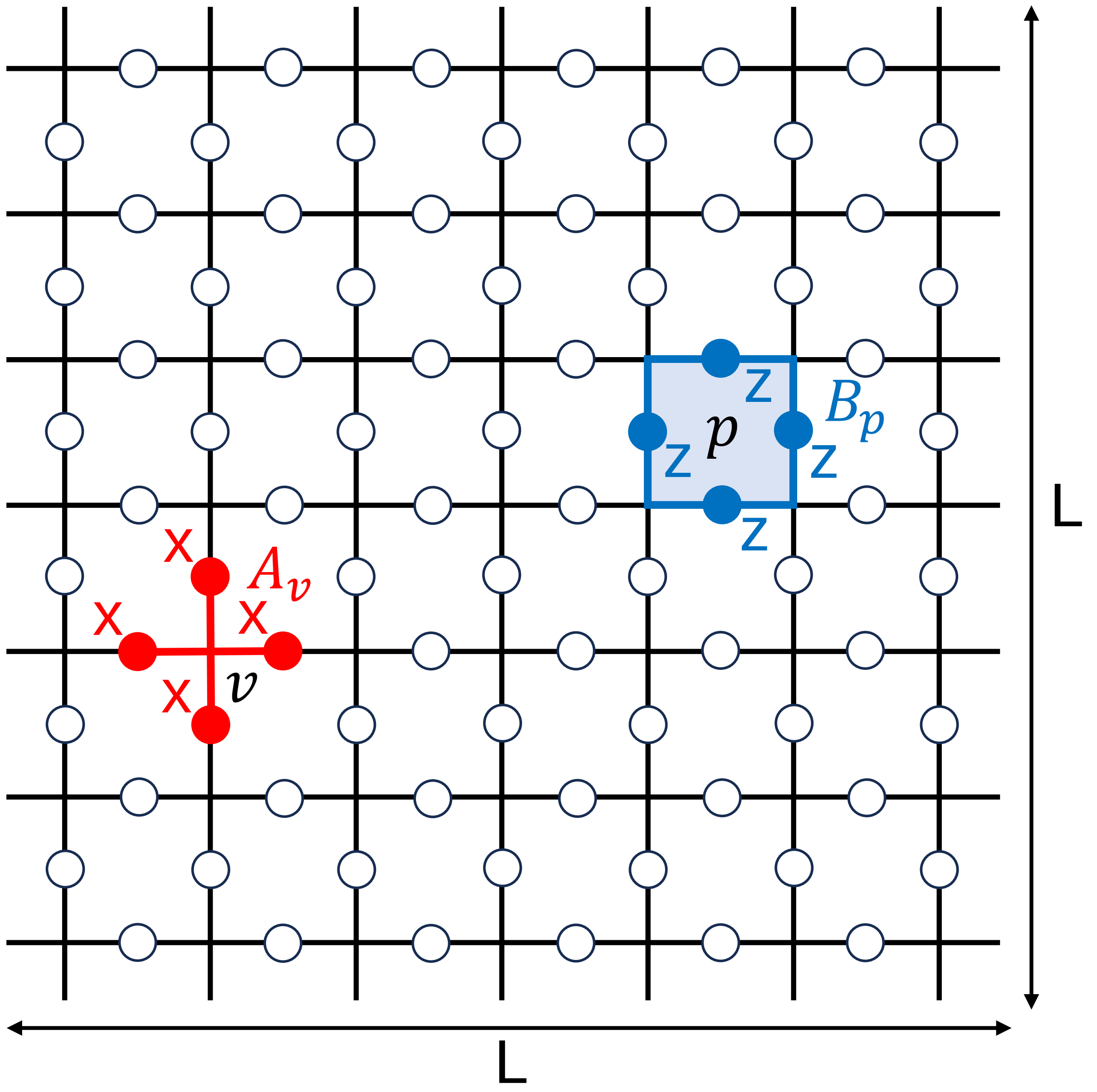
Constraints on stabilizers:

$$\prod_v A_v = \prod_j \left(\sigma_x^j\right)^2 = 1,$$

$$\prod_p B_p = \prod_j \left(\sigma_z^j\right)^2 = 1.$$

# of independent stabilizers:

$$N_A + N_B - 2 = 2(L^2 - 1)$$



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The ground states

Hamiltonian: $H = -\sum_v A_v - \sum_p B_p$.

$\Longrightarrow$ If $\exists$ a state $|\psi\rangle$ such that

$$\begin{cases} A_v|\psi\rangle = +|\psi\rangle \\ B_p|\psi\rangle = +|\psi\rangle \end{cases}$$, for all v, p,

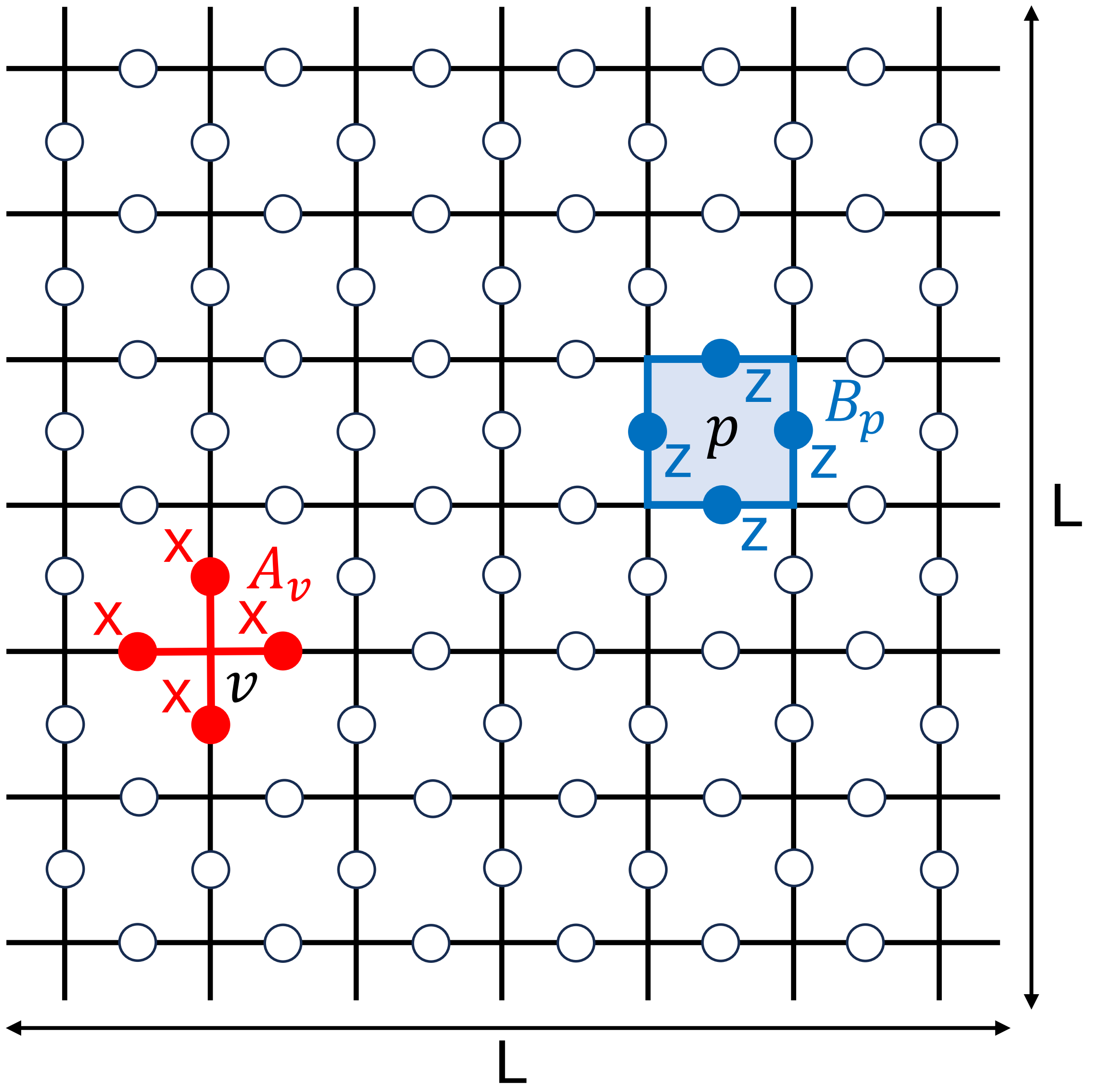
then $|\psi\rangle$ must be a ground state.

Suppose such state $|\psi_{GS}\rangle$ exists.

$\Longrightarrow E_{GS}|\psi_{GS}\rangle = H|\psi_{GS}\rangle = -2L^2|\psi_{GS}\rangle$.

Guess the degeneracy of such ground states: $2^{2L^2-2(L^2-1)} = 4$.

$$H = -\sum_v A_v - \sum_p B_p$$



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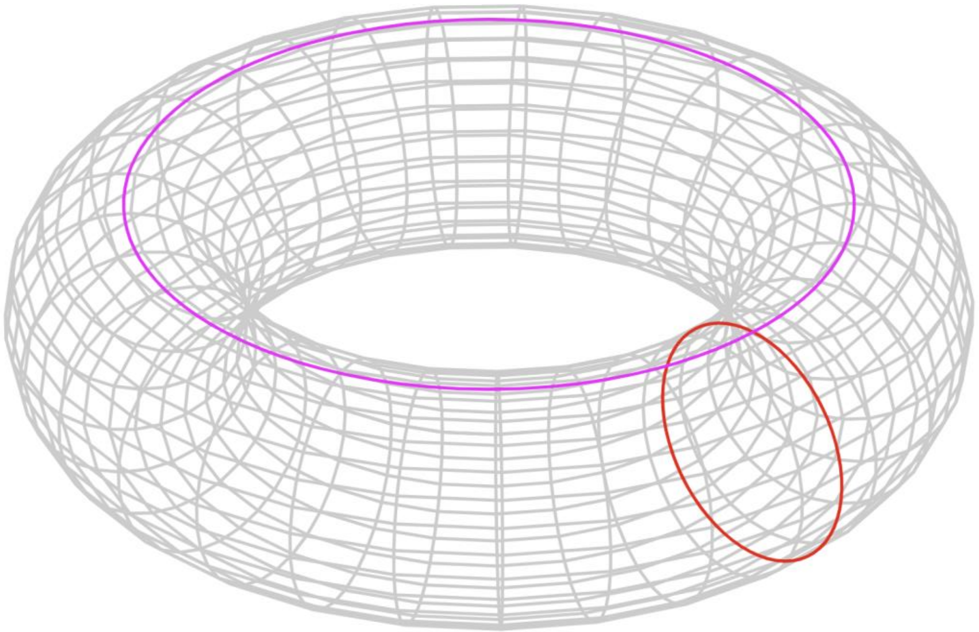
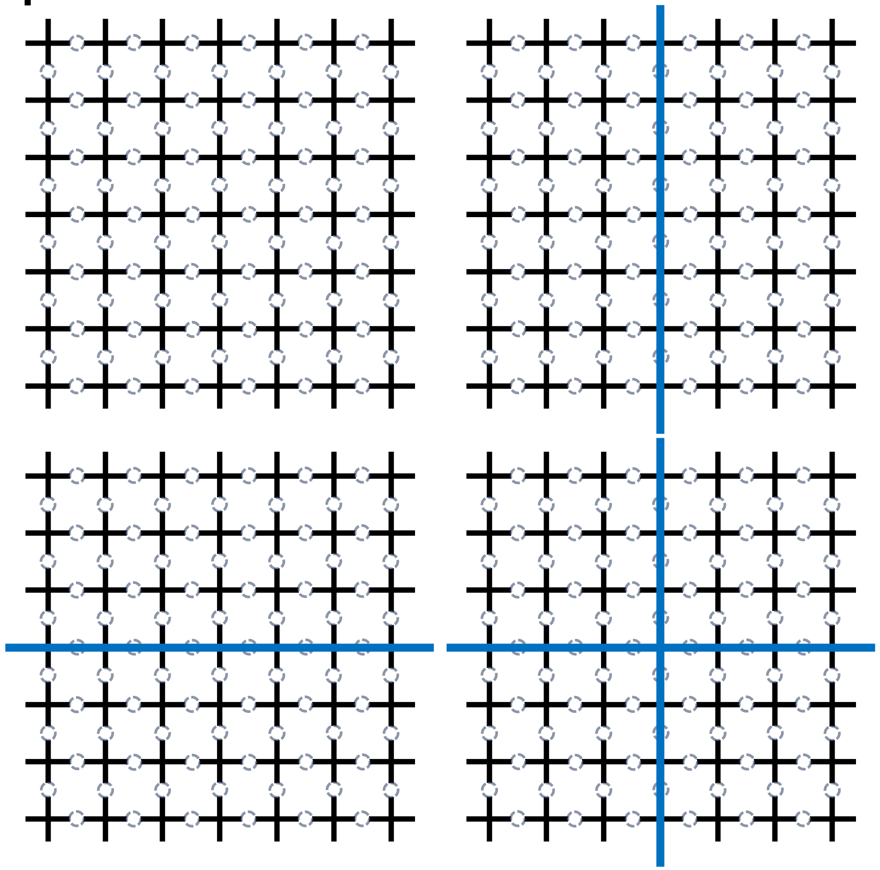
The ground states

The ground state degeneracy is related to the topological property of the system.

For torus, we have two non-contractive loops, which is related to four-fold ground states.





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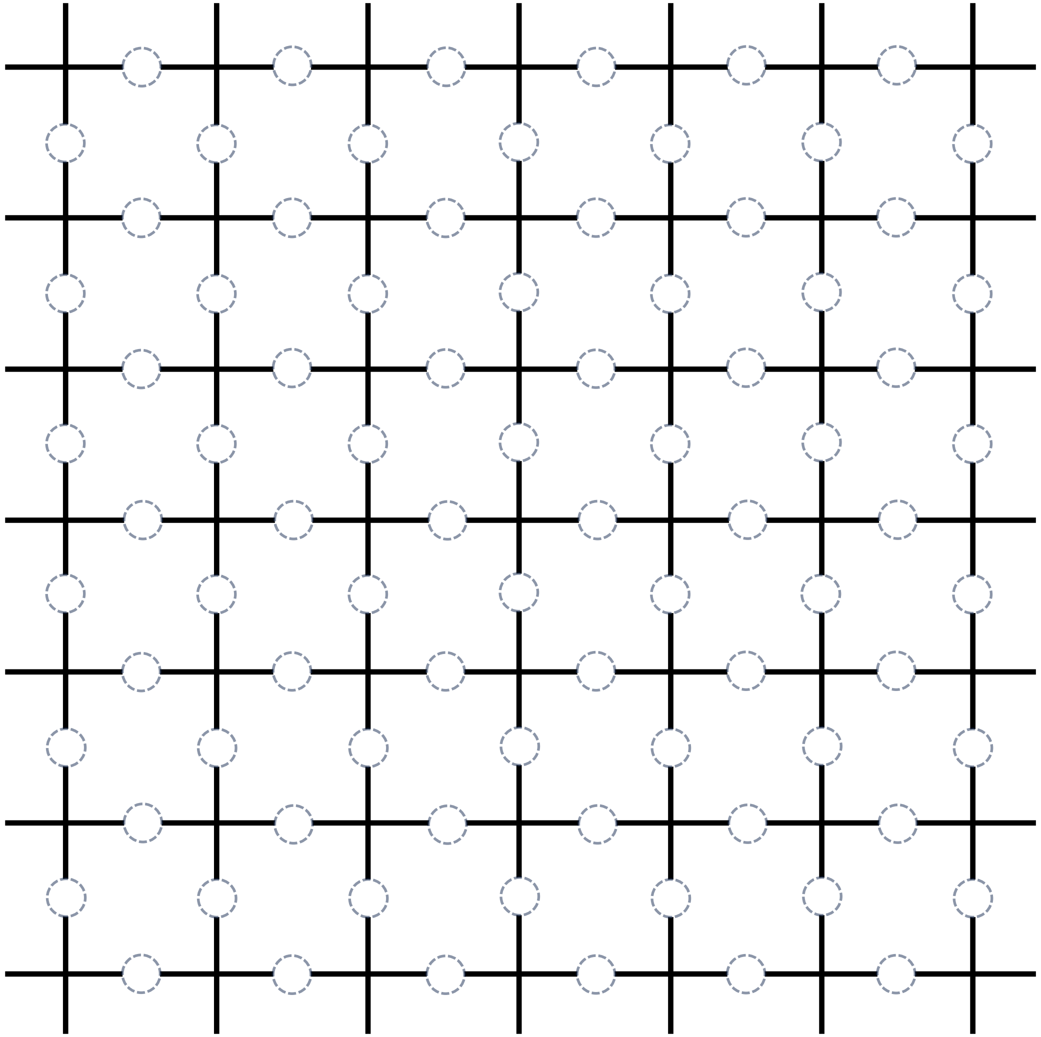
The excited states

$$H = -\sum_v A_v - \sum_p B_p$$

To obtain the excited states, we start from the ground state $|\psi_{GS}\rangle$.



