Capitolo con i dettagli della Relatività Speciale applicata all'astrofisica, ovvero trasformazioni per corpi estesi, misurazione di intervalli di tempo e lunghezze effettuati grazie alla luce (e non righelli), collimazione relativistica

Il capitolo è tratto dalle dispense di un corso di qualche anno fa di Processi Radiativi per l'Astrofisica, tenuto da Gabriele Ghisellini per l'Università di Milano-Bicocca. Questa versione delle dispense si può scaricare qui: https://arxiv.org/abs/1202.5949 (link al pdf in alto a destra). L'ultimo capitolo riguarda i Nuclei Galattici Attivi, ovvero i buchi neri supermassicci in accrescimento e tutte le firme osservative della materia circostante influenzata dalla loro presenza, non soltanto il disco di accrescimento. Segue il modello unificato che è stato menzionato durante la nostra lezione.

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Chapter 3

Beaming

3.1 Rulers and clocks

Special relativity taught us two basic notions: comparing dimensions and flow of times in two different reference frames, we find out that they differ. If we measure a ruler at rest, and then measure the same ruler when is moving, we find that, when moving, the ruler is shorter. If we syncronize two clocks at rest, and then let one move, we see that the moving clock is delaying. Let us see how this can be derived by using the Lorentz transformations, connecting the two reference frames K (that sees the ruler and the clock moving) and K' (that sees the ruler and the clock at rest). For semplicity, but without loss of generality, consider a a motion along the x axis, with velocity $v \equiv \beta c$ corresponding to the Lorentz factor Γ. Primed quantities are measured in K'. We have:

$$\begin{aligned} x' &= \Gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \Gamma\left(t - \beta\frac{x}{c}\right) \end{aligned} \tag{3.1}$$

with the inverse relations given by

$$\begin{aligned} x &= \Gamma(x' + vt') \\ y &= y' \\ z &= z' \\ t &= \Gamma\left(t' + \beta\frac{x'}{c}\right). \end{aligned} \tag{3.2}$$

The length of a moving ruler has to be measured through the position of its extremes *at the same time* t. Therefore, as $\Delta t = 0$, we have

$$x'_2 - x'_1 = \Gamma(x_2 - x_1) - \Gamma v \Delta t = \Gamma(x_2 - x_1) \tag{3.3}$$

i.e.

$$\Delta x = \frac{\Delta x'}{\Gamma} \rightarrow \text{contraction} \tag{3.4}$$

Similarly, in order to determine a time interval a (lab) clock has to be compared with one in the comoving frame, which has, in this frame, *the same position* x'. Then

$$\Delta t = \Gamma \Delta t' + \Gamma \beta \Delta \frac{x'}{c} = \Gamma \Delta t' \rightarrow \text{dilation} \tag{3.5}$$

An easy way to remember the transformations is to think to mesons produced in collisions of cosmic rays in the high atmosphere, which can be detected even if their lifetime (in the comoving frame) is much shorter than the time needed to reach the earth's surface. For us, on ground, relativistic mesons live longer (for the meson's point of view, instead, it is the length of the travelled distance which is shorter).

All this is correct if we measure lengths by comparing rulers (at the same time in K) and by comparing clocks (at rest in K') – the meson lifetime *is* a clock. In other words, **if we do not use photons** for the measurement process.

3.2 Photographs and light curves

If we have an extended moving object and if the information (about position and time) are carried by photons, we **must** take into account their (different) travel paths. When we take a picture, we detect photons arriving at the same time to our camera: if the moving body which emitted them is extended, we must consider that these photons have been emitted at different times, when the moving object occupied different locations in space. This may seem quite obvious. And it is. Nevertheless these facts were pointed out in 1959 (Terrel 1959; Penrose 1959), more than 50 years after the publication of the theory of special relativity.

3.2.1 The moving bar

Let us consider a moving bar, of proper dimension ℓ', moving in the direction of its length at velocity βc and at an angle θ with respect to the line of sight (see Fig. 3.1). The length of the bar in the frame K (according to relativity "without photons") is $\ell = \ell'/\Gamma$. The photon emitted in A_1 reaches the point H in the time interval Δt_e. After Δt_e the extreme B_1 has reached the position B_2, and by this time, photons emitted by the other extreme of the bar can reach the observer simultaneously with the photons emitted by A_1, since the travel paths are equal. The length $B_1 B_2 = \beta c \Delta t_e$, while $A_1 H = c \Delta t_e$. Therefore

$$A_1 H = A_1 B_2 \cos\theta \rightarrow \Delta t_e = \frac{\ell' \cos\theta}{c\Gamma(1-\beta\cos\theta)}. \tag{3.6}$$

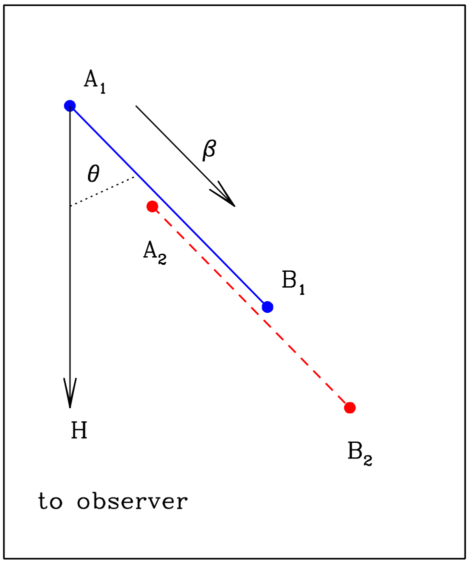

Figure 3.1: A bar moving with velocity βc in the direction of its length. The path of the photons emitted by the extreme A is longer than the path of photons emitted by B. When we make a picture (or a map) of the bar, we collect photons reaching the detector simultaneously. Therefore the photons from A have to be emitted before those from B, when the bar occupied another position.

Note the appearance of the term $\delta = 1/[\Gamma(1-\beta\cos\theta)]$ in the transformation: this accounts for both the relativistic length contraction $(1/\Gamma)$, and the Doppler effect $[1/(1-\beta\cos\theta)]$ (see below, Eq. 3.15). The length A_1B_2 is then given by

$$A_1B_2 \;=\; \frac{A_1H}{\cos\theta} \;=\; \frac{\ell'}{\Gamma(1-\beta\cos\theta)} \;=\; \delta\ell'. \tag{3.7}$$

In a real picture, we would see the projection of A_1B_2, i.e.:

$$HB_2 \;=\; A_1B_2\sin\theta \;=\; \ell'\,\frac{\sin\theta}{\Gamma(1-\beta\cos\theta)} \;=\; \ell'\delta\sin\theta, \tag{3.8}$$

The observed length depends on the viewing angle, and reaches the maximum (equal to ℓ') for $\cos\theta = \beta$.

3.2.2 The moving square

Now consider a square of size ℓ' in the comoving frame, moving at $90°$ to the line of sight (Fig. 3.2). Photons emitted in A, B, C and D have to arrive