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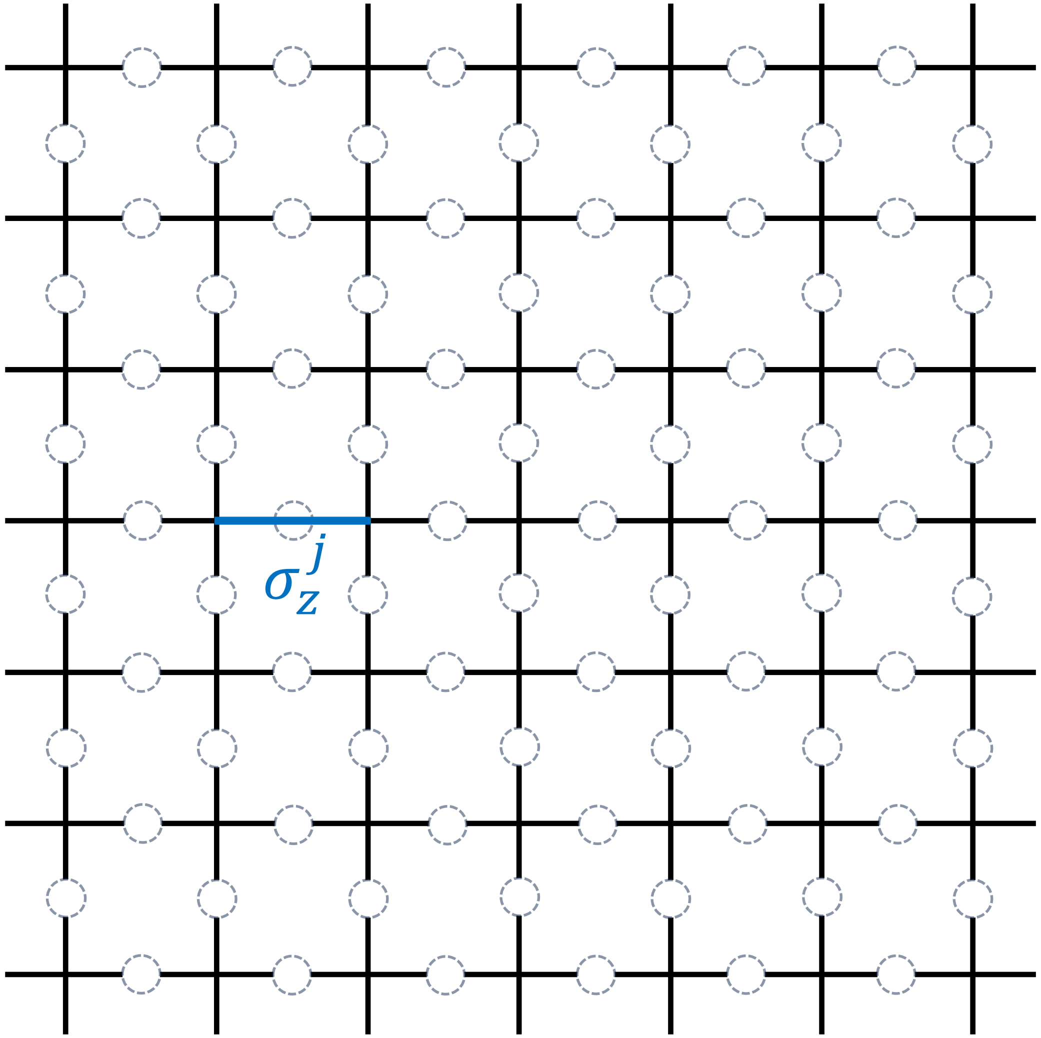

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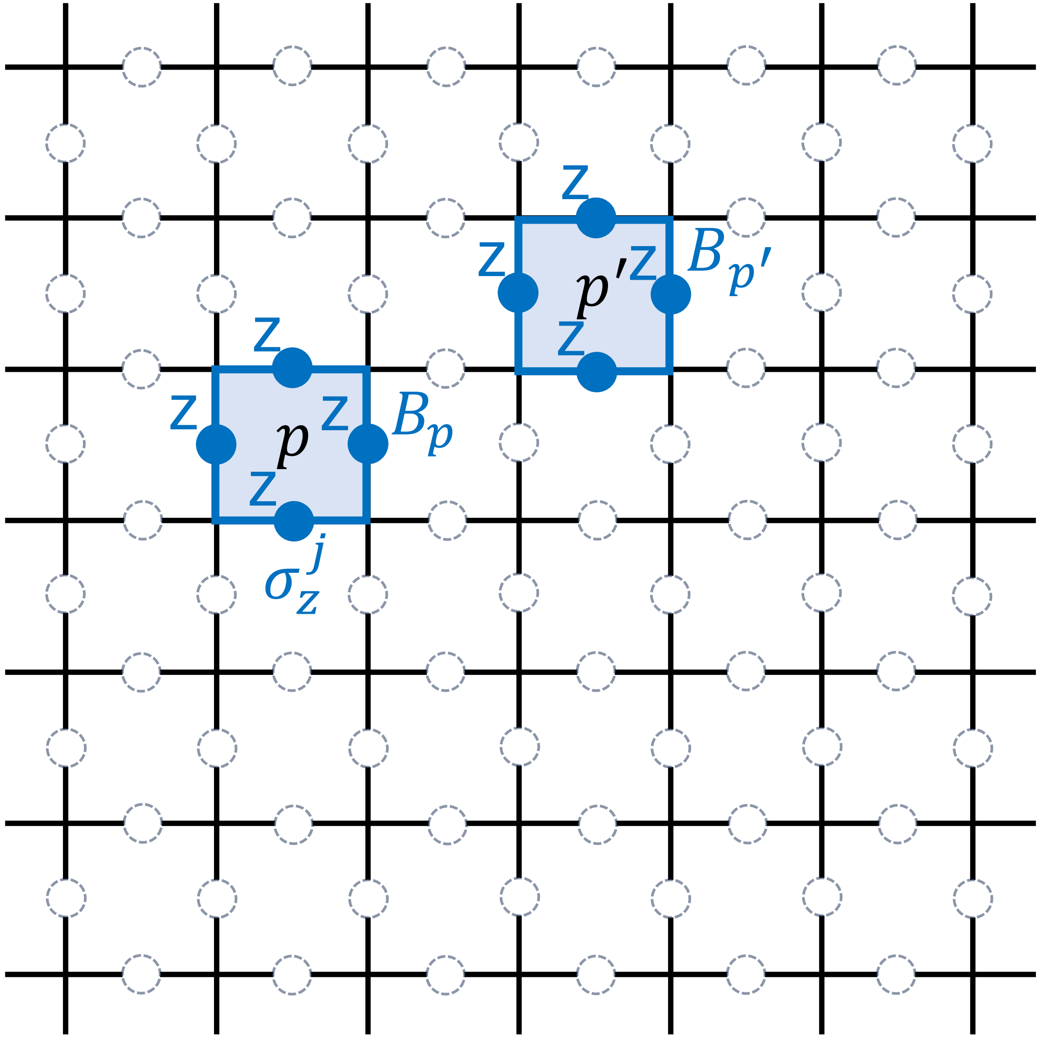

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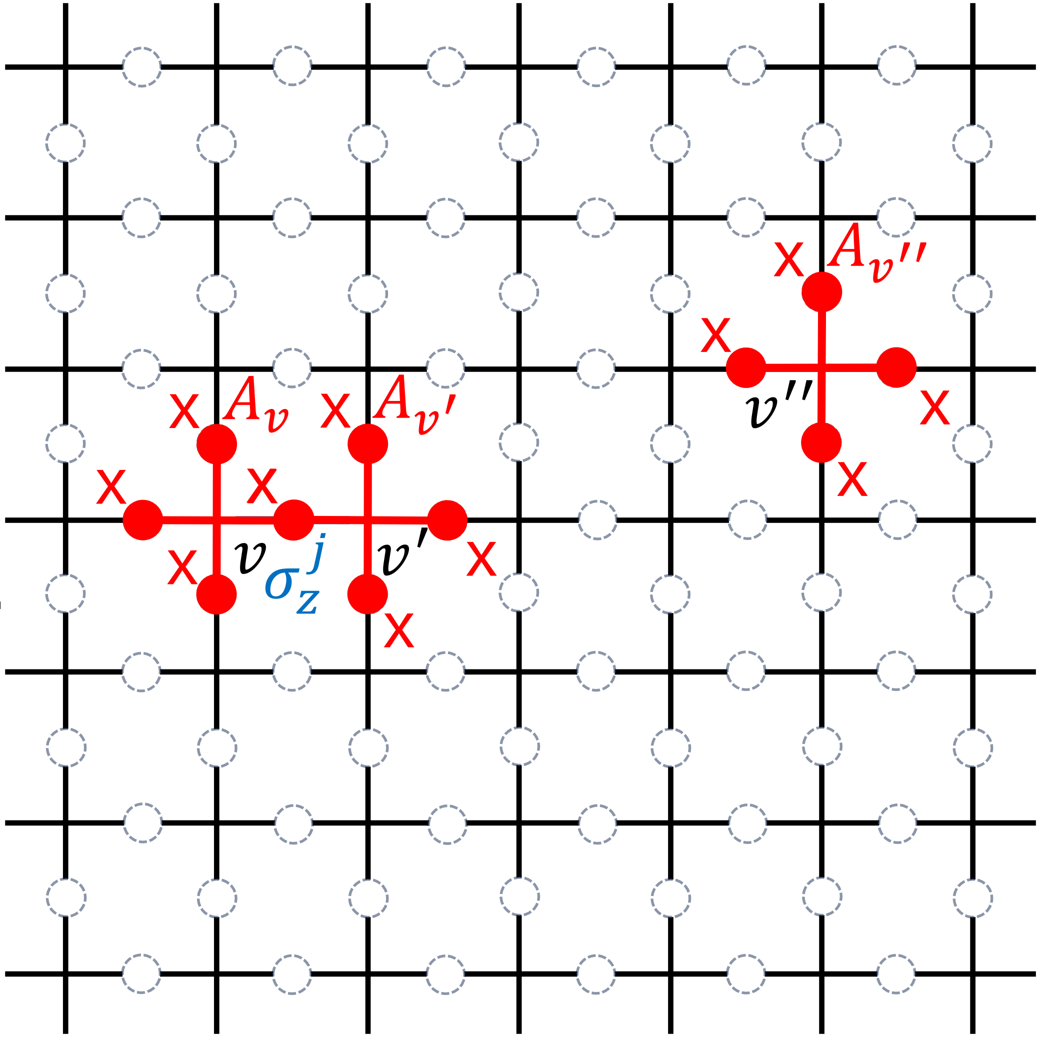

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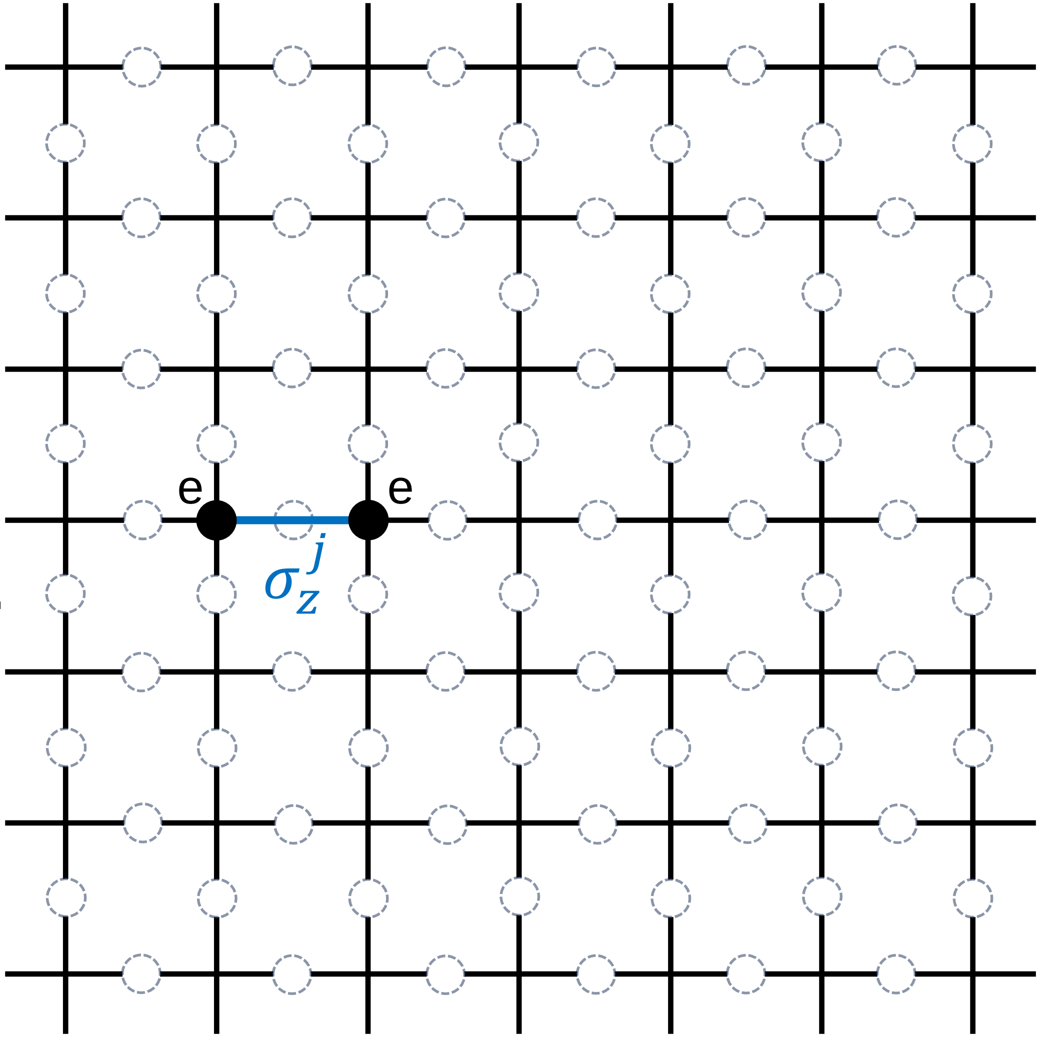
$$\Longrightarrow \mathrm{H}\sigma_z^j|\psi_{GS}\rangle = (-2\mathrm{L}^2+2)\sigma_z^j|\psi_{GS}\rangle.$$



We denote these two excitation modes by “electric defect” e.

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Similarly, consider $\sigma_x^{j'}|\psi_{GS}\rangle$.

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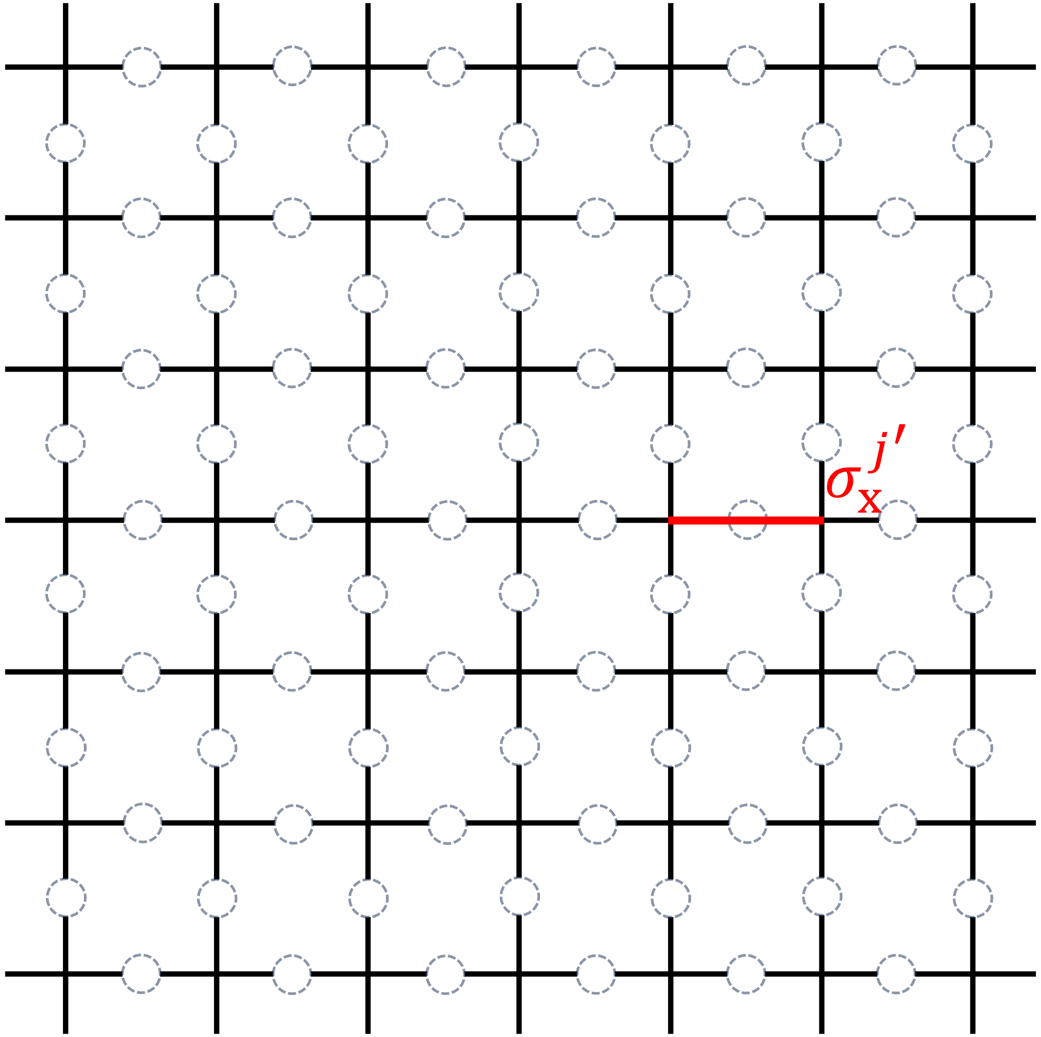

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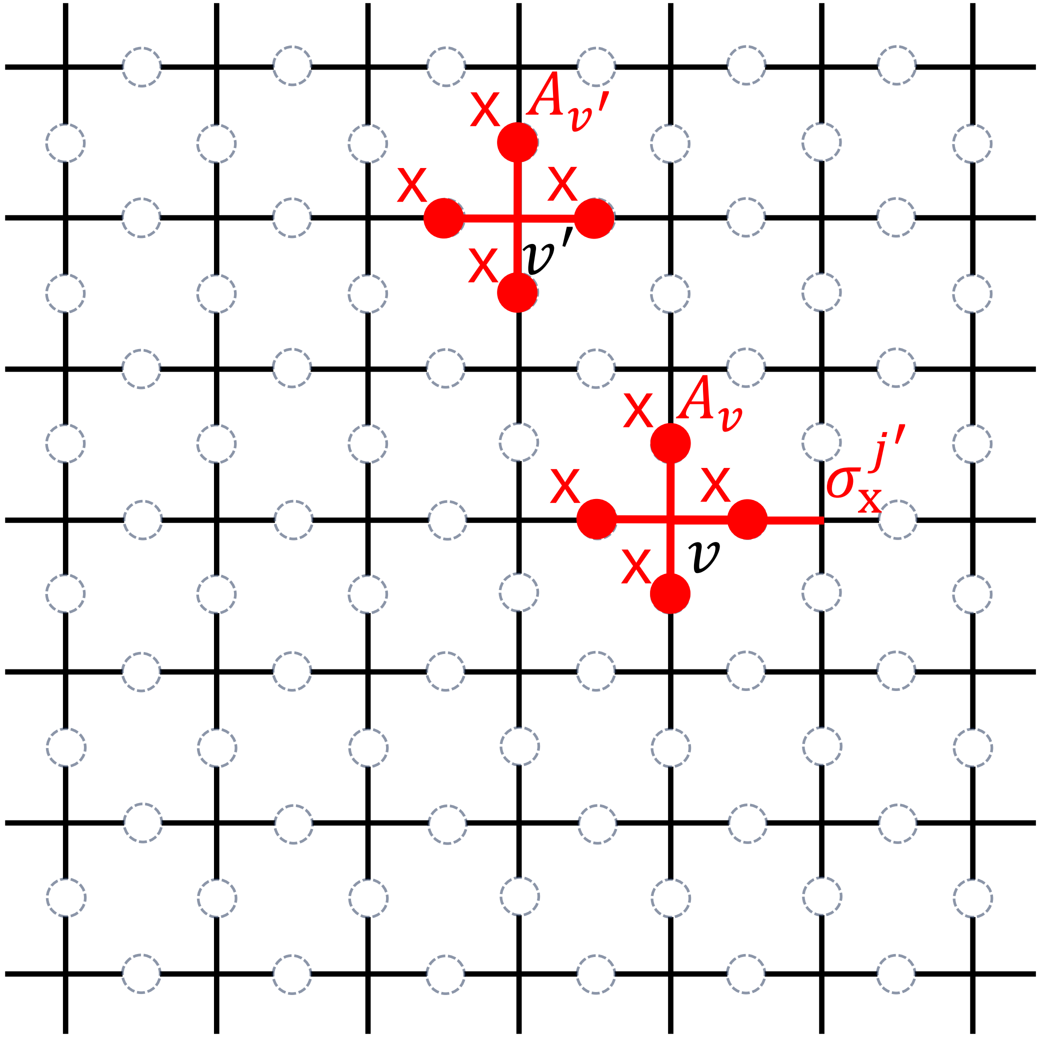