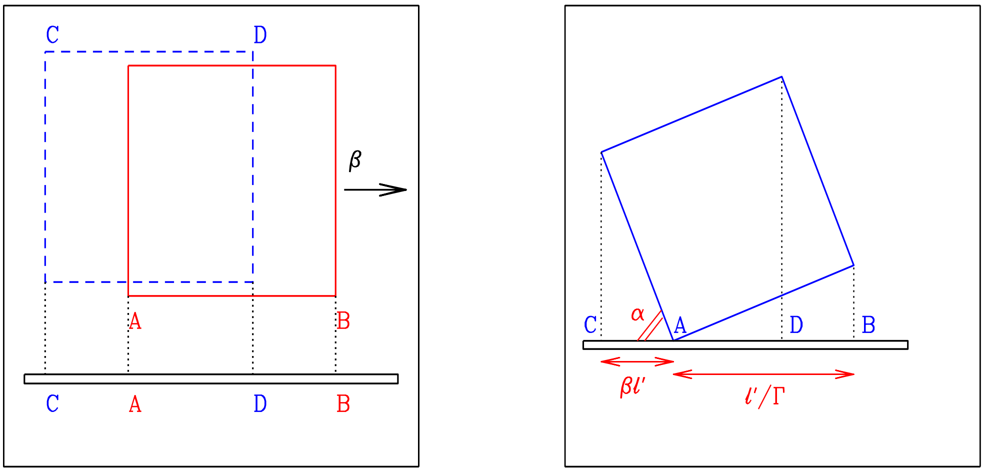

Figure 3.2: *Left:* A square moving with velocity βc seen at 90°. The observer can see the left side (segment CA). Light rays are assumed to be parallel, i.e. the square is assumed to be at large distance from the observer. *Right:* The moving square is seen as *rotated* by an angle α given by $\cos\alpha = \beta$.

to the film plate at the same time. But the paths of photons from C and D are longer $\rightarrow$ they have to be emitted earlier than photons from A and B: when photons from C and D were emitted, the square was in another position.

The interval of time between emission from C and from A is ℓ'/c. During this time the square moves by $\beta\ell'$, i.e. the length CA. Photons from A and B are emitted and received at the same time and therefore $AB = \ell'/\Gamma$. The total observed length is given by

$$CB = CA + AB = \frac{\ell'}{\Gamma}(1 + \Gamma\beta). \tag{3.9}$$

As β increases, the observer sees the side AB increasingly shortened by the Lorentz contraction, but at the same time the length of the side CA increases. The maximum total length is observed for $\beta = 1/\sqrt{2}$, corresponding to $\Gamma = \sqrt{2}$ and to $CB = \ell'\sqrt{2}$, i.e. equal to *the diagonal* of the square. Note that we have considered the square (and the bar in the previous section) to be at large distances from the observer, so that the emitted light rays are all parallel. If the object is near to the observer, we must take into account that different points of one side of the square (e.g. the side AB in Fig. 3.2) have different travel paths to reach the observer, producing additional distortions. See the book by Mook and Vargish (1991) for some interesting illustrations.

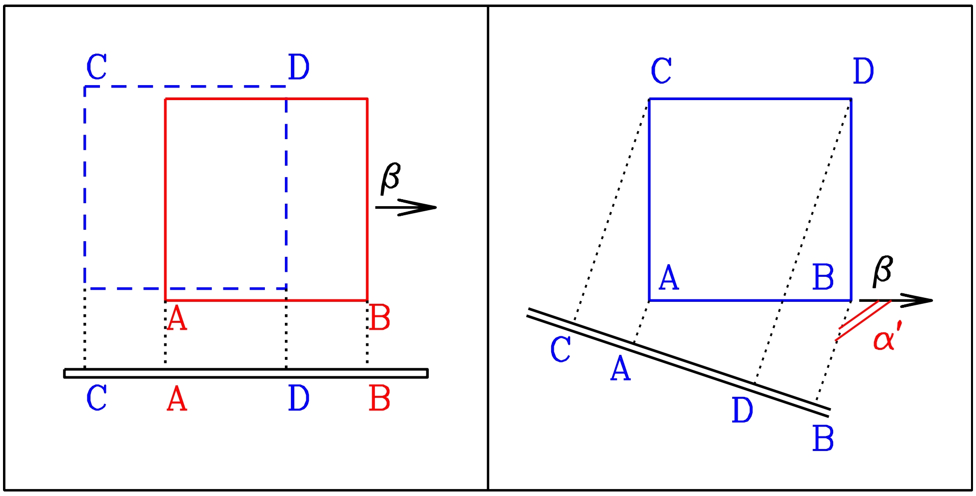

Figure 3.3: An observer that sees the object at rest at a viewing angle given by $\sin\alpha' = \delta\sin\alpha$, will take the same picture as the observer that sees the object moving and making an angle α with his/her line of sight. Note that $\sin\alpha' = \sin(\pi - \alpha')$.

3.2.3 Rotation, not contraction

The net result (taking into account **both** the length contraction **and** the different paths) is an apparent **rotation** of the square, as shown in Fig. 3.2 (right panel). The rotation angle α can be simply derived (even geometrically) and is given by

$$\cos\alpha = \beta \tag{3.10}$$

A few considerations follow:

- If you rotate a sphere you still get a sphere: you **do not** observe a contracted sphere.