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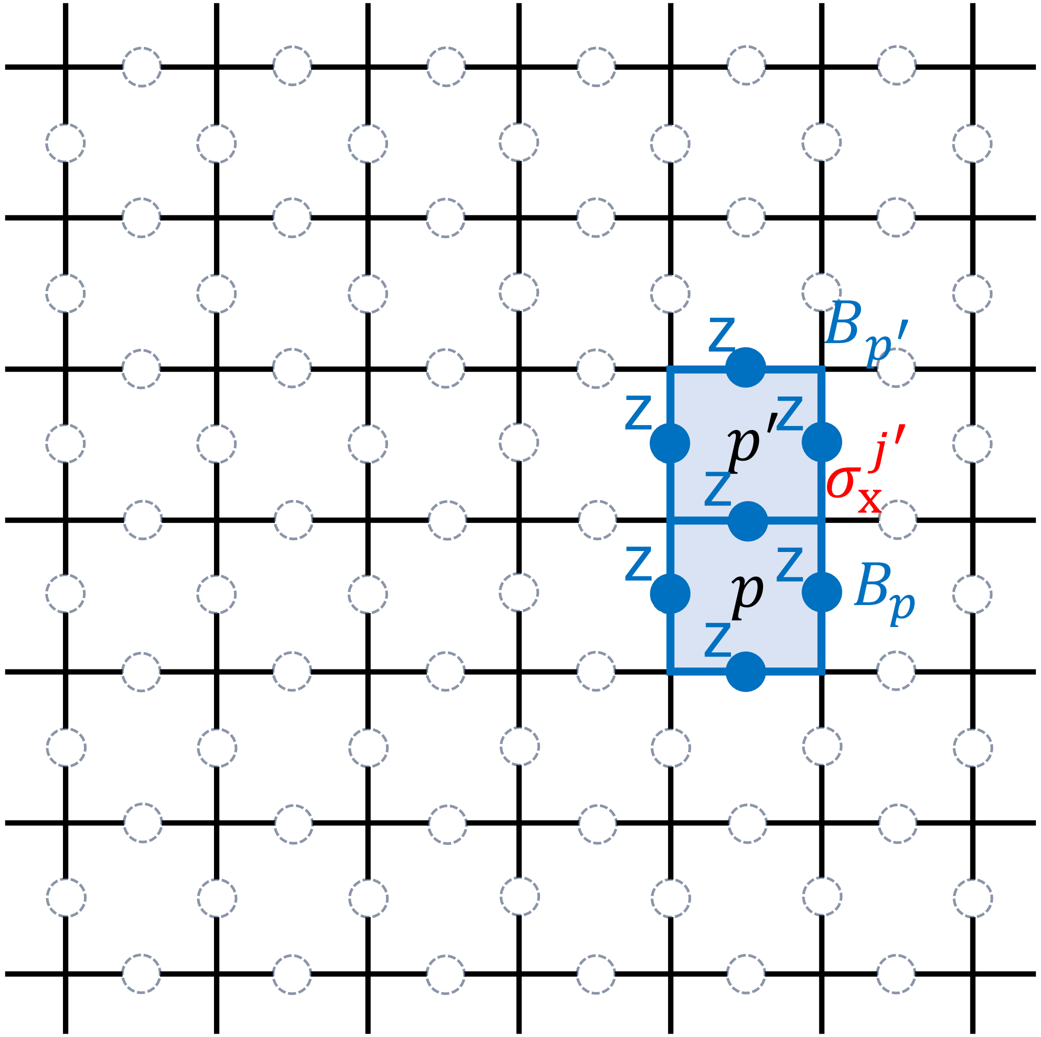

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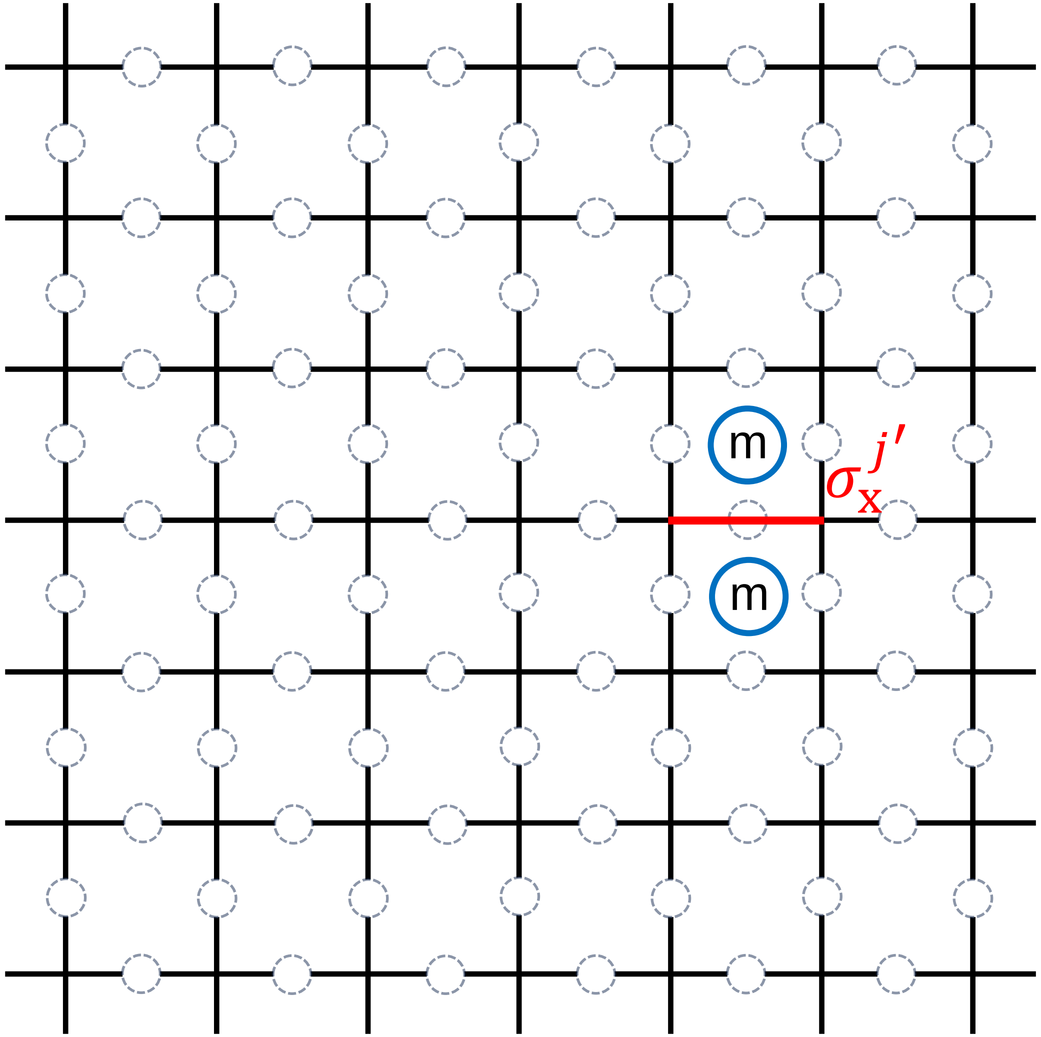
We denote these two excitation modes by "magnetic vortex" m.



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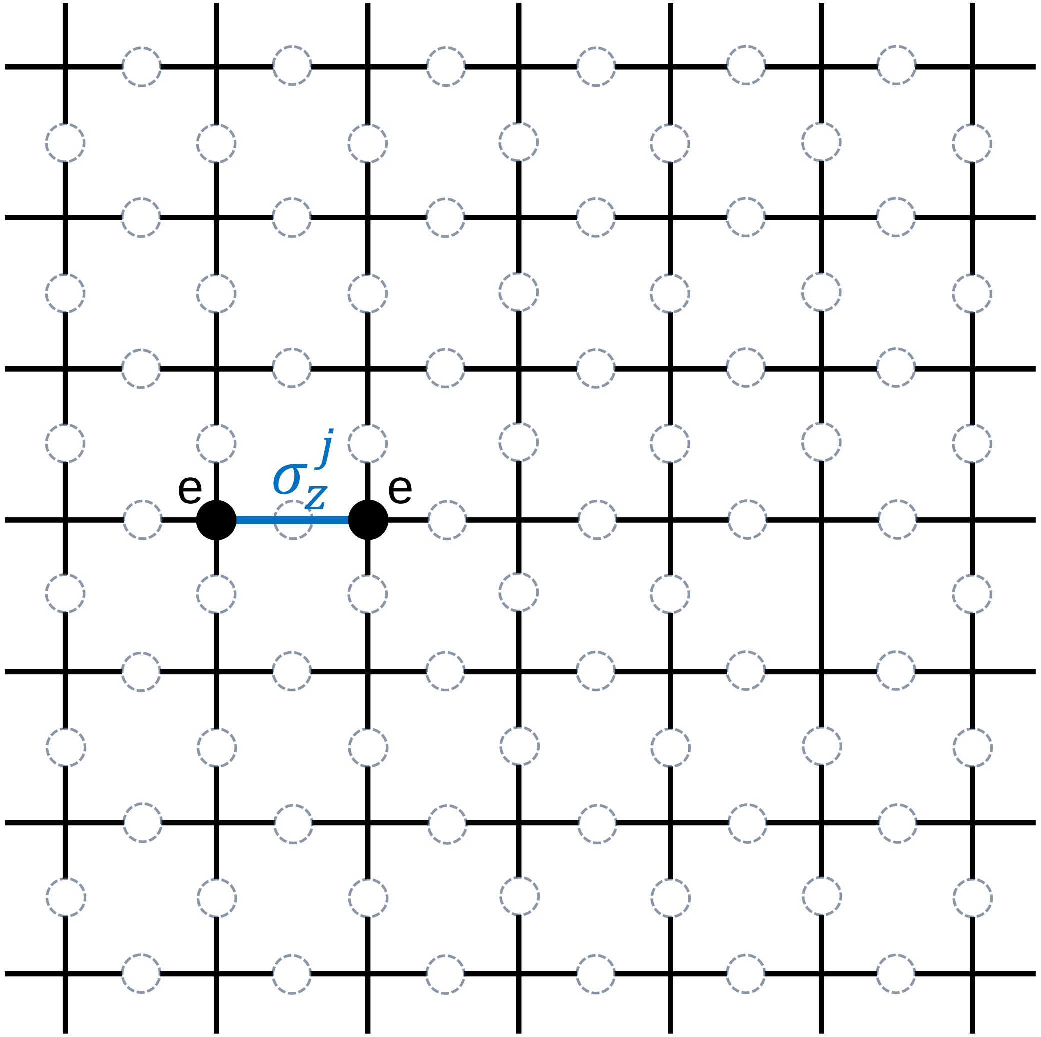
$$H = -\sum_v A_v - \sum_p B_p$$

σ_z^j: creator of two “electric defect” e at both vertices adjacent to the site j.



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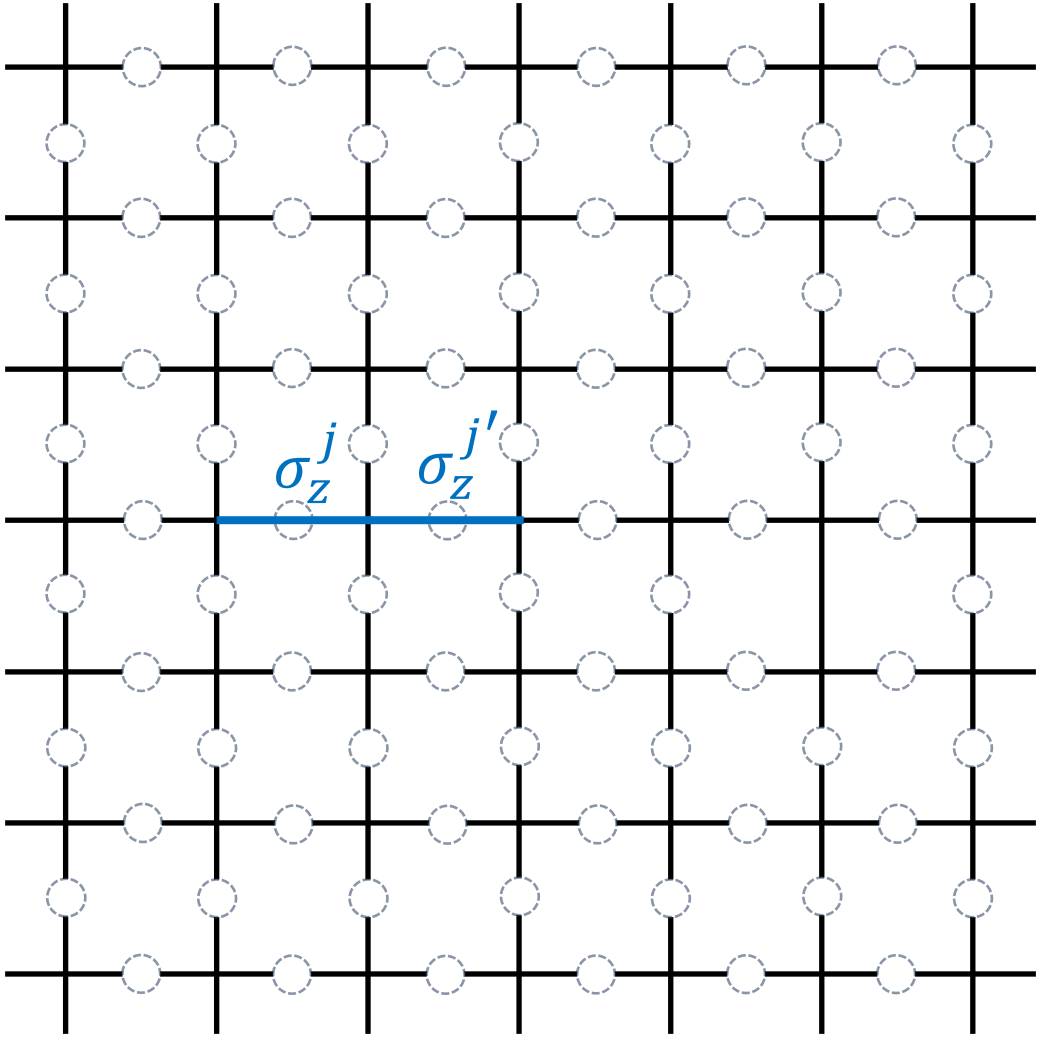

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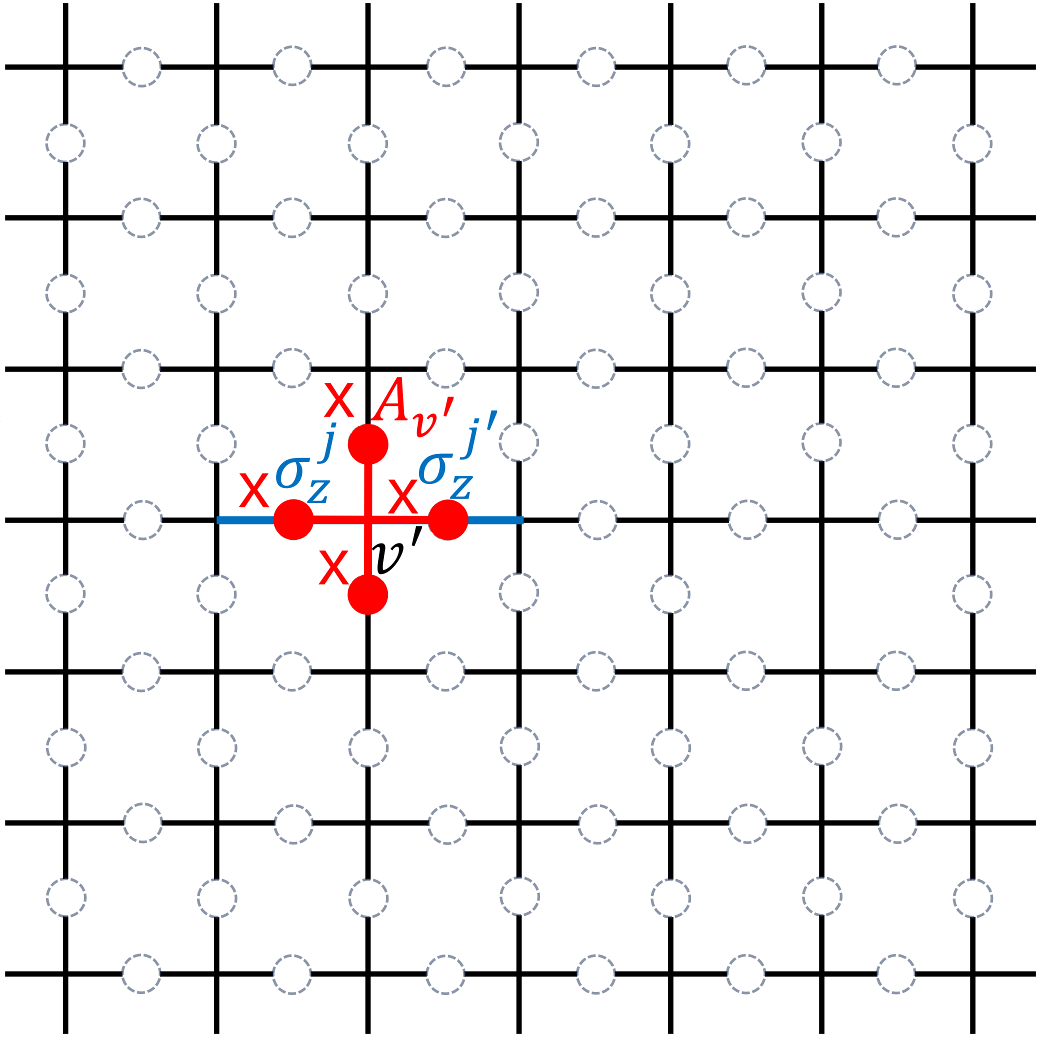
$A_{v'}$ which originally anti-commuted now commute with $\sigma_z^j \sigma_z^{j'}$.



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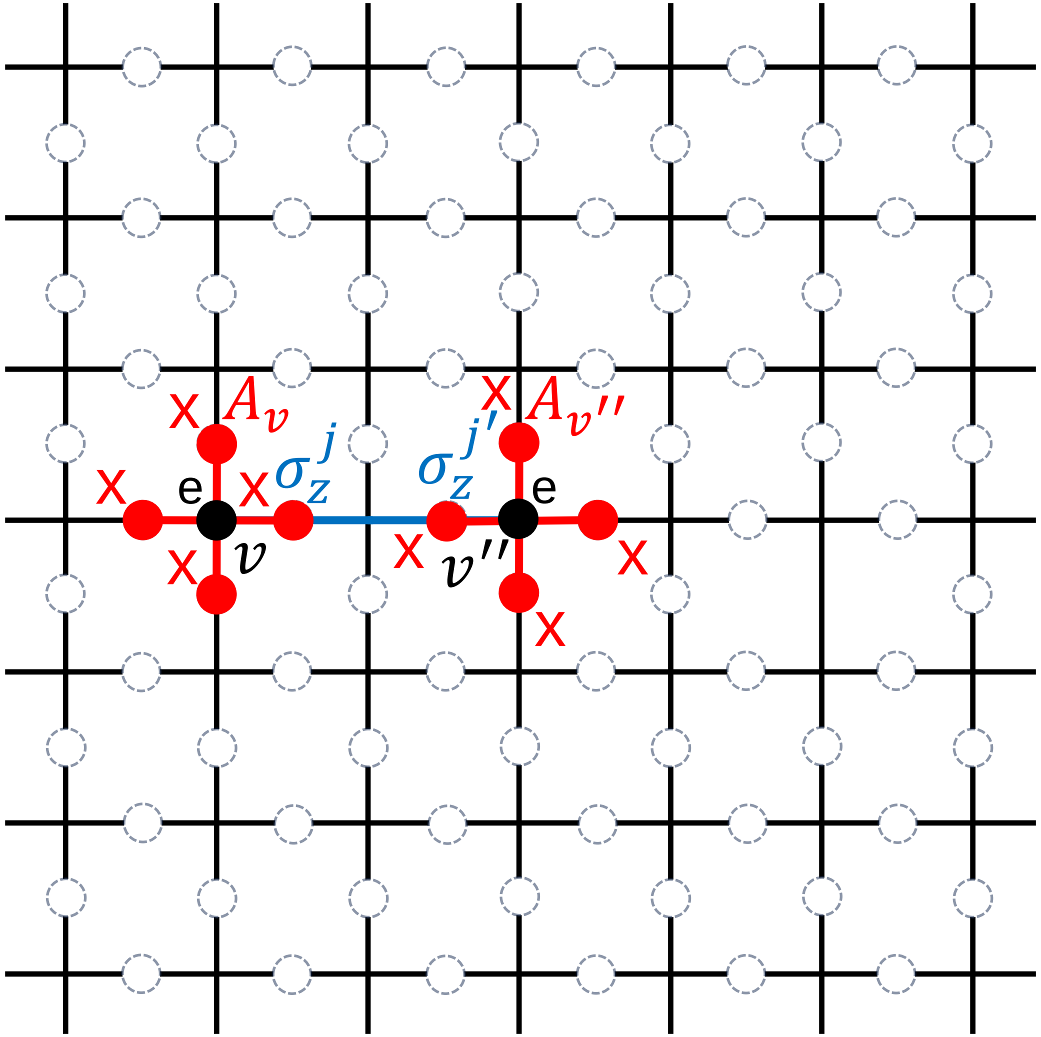
since only two stabilizers A_v on the end of the operator-acting edges does not commute with $\sigma_z^j \sigma_z^{j'}$, we still have the same excitation by 2.