- The total length of the projected square, appearing on the film, is $\ell'(\beta + 1/\Gamma)$. It is maximum when the "rotation angle" $\alpha = 45° \to \beta = 1/\sqrt{2} \to \Gamma = \sqrt{2}$. This corresponds to the diagonal.

- The appearance of the square *is the same as what seen in a comoving frame for a line of sight making an angle α' with respect to the velocity vector, where α' is the aberrated angle* given by

$$\sin\alpha' = \frac{\sin\alpha}{\Gamma(1-\beta\cos\alpha)} = \delta\sin\alpha \tag{3.11}$$

  See Fig. 3.3 for a schematic illustration.

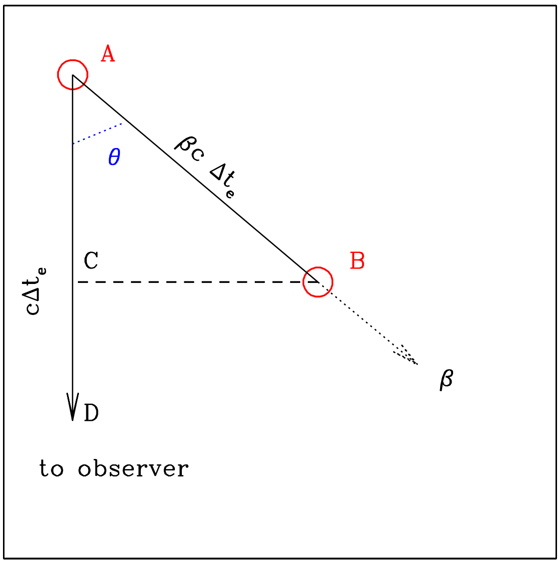

Figure 3.4: Difference between the proper time and the photons arrival time. A lamp, moving with a velocity βc, emits photons for a time interval $\Delta t'_{\rm e}$ in its frame K'. The corresponding time interval measured by an observed at an angle θ, who receives the photons produced by the lamp is $\Delta t_{\rm a} = \Delta t'_{\rm e}/\delta$.

The last point is particularly important, because it introduces a great simplification in calculating not only the appearance of bodies with a complex shape but also the light curves of varying objects.

3.2.4 Time

Consider a lamp moving with velocity $v = \beta c$ at an angle θ from the line of sight. In K', the lamp remains on for a time $\Delta t'_{\rm e}$. According to special relativity ("without photons") the measured time in frame K should be $\Delta t_{\rm e} = \Gamma \Delta t'_{\rm e}$ (time dilation). However, if we use photons to measure the time interval, we once again must consider that the first and the last photons have been emitted in different location, and their travel path lengths are different. To find out $\Delta t_{\rm a}$, the time interval between the arrival of the first and last photon, consider Fig. 3.4. The first photon is emitted in A, the last in B. If these points are measured in frame K, then the path AB is

$$AB = \beta c \Delta t_{\rm e} = \Gamma \beta c \Delta t'_{\rm e} \tag{3.12}$$

While the lamp moved from A to B, the photon emitted when the lamp was in A has travelled a distance $AC = c\Delta t_{\rm e}$, and is now in point D. Along the direction of the line of sight, the first and the last photons (the ones emitted in A and in B) are separated by CD. The corresponding time interval, CD/c, is the interval of time $\Delta t_{\rm a}$ between the arrival of the first and the last photon:

$$\begin{aligned}\Delta t_{\rm a} &= \frac{CD}{c} = \frac{AD-AC}{c} = \Delta t_{\rm e} - \beta\Delta t_{\rm e}\cos\theta \\ &= \Delta t_{\rm e}(1-\beta\cos\theta) \\ &= \Delta t'_{\rm e}\Gamma(1-\beta\cos\theta) \\ &= \frac{\Delta t'_{\rm e}}{\delta}\end{aligned} \tag{3.13}$$

If θ is small and the velocity is relativistic, then $\delta > 1$, and $\Delta t_{\rm a} < \Delta t_{\rm s}$, i.e. we measure a *time contraction* instead of time dilation. Note also that we recover the usual time dilation (i.e. $\Delta t_{\rm a} = \Gamma\Delta t'_{\rm e}$) if $\theta = 90°$, because in this case all photons have to travel the same distance to reach us.