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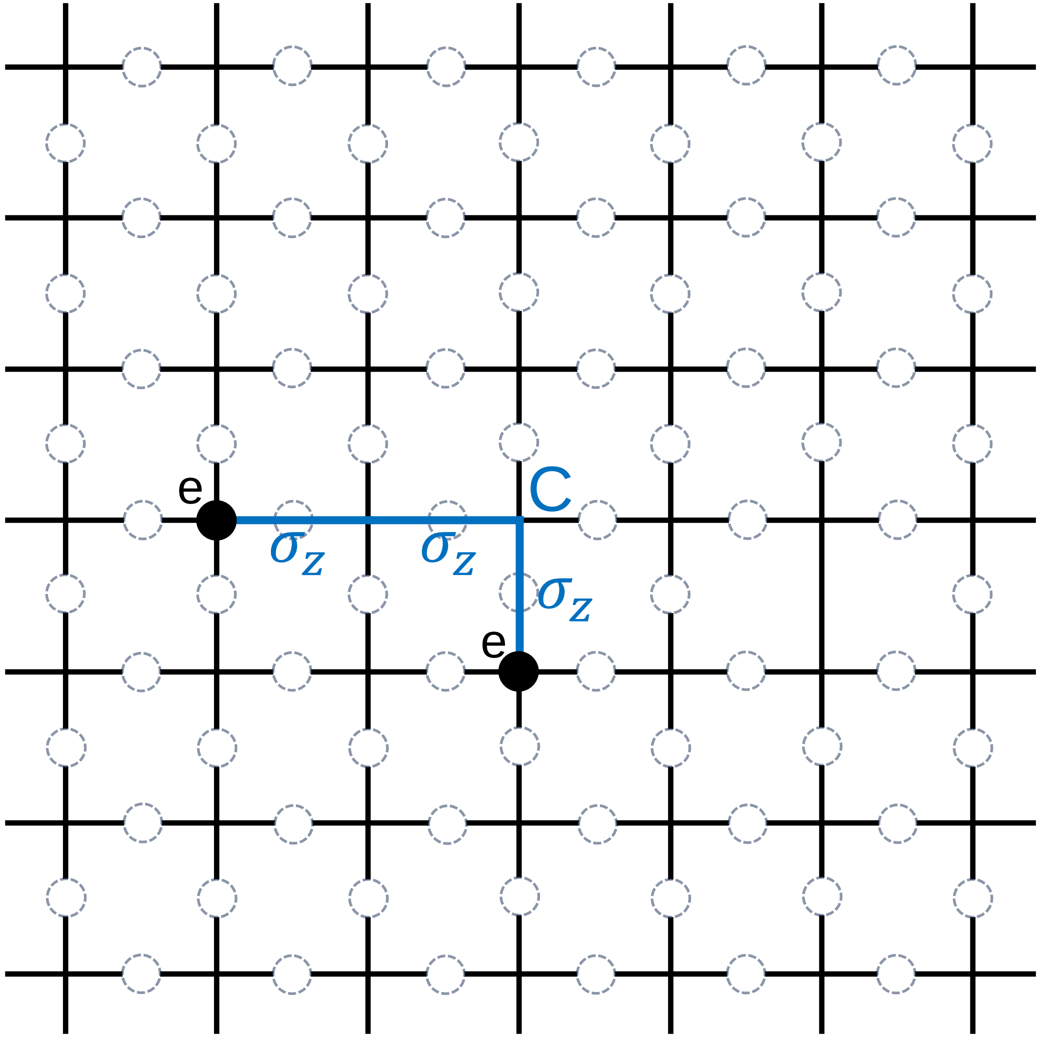
Define a string operator:

$S_z(C) = \prod_{j \in C} \sigma_z^j$ for a path C.

$\Longrightarrow S_z(C)|\psi_{GS}\rangle$: excited state with 2 electric defect at the end of the path C.



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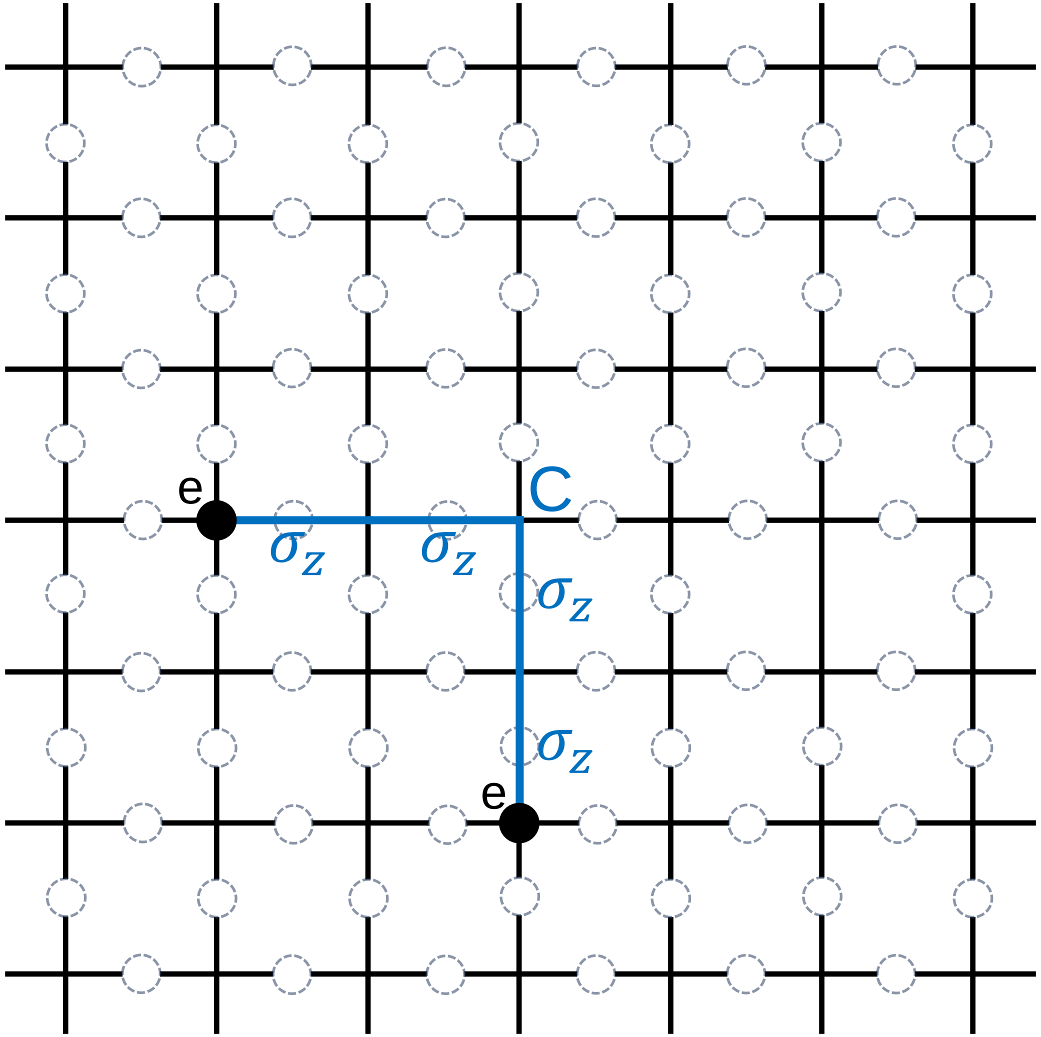
By acting a string operator, we can generate or move electric defects e.



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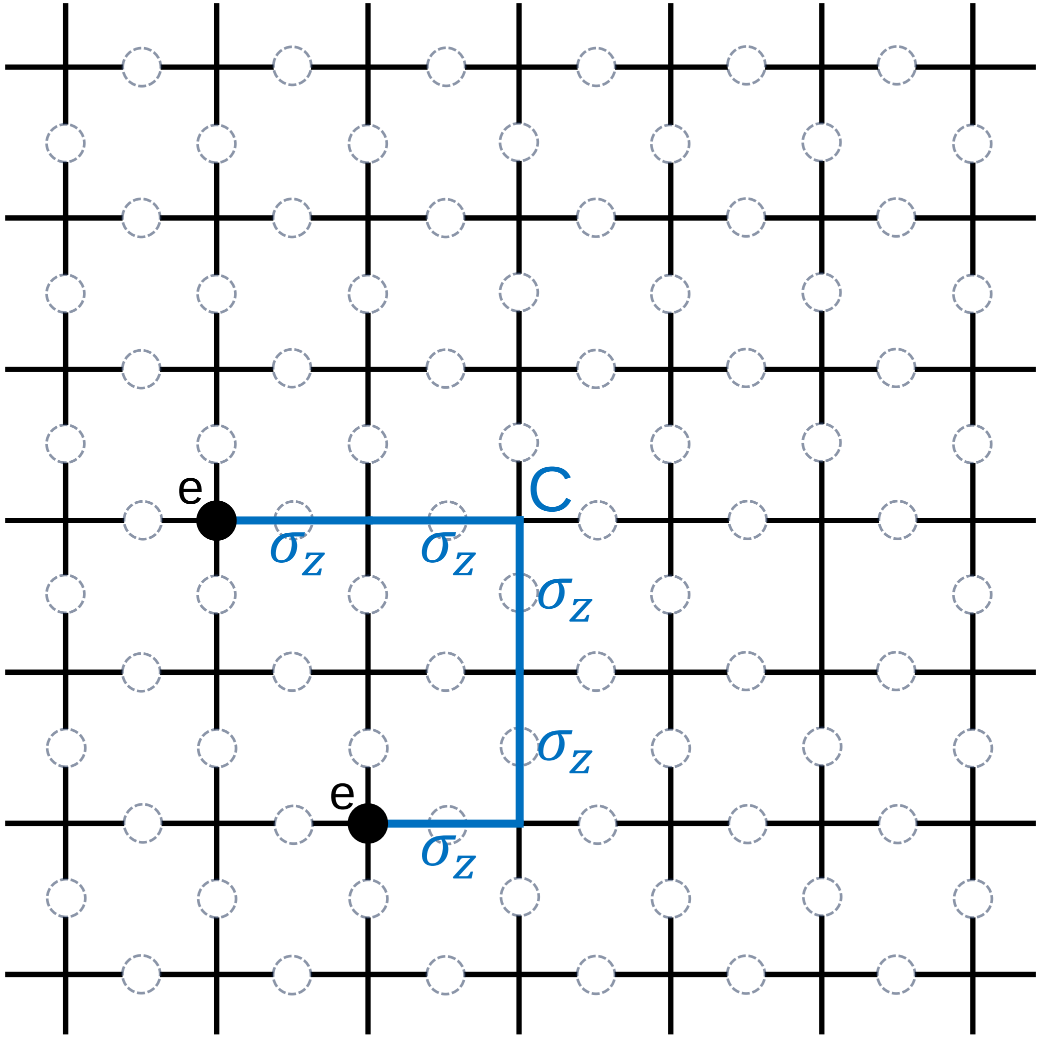

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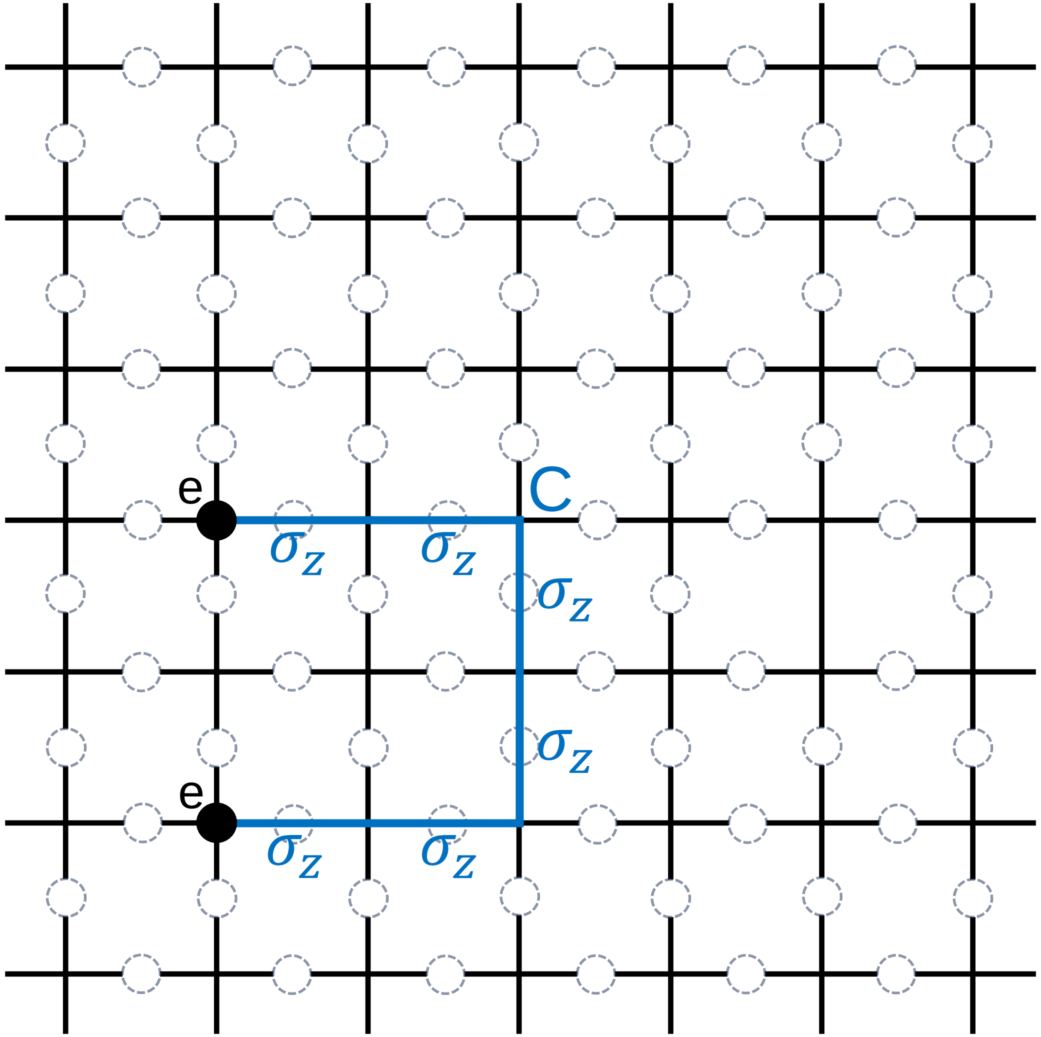

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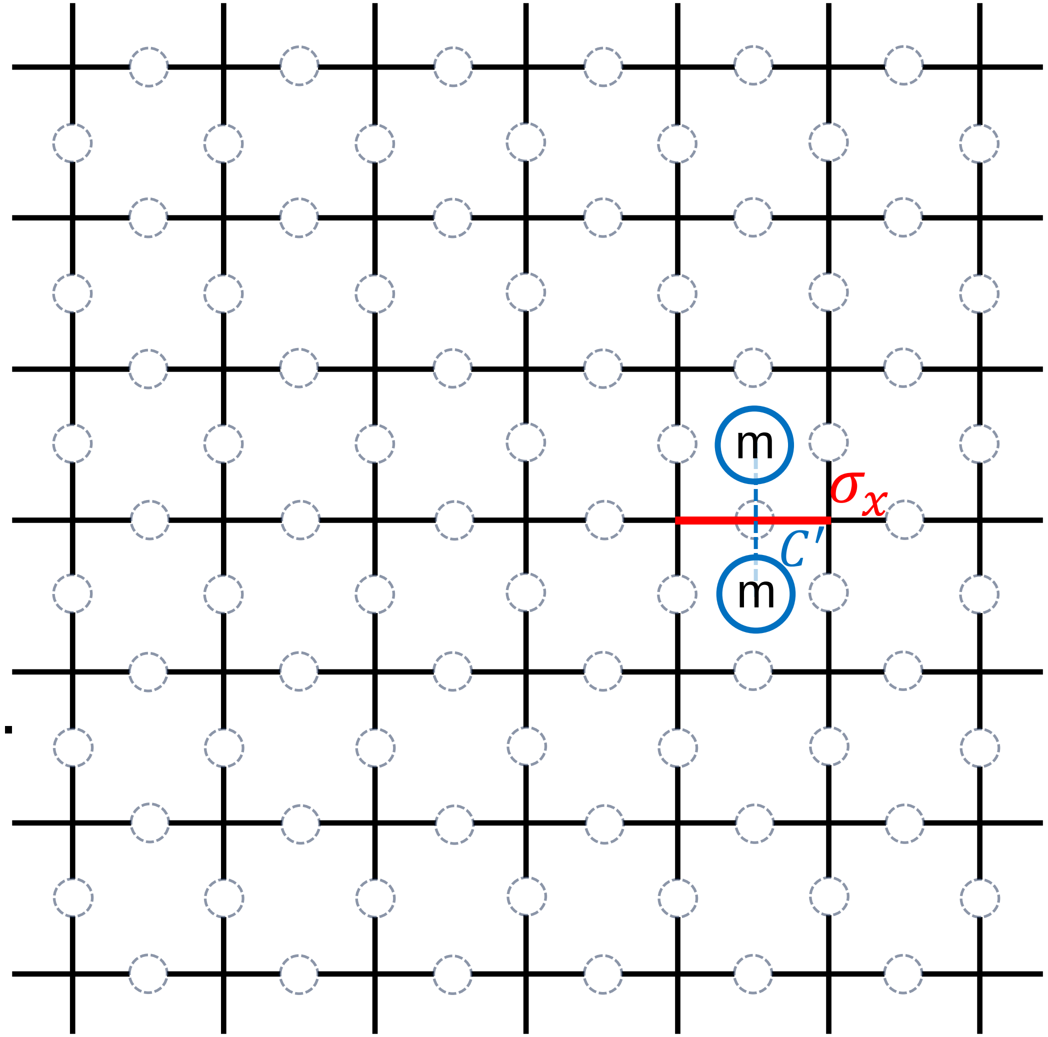
Similarly, we can define the string operator for σ_x^j as following:

$S_x(C') = \prod_{j \in C} \sigma_x^j$ for a path C' on the dual lattice.

Then, $S_x(C')|\psi_{GS}\rangle$ is an excited state with 2 electric defect at the end of C'.

By acting a string operator, we can generate or move magnetic vortexes m.



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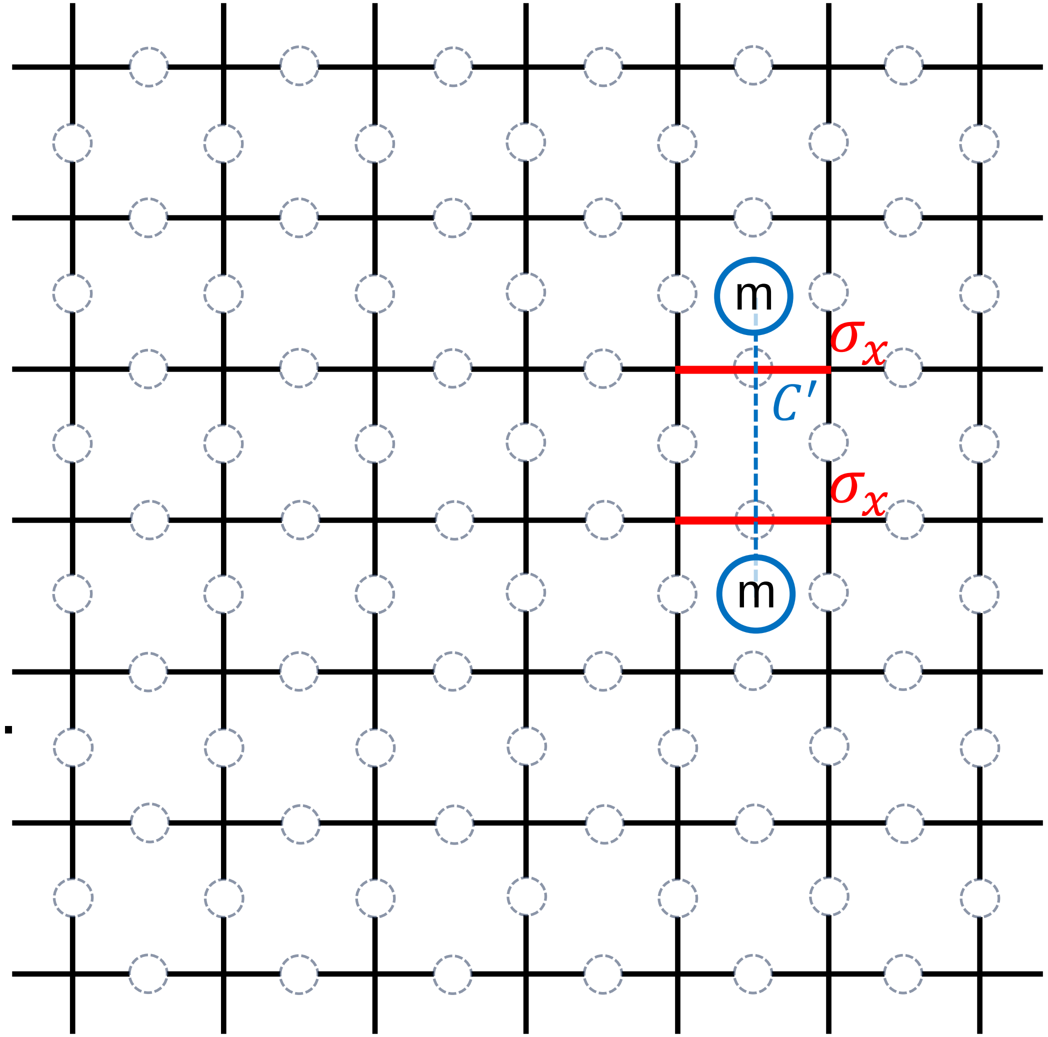

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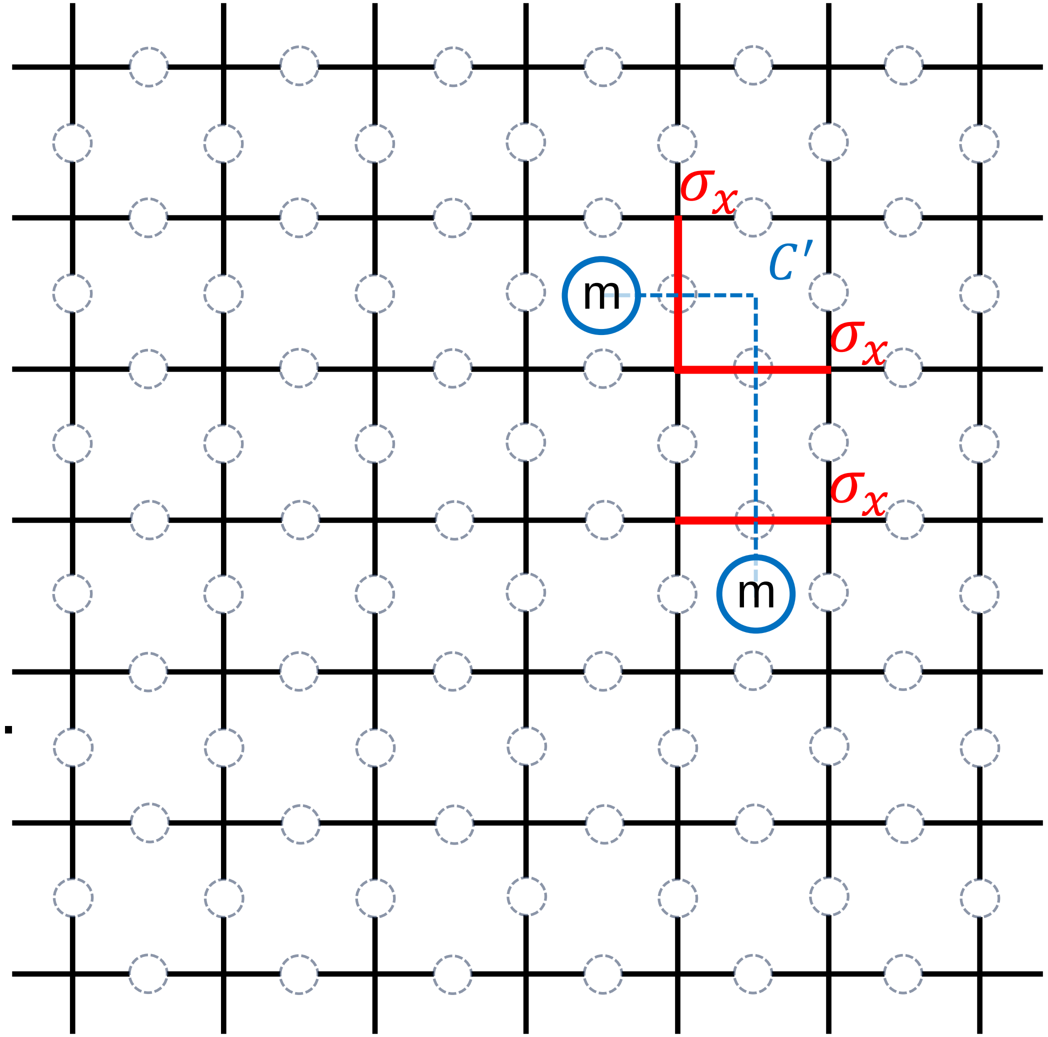

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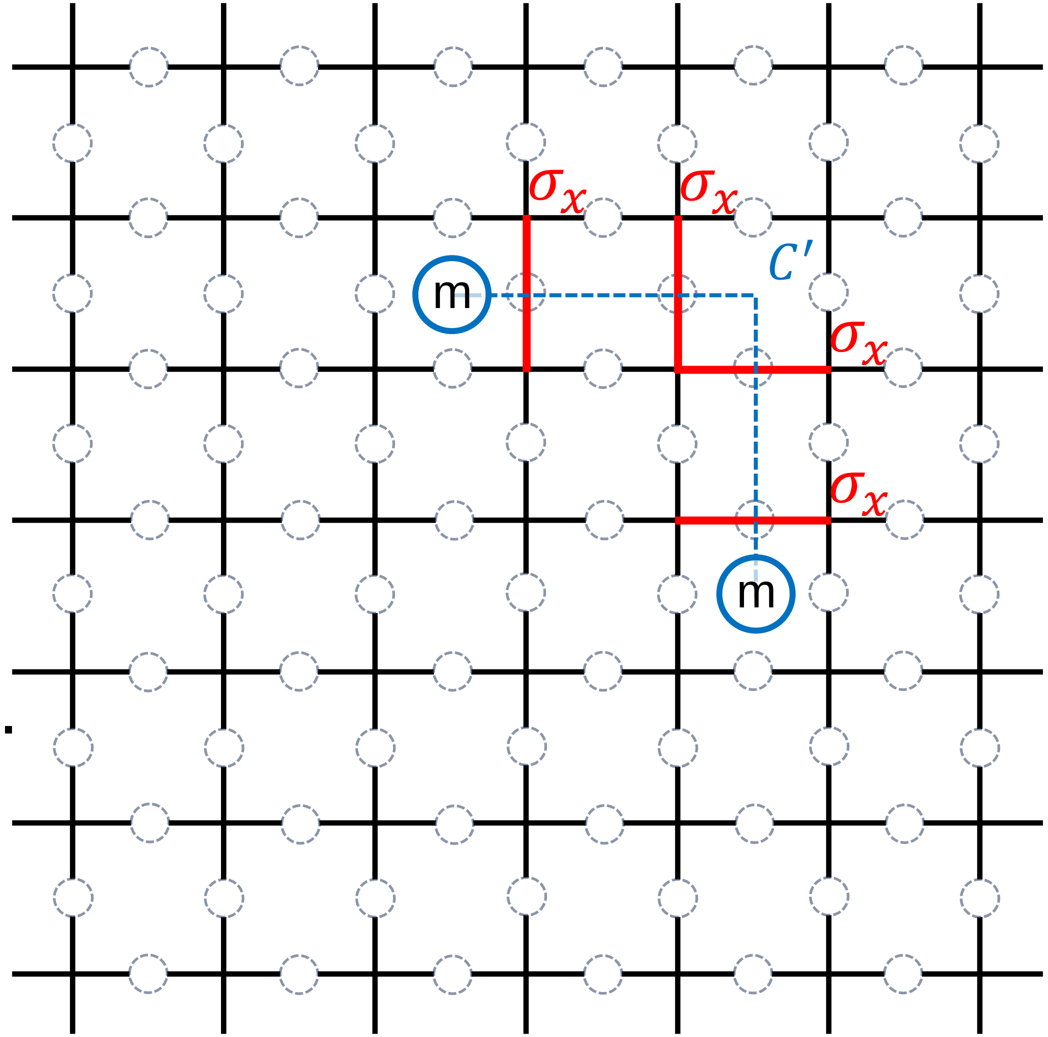

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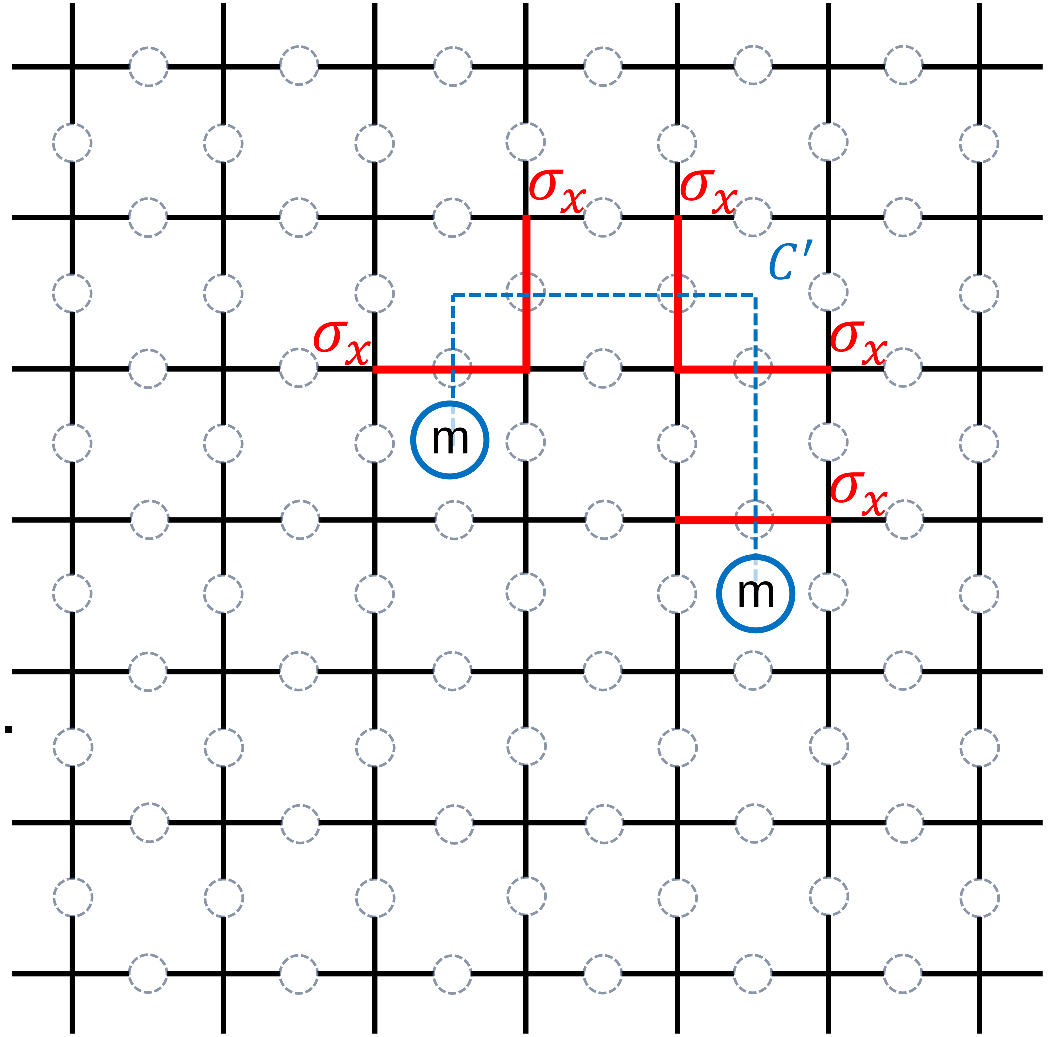

Kitaev, A. Y. (2003). "Fault-tolerant quantum computation by anyons." Annals of Physics **303(1): 2-30.**

The excited states

$$H = -\sum_v A_v - \sum_p B_p$$

Similarly, we can define the string operator for σ_x^j as following:

$S_x(C') = \prod_{j \in C} \sigma_x^j$ for a path C' on the dual lattice.

Then, $S_x(C')|\psi_{GS}\rangle$ is an excited state with 2 electric defect at the end of C'.

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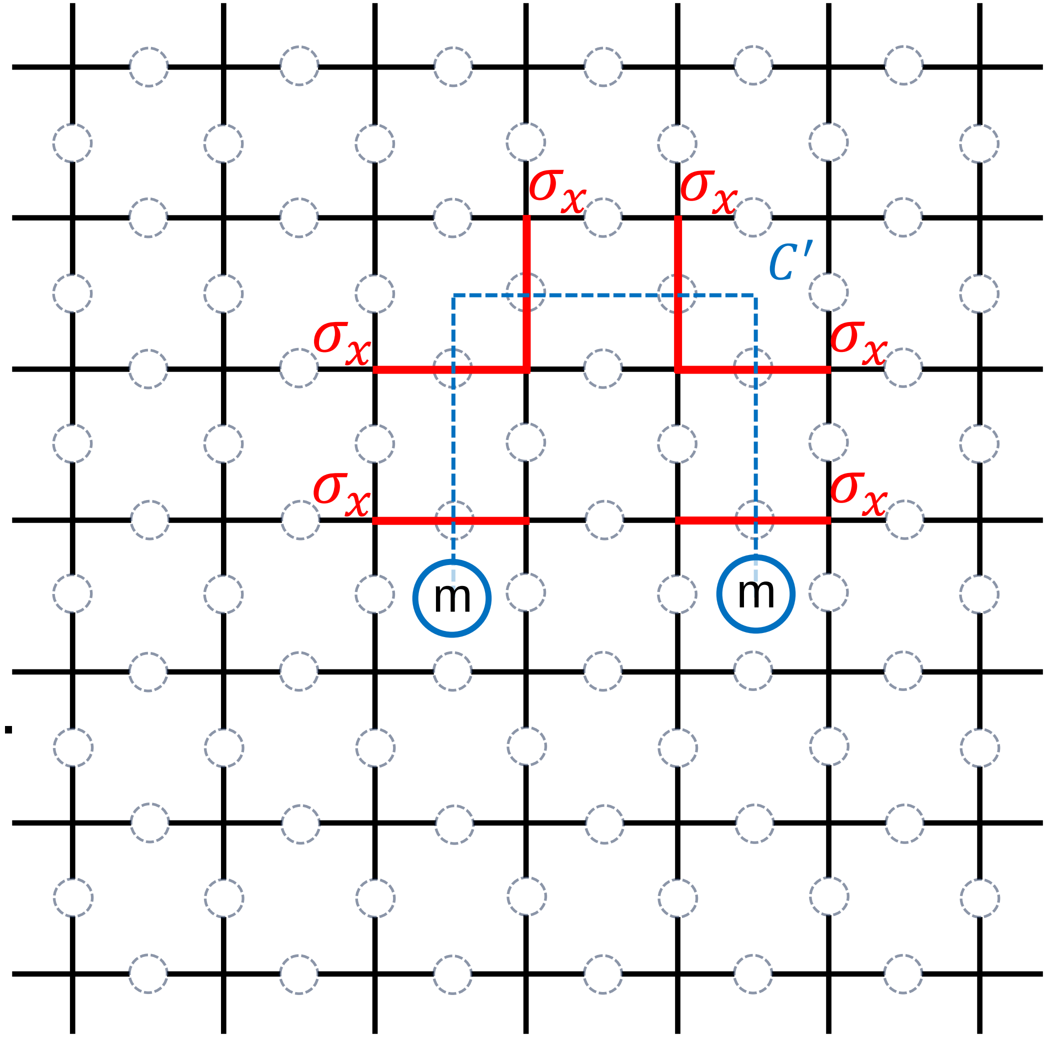

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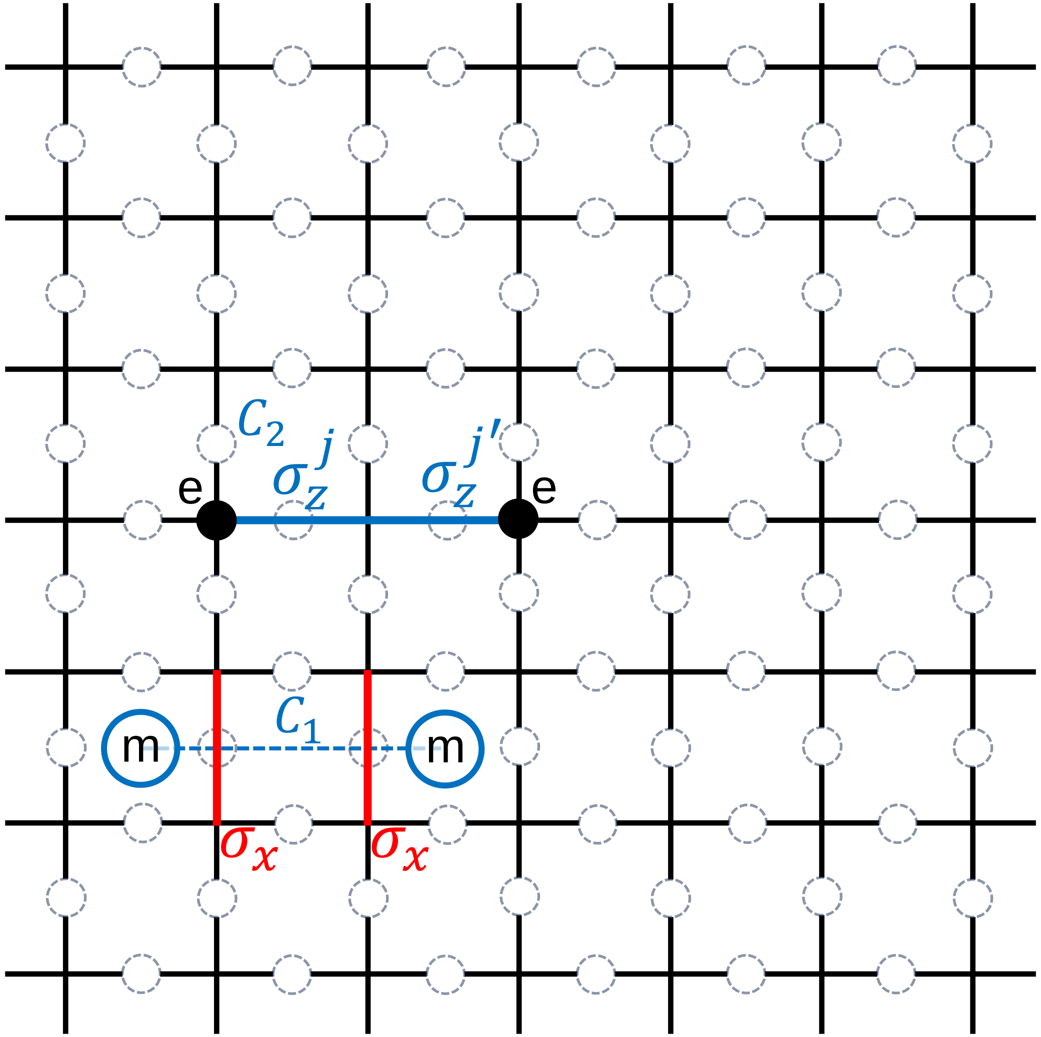
Quasiparticle excitation

$$H = -\sum_v A_v - \sum_p B_p$$

So, what are these excitations?