Since a frequency is the inverse of time, it will transform as

$$\nu = \nu'\,\delta \tag{3.14}$$

It is because of this that the factor δ is called the relativistic Doppler factor. Its definition is then

$$\delta = \frac{1}{\Gamma(1-\beta\cos\theta)} \tag{3.15}$$

Note the two terms:

- The term $1/\Gamma$: this corresponds to the usual special relativity term.
- The term $1/(1-\beta\cos\theta)$: this corresponds to the usual Doppler effect.

The δ factor is the result of the competition of these two terms: for $\theta = 90°$ the usual Doppler term is unity, and only "special relativity" remains: $\delta = 1/\Gamma$. For small θ the term $1/(1-\beta\cos\theta)$ becomes very large, more than compensating for the $1/\Gamma$ factor. For $\cos\theta = \beta$ (i.e. $\sin\theta = 1/\Gamma$) we have $\delta = \Gamma$. For $\theta = 0°$ we have $\delta = \Gamma(1+\beta)$.

3.2.5 Aberration

Another very important effect happening when a source is moving is the aberration of light. It is rather simple to understand, if one looks at Fig. 3.5. A source of photons is located perpendilarly to the right wall of a lift. If the lift is not moving, and there is a hole in its right wall, then the ligth ray enters in A and ends its travel in B. If the lift is not moving, A and B are at the same heigth. If the lift is moving with a constant velocity v to the top, when the photon smashes the left wall it has a different location, and the point B will have, for a comoving observer, a smaller height than A. The light ray path now appears oblique, tilted. Of course, the greater v, the more tilted the light ray path appears. This immediately stimulate the question: what happens if the lift, instead to move with a constant velocity, is accelerating? With this example one can easily convince him/herself that the “trajectory” of the photon would appear curved. Since, by the equivalence principle, the accelerating lift cannot tell if there is an engine pulling him up or if there is a planet underneath it, we can then say that gravity bends the light rays, and make the space curved.

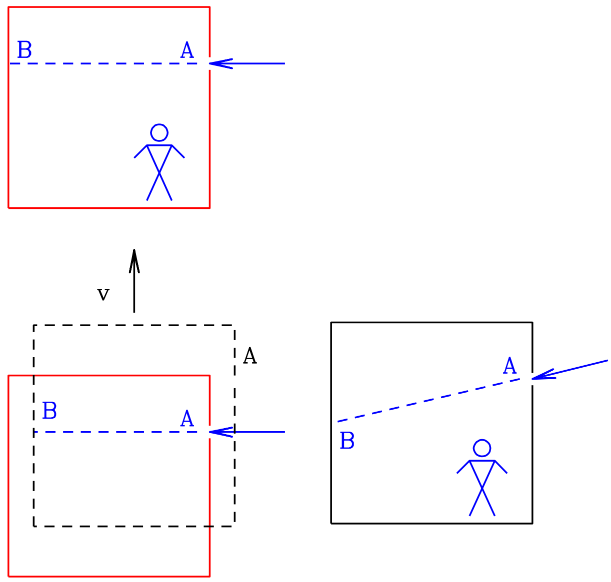

Figure 3.5: The relativistic lift, to explain relativistic aberration of light. Assume first a non–moving lift, with a hole on the right wall. A light ray, coming perpendicularly to the right wall, enter through the wall in A and ends its travel in B. If the lift is moving with a constant velocity v to the top, its position is changed when the photon arrives to the left wall. For the comoving observer, therefore, it appears that the light path is tilted, since the point B where the photon smashes into the left wall is below the point A. What happens if the lift, instead to move with a constant velocity, is accelerating?