Consider the initial state:

$|\psi_{initial}\rangle = S_z(C_2)S_x(C_1)|\psi_{GS}\rangle.$



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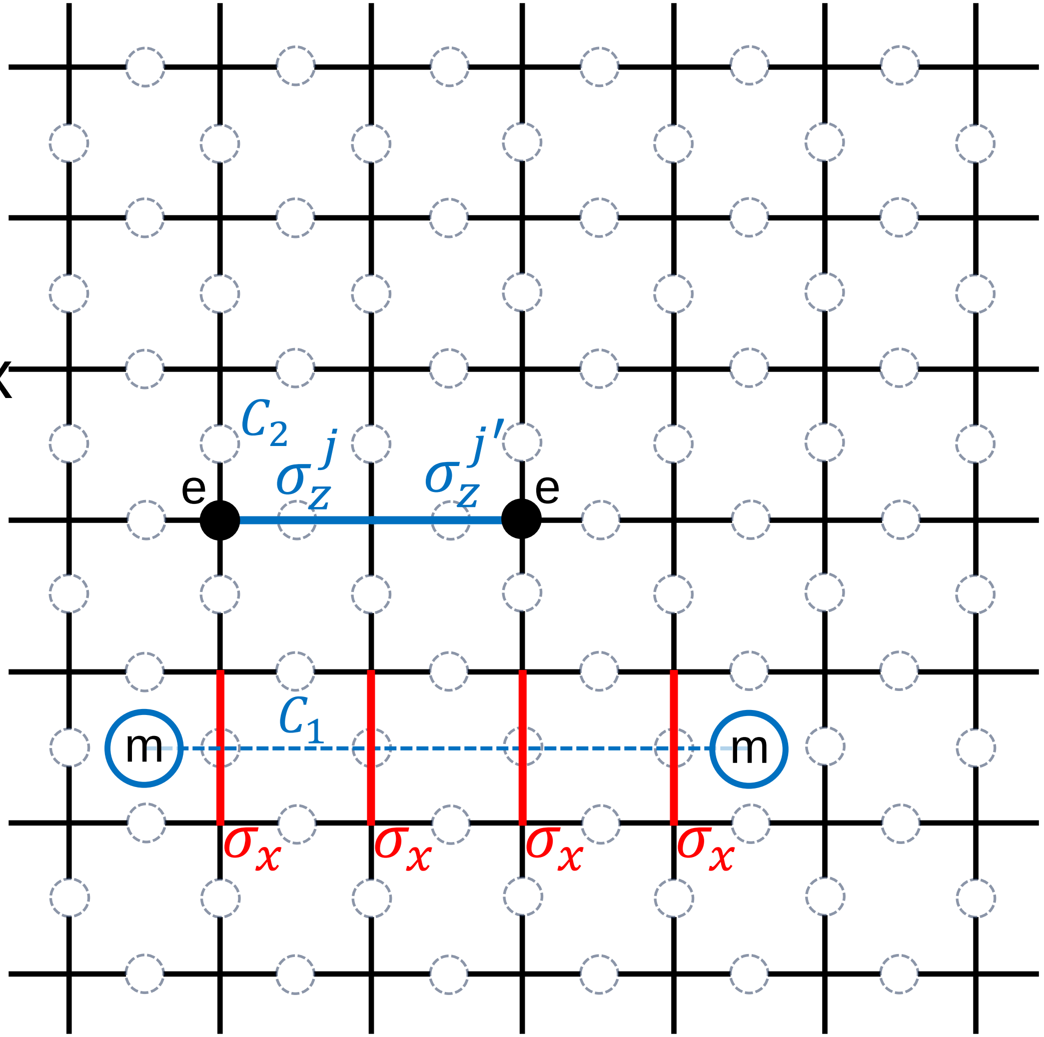

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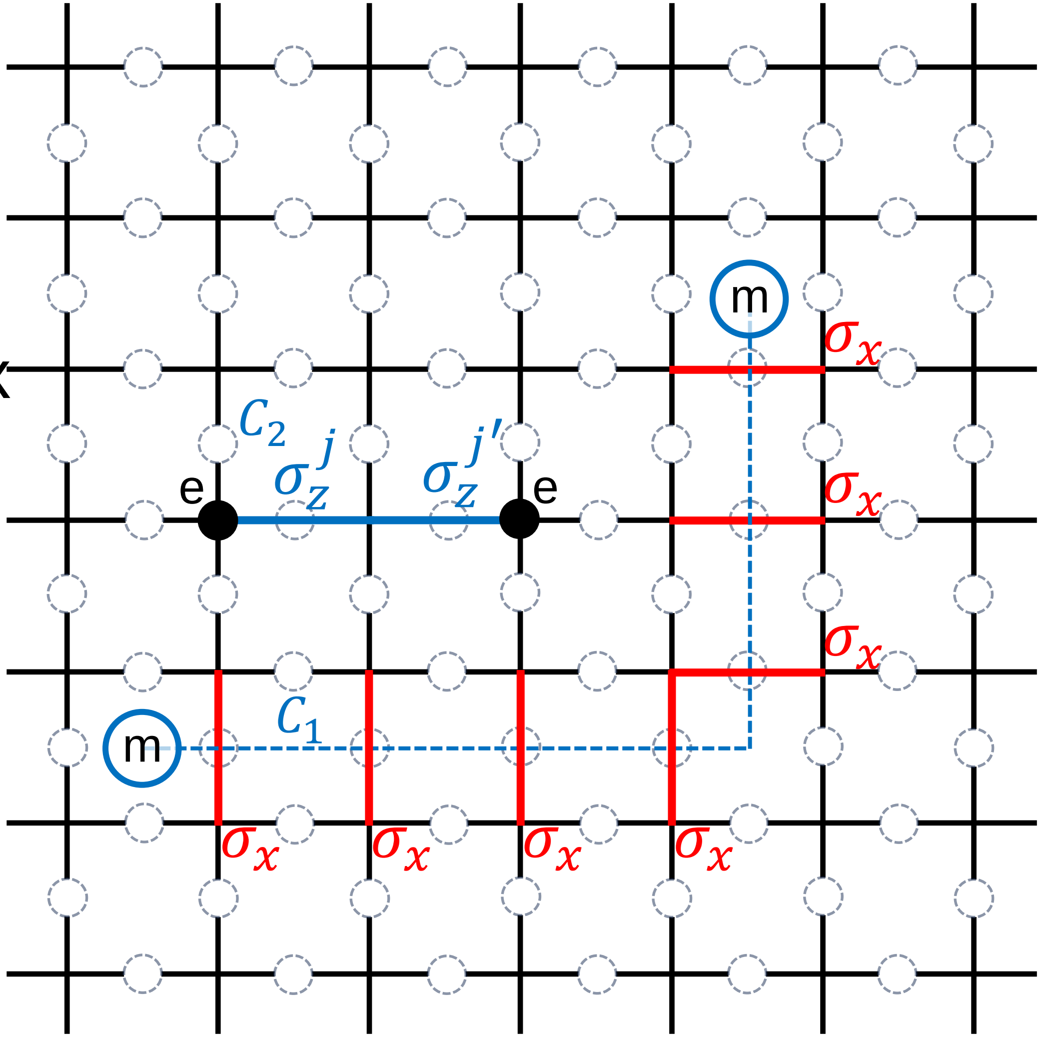

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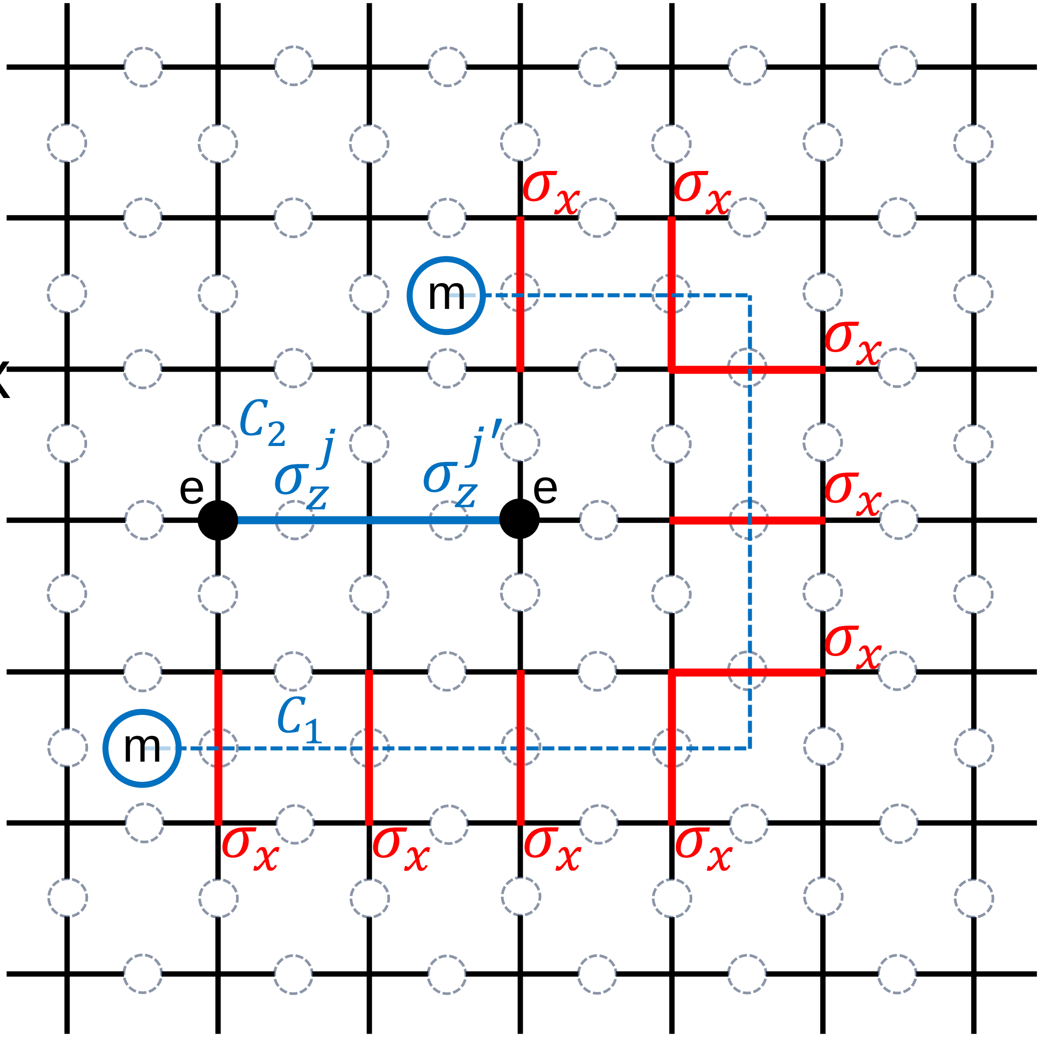

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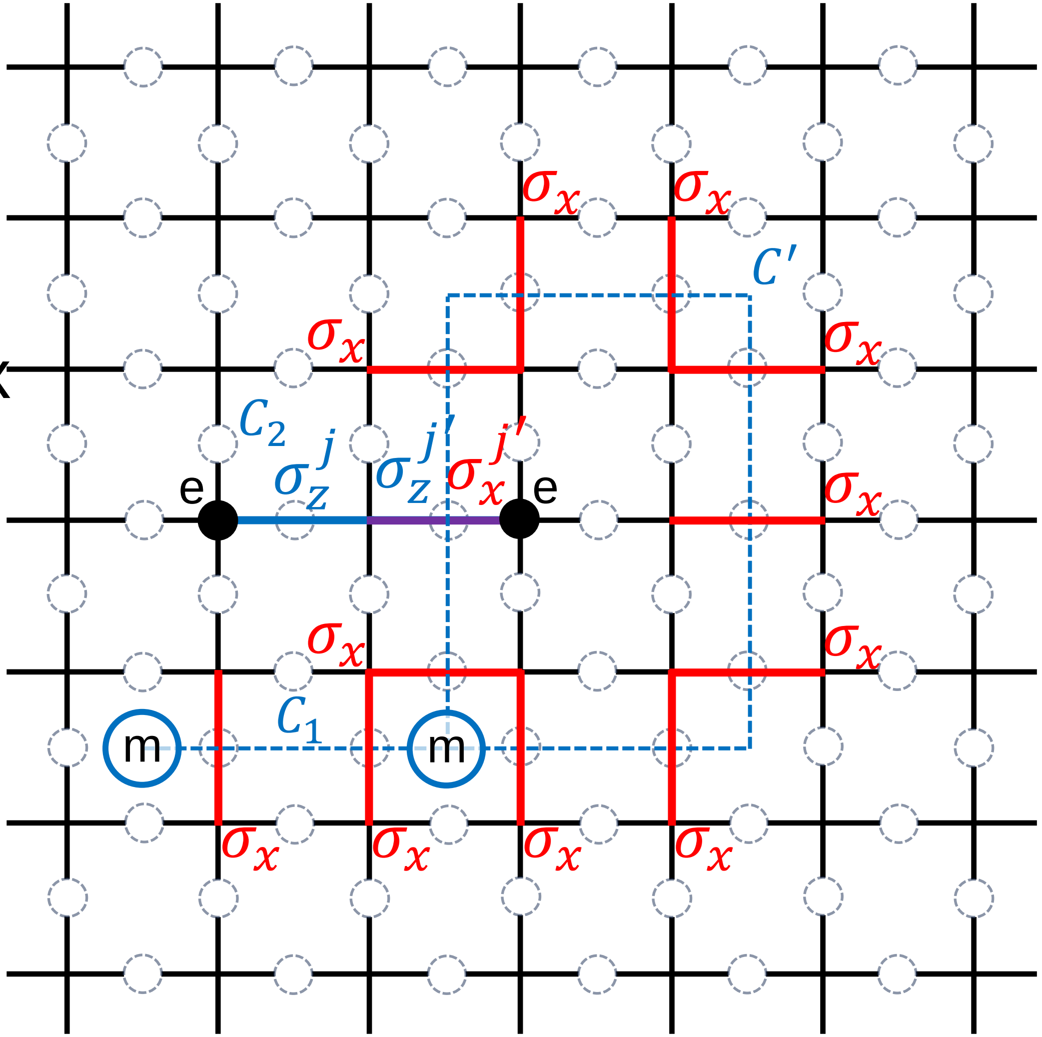
Consider the initial state:

$|\psi_{initial}\rangle = S_z(C_2)S_x(C_1)|\psi_{GS}\rangle.$

With string operators, a magnetic vortex is moved around an electric defect.

Final state:

$|\psi_{final}\rangle = S_x(C')|\psi_{initial}\rangle$
$= S_x(C')S_z(C_2)S_x(C_1)|\psi_{GS}\rangle$
$= -S_z(C_2)S_x(C')S_x(C_1)|\psi_{GS}\rangle$
$= -S_z(C_2)S_x(C_1)(S_x(C')|\psi_{GS}\rangle)$
$= -|\boldsymbol{\psi}_{initial}\rangle$



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Braidings: exchange of quasiparticles

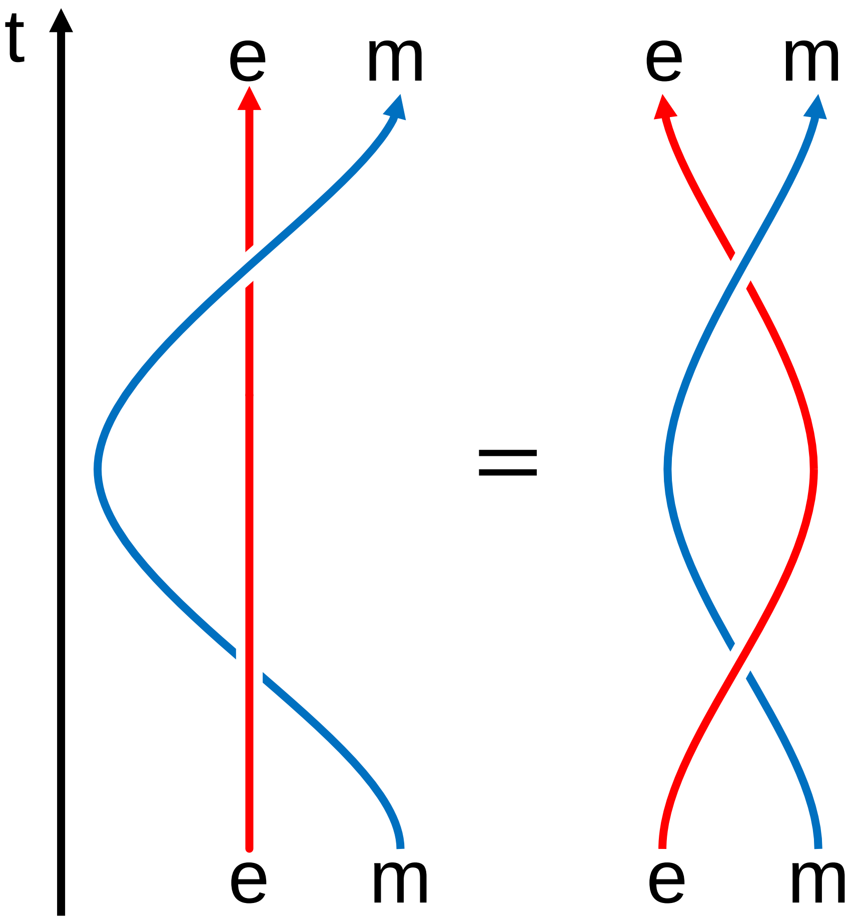
Under the rotation of “m” around “e”,

$|\psi_{final}\rangle = -|\psi_{initial}\rangle$.

Topologically, rotating a particle around the other is equivalent to double exchanges of their position.

Exchange leads to $R_{em} = e^{\frac{i\pi}{2}}$ of phase change.

$\Longrightarrow (R_{em})^2 = e^{i\pi} = -1$ for a whole rotation.



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Anyonic statistics

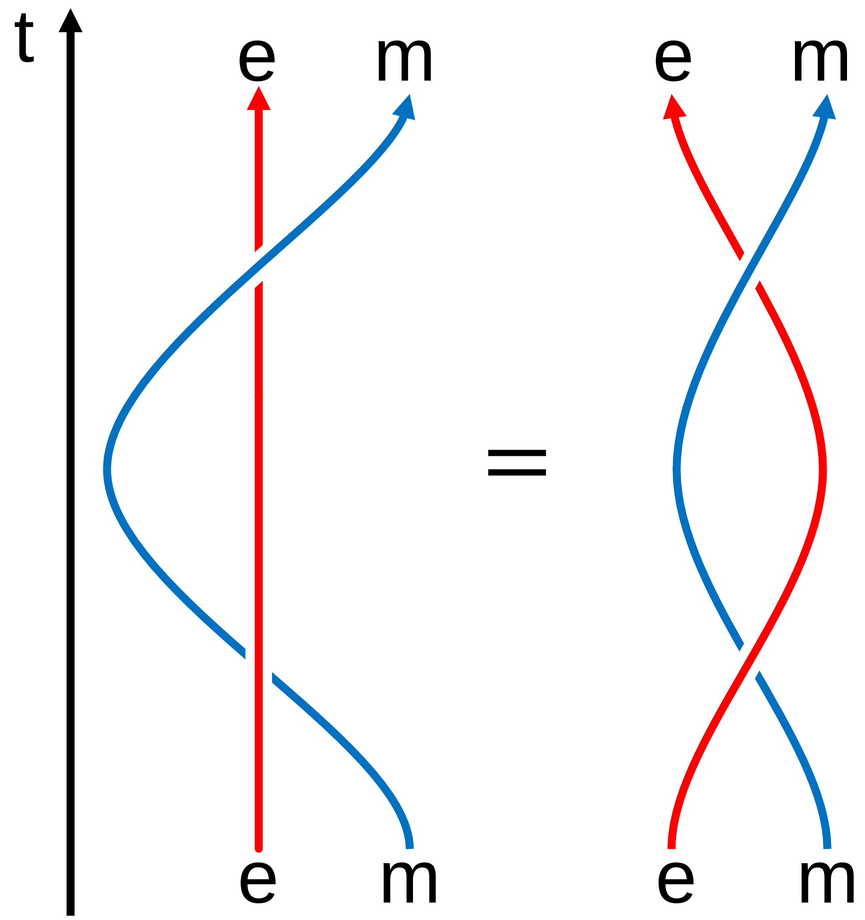
Phase factor due to exchange:

$$R_{em} = e^{\frac{i\pi}{2}},$$

which is neither bosonic nor fermionic.

Thus, we can conclude that these are quasiparticles which follow statistics completely different from the ones of ordinary particles.

We call those quasiparticles as Anyon.



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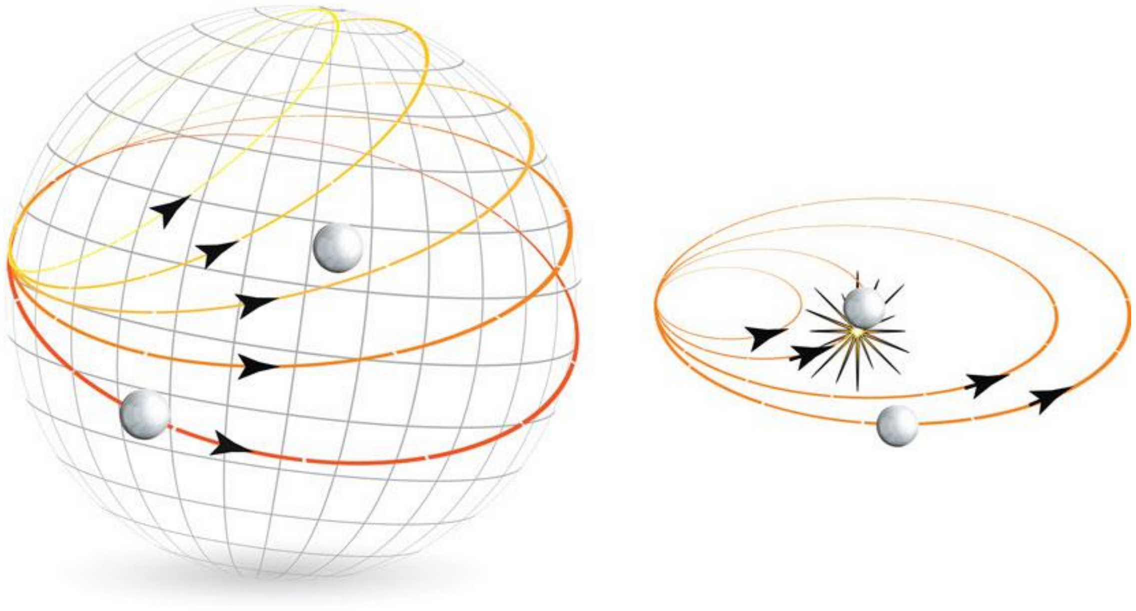
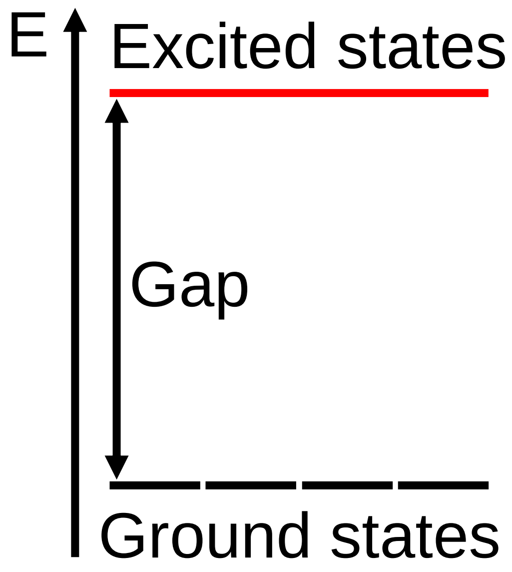
Conclusion

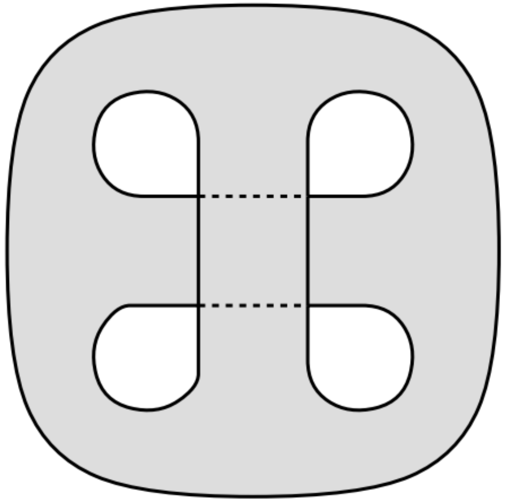
Toric Code system has excitation states following anyonic statistics: exchange of particles leads to phase change by $e^{\frac{i\pi}{2}}$.

These anyons can be used to store quantum information: braidings of anyonic particles store the record of event that they exchanged their positions.

The degenerate ground states of the Toric Code themselves can be used as qubits.







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- Kitaev, A. Y. (1997). "Quantum computations: algorithms and error correction." Russian Mathematical Surveys **52(6): 1191.**
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- Valenti, A., E. van Nieuwenburg, S. Huber and E. Greplova (2019). "Hamiltonian learning for quantum error correction." Physical Review Research **1(3): 033092.**

Toric Code and Anyons
Q&A Session

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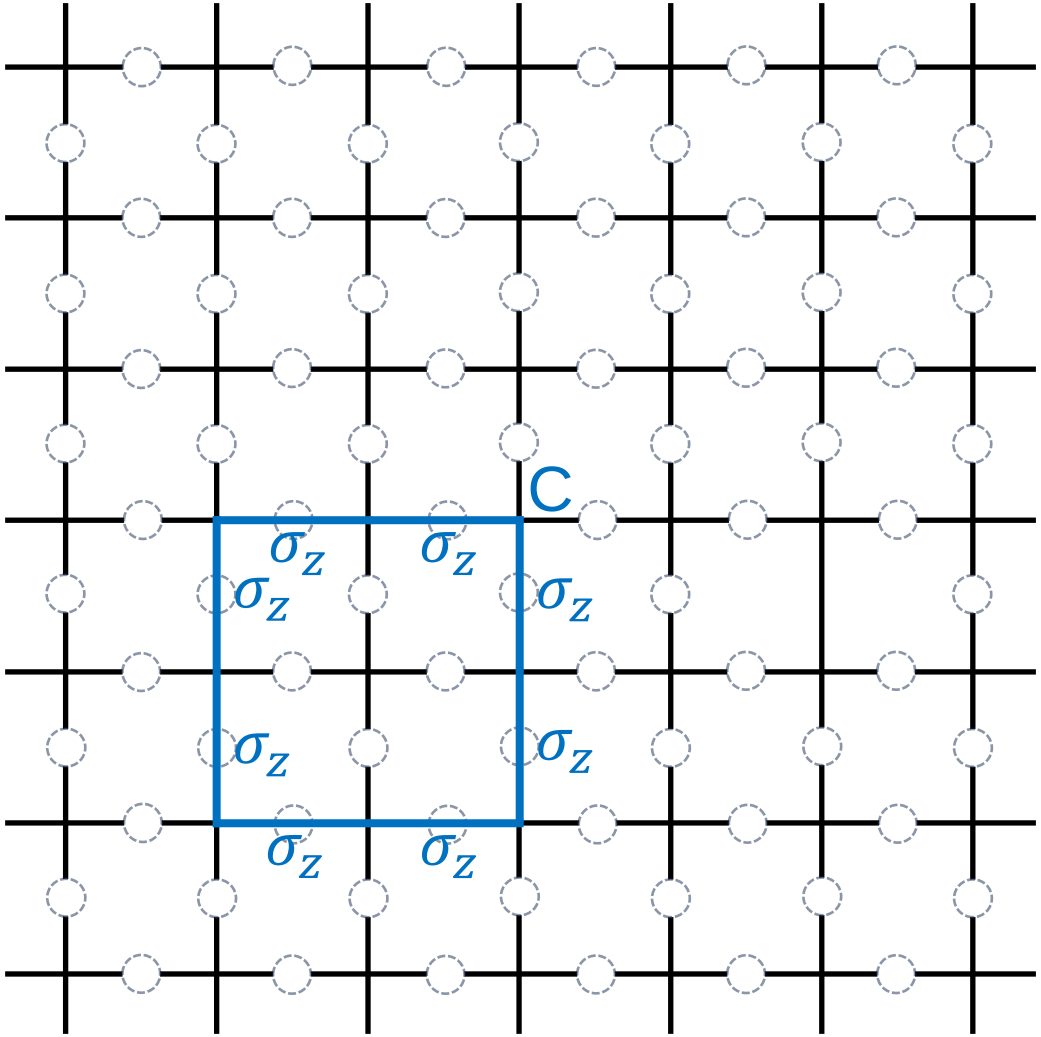
Kitaev, A. Y. (2003). "Fault-tolerant quantum computation by anyons." Annals of Physics **303(1): 2-30.**

Appendix. Ground state degeneracy

For a closed path C, then $S_z(C)$ commutes with every stabilizer.

$\Longrightarrow S_z(C)|\psi_{GS}\rangle$: ground state for a closed path C.



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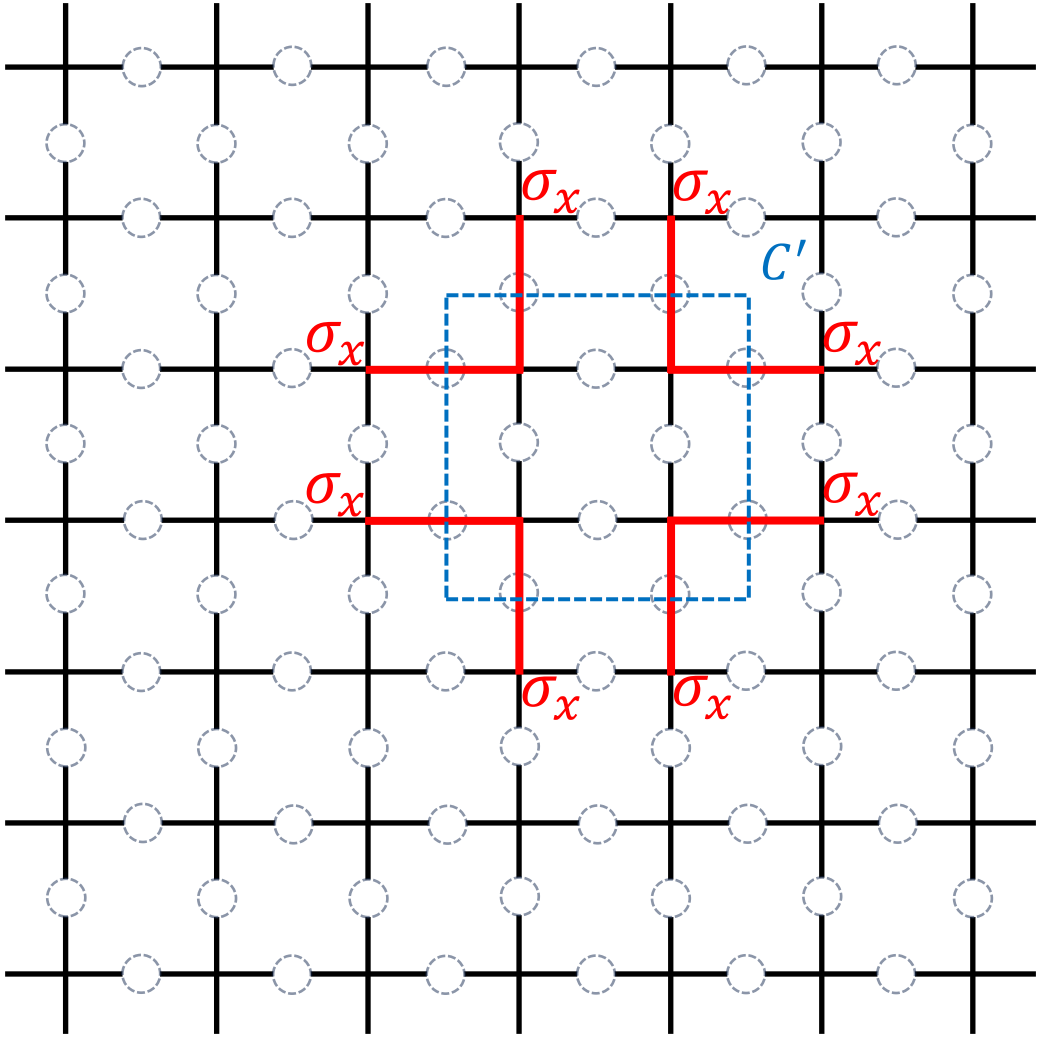
Appendix. Ground state degeneracy

For a closed path C, then $S_z(C)$ commutes with every stabilizer.

$\Longrightarrow S_z(C)|\psi_{GS}\rangle$: ground state for a closed path C.

Similarly, for a closed path C' defined on the dual lattice, $S_x(C')$ commutes with every stabilizer.

$\Longrightarrow S_x(C')|\psi_{GS}\rangle$: ground state for a closed path C'.



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Appendix. Ground state degeneracy

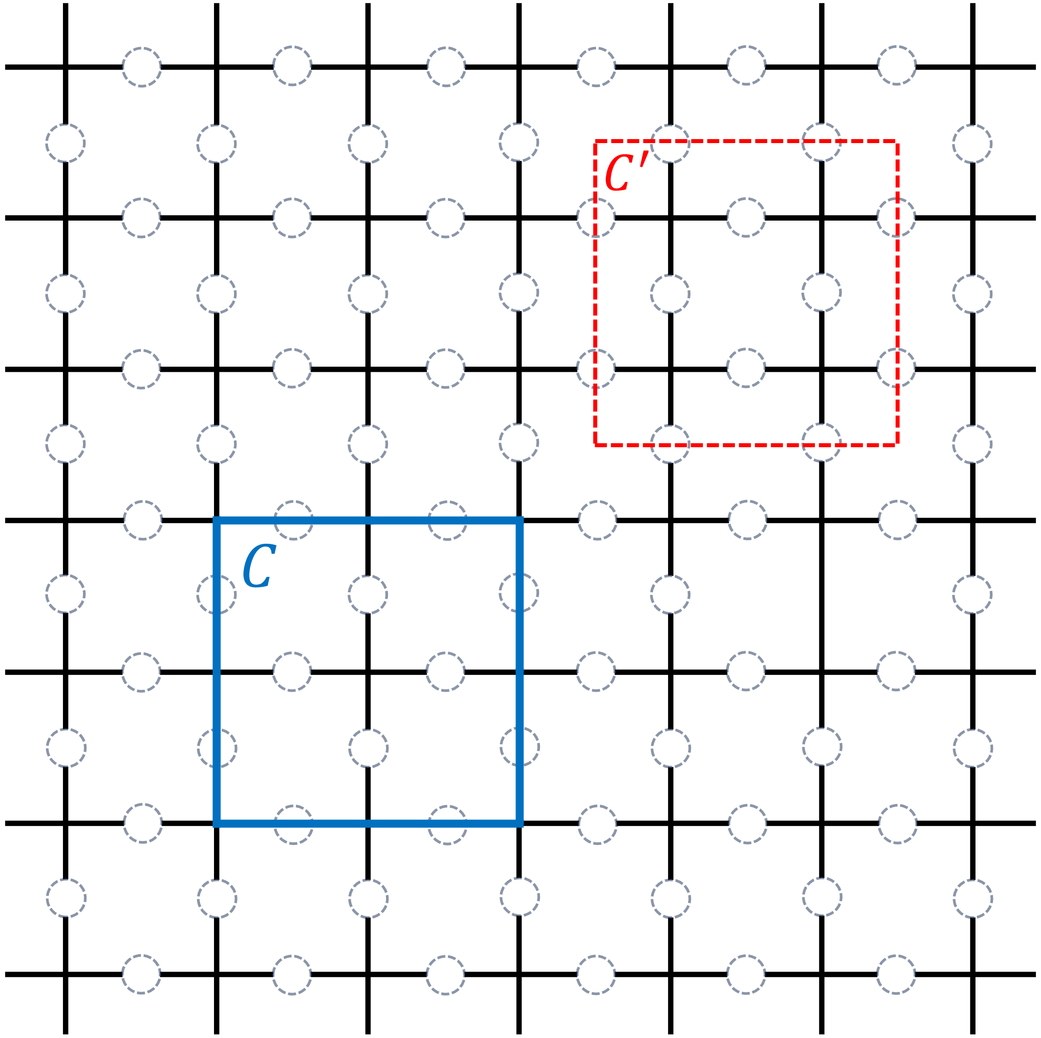
We assumed that $|\psi_{GS}\rangle$ such that

$$\begin{cases} A_v|\psi_{GS}\rangle = +|\psi_{GS}\rangle \\ B_p|\psi_{GS}\rangle = +|\psi_{GS}\rangle \end{cases}$$, for all v, p, exists,

and speculated that there would be **4-fold degeneracy**.

However, it appears that there exists **infinite-fold degeneracy** for each possible form of closed loop:

for any closed loop C in lattice or C' in dual lattice, $S_z(C)|\psi_{GS}\rangle$ and $S_x(C')|\psi_{GS}\rangle$ are also ground states.



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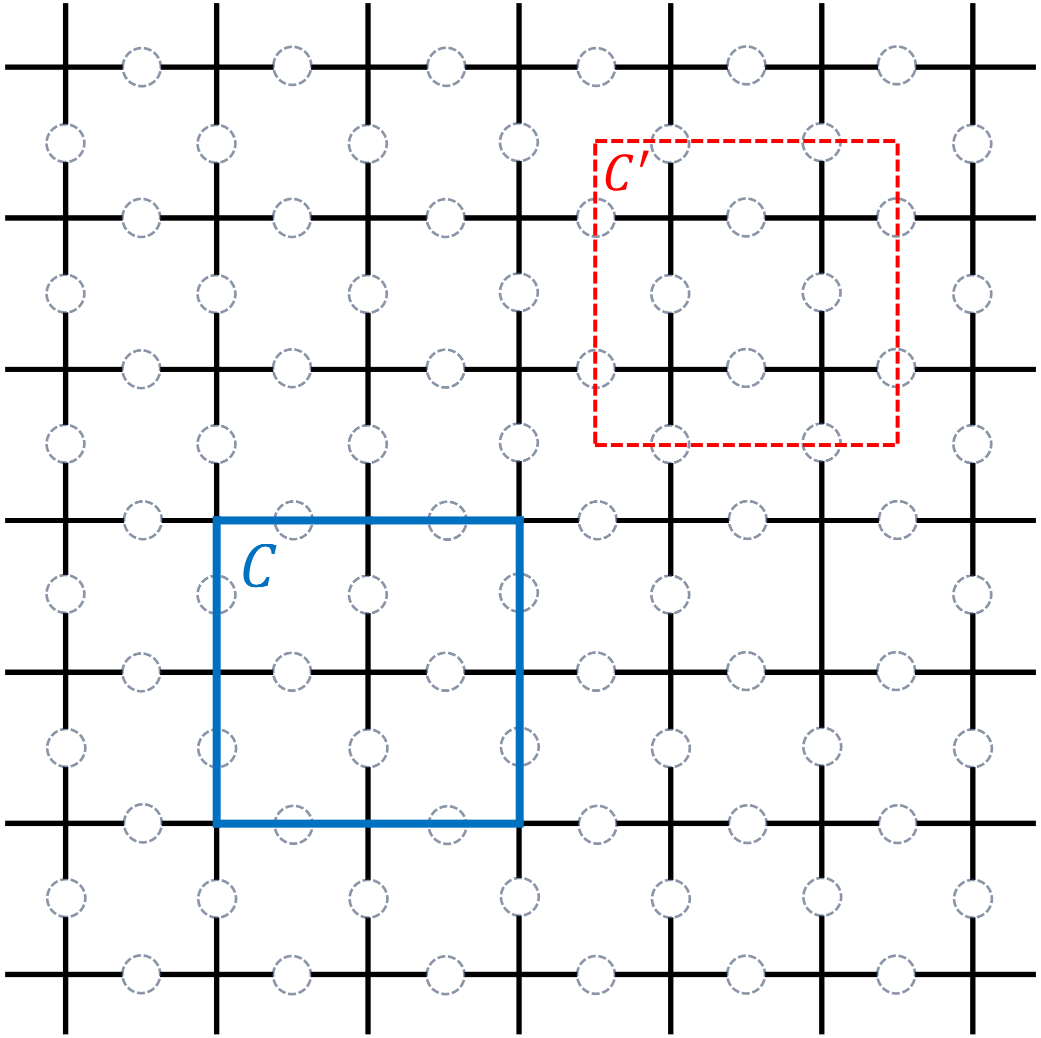
Appendix. Ground state degeneracy

Then, what is wrong with our logics?

In fact, $S_z(C)|\psi_{GS}\rangle$ and $S_x(C')|\psi_{GS}\rangle$ all stands for the same state.

We claim that **there exists exactly 4-fold degenerate ground states**.



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Appendix. Ground state degeneracy

Claim 1: $|\psi\rangle = \prod_v (1 + A_v)\,|0\rangle$, where $|0\rangle$ is a tensor product of up-spin state for each site, is a ground state.

$B_p|\psi\rangle = \left(\prod_{v'}(1 + A_{v'})\right) B_p|0\rangle = |\psi\rangle,$

$A_v|\psi\rangle = A_v \prod_{v'}(1 + A_{v'})\,|0\rangle$

$= \prod_{v'} C_{v'}\,|0\rangle$, where $C_{v'} = \begin{cases} 1 + A_{v'}, & \text{if } v' \neq v, \\ A_v(1 + A_v), & \text{if } v' = v, \end{cases} = 1 + A_{v'}, \text{for any } v'$

$= |\psi\rangle.$

$\Longrightarrow \mathrm{H}|\psi\rangle = -2L^2|\psi\rangle.$

Thus, we may take $|\psi_{GS}\rangle = c|\psi\rangle$ for proper normalization const. c.

Valenti, A., E. van Nieuwenburg, S. Huber and E. Greplova (2019). "Hamiltonian learning for quantum error correction." Physical Review Research **1(3): 033092.**

Appendix. Ground state degeneracy

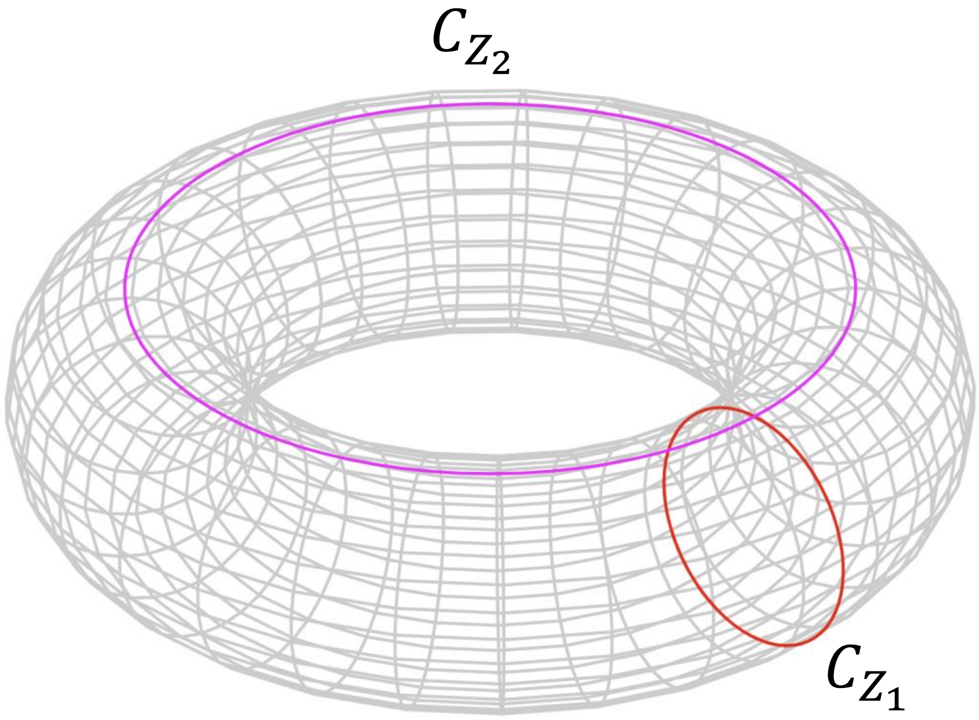
Definition.

We define Z_i as a string operator on the non-contractible loop C_{Z_i}:

$Z_i = S_z(C_{Z_i}) = \prod_{j \in C_{Z_i}} \sigma_z^j$, for $i = 1,2$.

Similarly, define X_i on the non-contractible loop C_{X_i} in dual lattice:

$X_i = S_x(C_{X_i}) = \prod_{j \in C_{X_i}} \sigma_x^j$, for $i = 1,2$.



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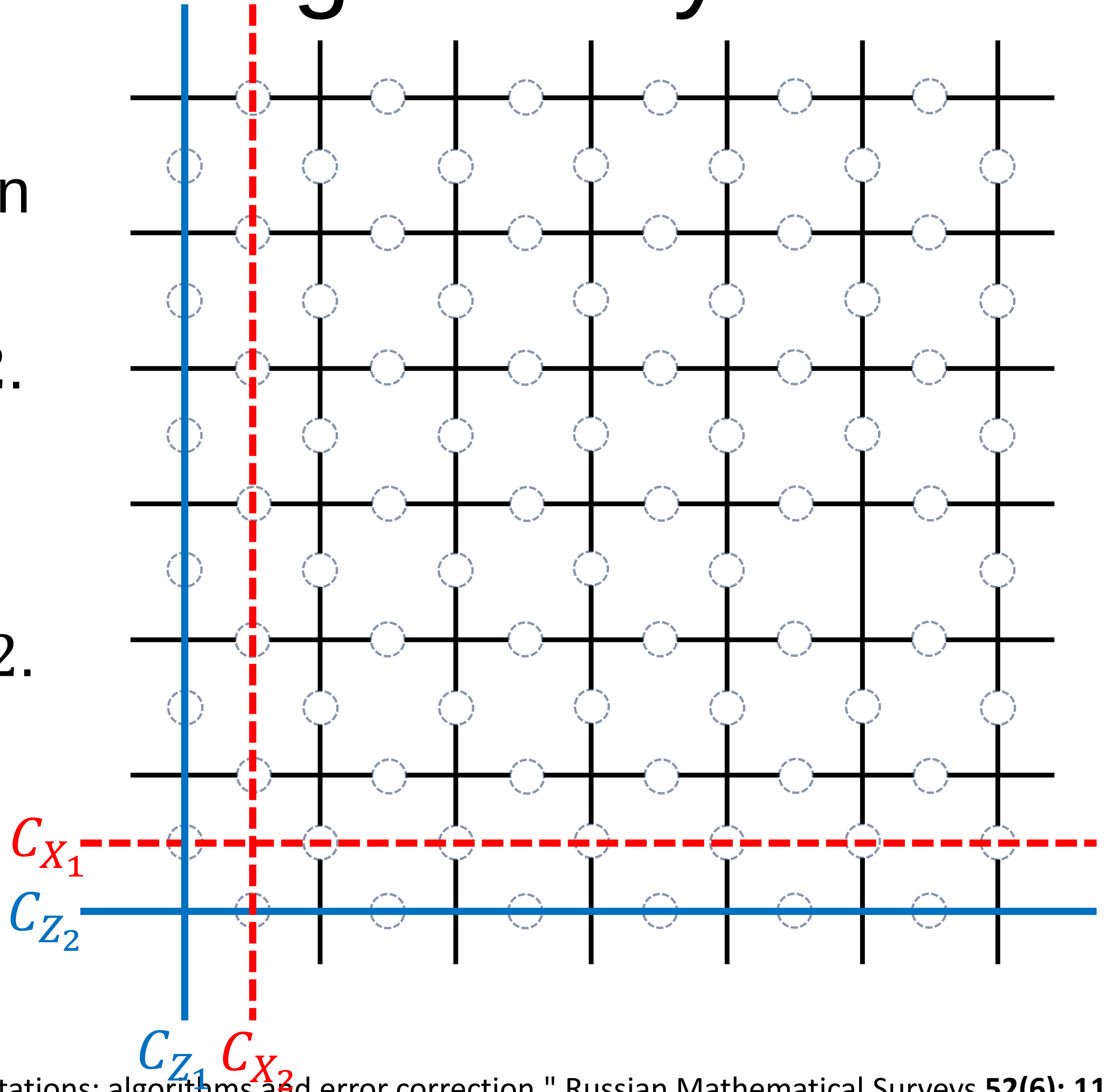


Kitaev, A. Y. (1997). "Quantum computations: algorithms and error correction." Russian Mathematical Surveys **52(6): 1191.**

Appendix. Ground state degeneracy

$Z_i = \prod_{j \in C_{Z_i}} \sigma_z^j, \; X_i = \prod_{j \in C_{X_i}} \sigma_x^j.$

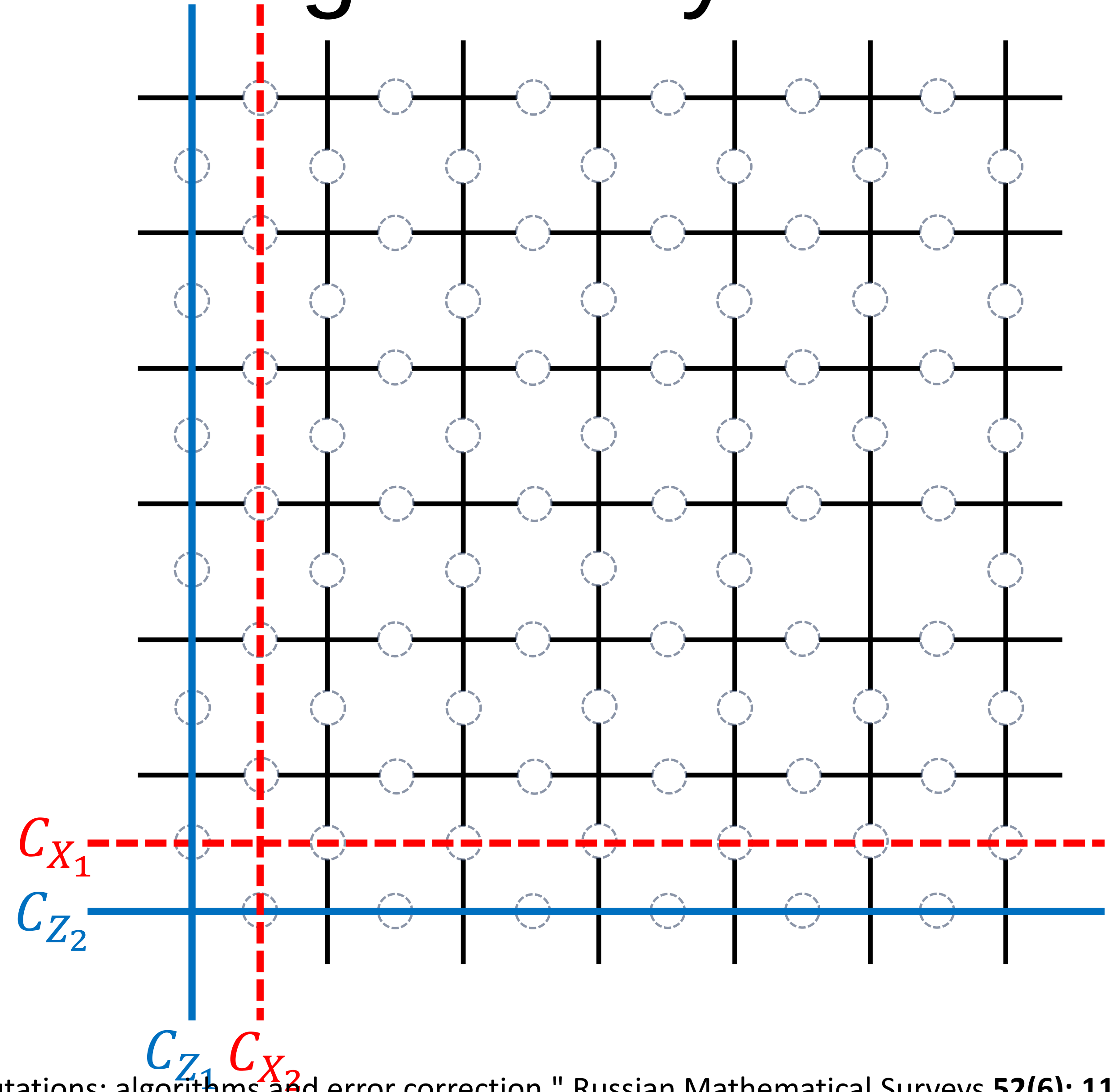
Commutation relations:

$[Z_1, Z_2] = [X_1, X_2] = 0,$

$[Z_1, X_2] = [Z_2, X_1] = 0,$

$Z_1 X_1 = -X_1 Z_1, \; Z_2 X_2 = -X_2 Z_2,$

$Z_1^2 = Z_2^2 = X_1^2 = X_2^2 = 1.$



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Appendix. Ground state degeneracy

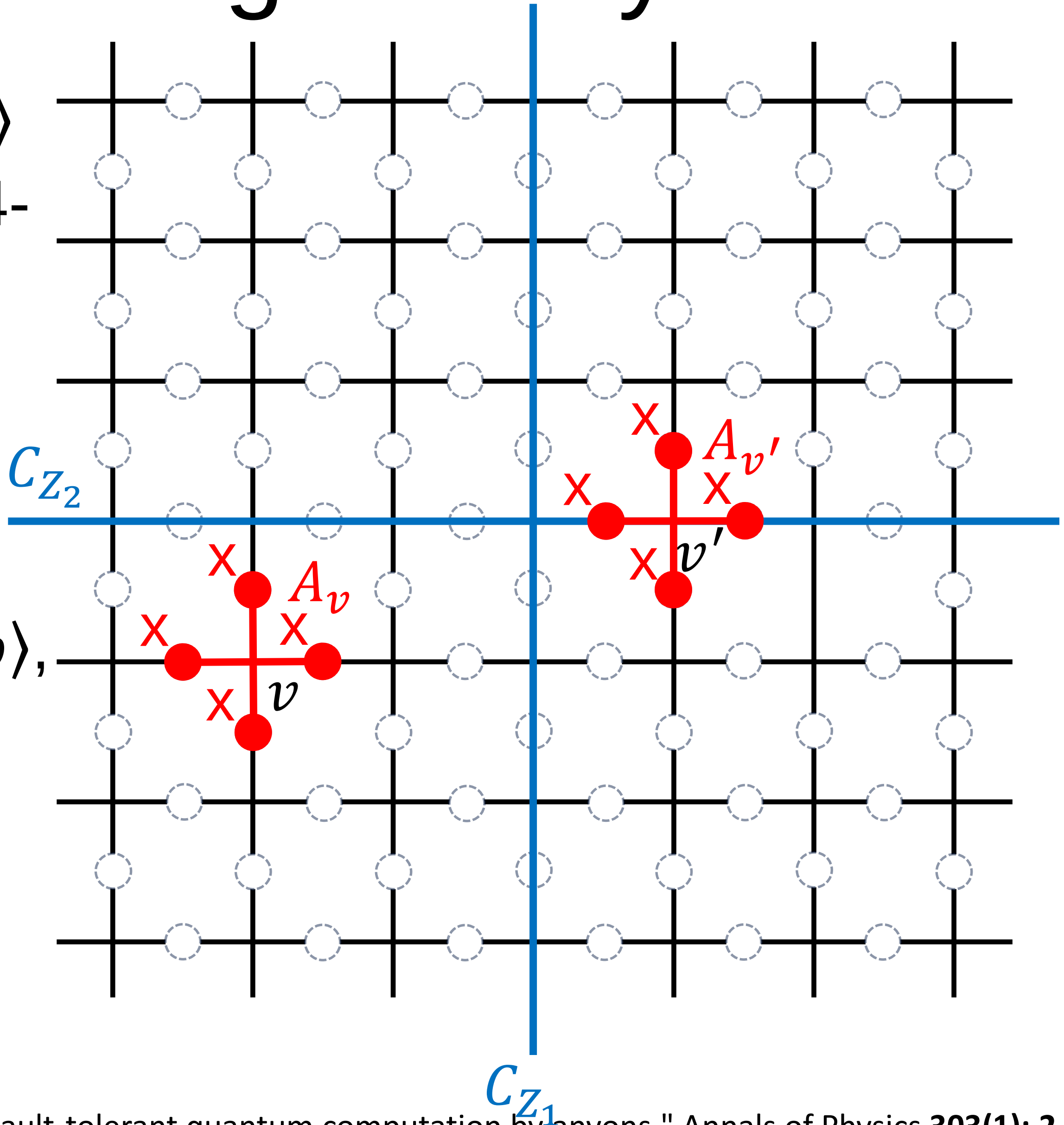
Claim 2: Take $|a,b\rangle = \frac{1+aZ_1}{2}\frac{1+bZ_2}{2}|\psi_{GS}\rangle$ for $a,b=\pm1$. Then, $|a,b\rangle$ composes 4-fold degenerate ground states.

$[A_v, Z_i] = \left[\prod_{j\in *(v)} \sigma_x^j, \prod_{j\in C_{Z_i}} \sigma_z^j\right] = 0,$

$[B_p, Z_i] = \left[\prod_{j\in \partial(p)} \sigma_z^j, \prod_{j\in C_{Z_i}} \sigma_z^j\right] = 0.$

$\therefore A_v|a,b\rangle = A_v\frac{1+aZ_1}{2}\frac{1+bZ_2}{2}|\psi_{GS}\rangle = |a,b\rangle,$

$B_p|a,b\rangle = B_p\frac{1+aZ_1}{2}\frac{1+bZ_2}{2}|\psi_{GS}\rangle = |a,b\rangle.$



Kitaev, A. Y. (2003). "Fault-tolerant quantum computation by anyons." Annals of Physics **303(1): 2-30.**

Remarks for Appendix

1. Toric Code system has four-fold degenerate ground states given by $|a,b\rangle = c\,\frac{1+aZ_1}{2}\frac{1+bZ_2}{2}\prod_v (1+A_v)\,|\uparrow\uparrow\uparrow\cdots\rangle$ for $a,b=\pm 1$.

2. Its degeneracy is based on the system’s topological property: if it were not for torus, say for 2D square lattice, the system has non-degenerate ground state, since every loop can be represented as a product of stabilizers. (i.e., every loop is contractible.)

3. Based on the four degenerate ground states, we can construct 2 qubits and thus can be used in the field of quantum computation.