This helps to understand why angles, between two inertial frames, change. Calling θ the angle between the direction of the emitted photon and the source velocity vector, we have:

$$\begin{aligned} \sin\theta &= \frac{\sin\theta'}{\Gamma(1+\beta\cos\theta')}; \qquad \sin\theta' = \frac{\sin\theta}{\Gamma(1-\beta\cos\theta)} \\ \cos\theta &= \frac{\cos\theta'+\beta}{1+\beta\cos\theta'}; \qquad \cos\theta' = \frac{\cos\theta-\beta}{1-\beta\cos\theta} \end{aligned} \tag{3.16}$$

Note that, if $\theta' = 90^\circ$, then $\sin\theta = 1/\Gamma$ and $\cos\theta = \beta$. Consider a source emitting isotropically in K'. Half of its photons are emitted in one emisphere, namely, with $\theta' \leq 90°$. Then, in K, the same source will appear to emit half of its photons into a cone of semiaperture $1/\Gamma$.

Assuming symmetry around the angle ϕ, the transformation of the solid angle $d\Omega$ is

$$d\Omega \;=\; 2\pi d\cos\theta = \frac{d\Omega'}{\Gamma^2(1+\beta\cos\theta')^2} = d\Omega'\,\Gamma^2(1-\beta\cos\theta)^2 = \frac{d\Omega'}{\delta^2} \quad (3.17)$$

3.2.6 Intensity

We now have all the ingredients necessary to calculate the transformation of the specific (i.e. monochromatic) and bolometric intensity. The specific intensity has the unit of energy per unit surface, time, frequency and solid angle. In cgs, the units are [erg cm^{-2} s^{-1} Hz^{-1} ster^{-1}]. We can then write the specific intensity as

$$\begin{aligned} I(\nu) &= h\nu\frac{dN}{dt\,d\nu\,d\Omega\,dA} \\ &= \delta h\nu'\frac{dN'}{(dt'/\delta)\,\delta d\nu'\,(d\Omega'/\delta^2)\,dA'} \\ &= \delta^3\,I'(\nu') \;=\; \delta^3 I'(\nu/\delta) \end{aligned} \quad (3.18)$$

Note that $dN = dN'$ because it is a number, and that $dA = dA'$. If we integrate over frequencies we obtain the bolometric intensity which transforms as

$$I \;=\; \delta^4 I' \quad (3.19)$$

The fourth power of δ can be understood in a simple way: one power comes from the transformation of the frequencies, one for the time, and two for the solid angle. They all add up. This transformation is at the base of our understanding of relativistic sources, namely radio–loud AGNs, gamma–ray bursts and galactic superluminal sources.

3.2.7 Luminosity and flux

The transformation of fluxes and luminosities from the comoving to the observer frames is not trivial. The most used formula is $L = \delta^4 L'$, but this assumes that we are dealing with a single, spherical blob. It can be simply derived by noting that $L = 4\pi d_{\rm L}^2 F$, where F is the observed flux, and by considering that the flux, for a distance source, is $F \propto \int_{\Omega_s} I d\Omega$. Since Ω_s is the source solid angle, which is the same in the two K and K' frames, we have that F transforms like I, and so does L. But the emission from jets may come not only by a single spherical blob, but by, for instance, many blobs, or even by a continuous distribution of emitting particles flowing in the jet. If we assume that the walls of the jet are fixed, then the concept of "comoving" frame is somewhat misleading, because if we are comoving with the flowing plasma, then we see the walls of the jet which are moving.

A further complication exists if the velocity is not uni–directional, but radial, like in gamma–ray bursts. In this case, assume that the plasma is contained in a conical narrow shell (width smaller than the distance of the shell from the apex of the cone). The observer which is moving together with a portion of the plasma, (the nearest case of a "comoving observer") will see the plasma close to her going away from her, and more so for more distant portions of the plasma. Indeed, there could be a limiting distance beyond which the two portions of the shells are causally disconnected.

Useful references are Lind & Blandford (1985) and Sikora et al. (1997).

3.2.8 Emissivity

The (frequency integrated) emissivity j is the energy emitted per unit time, solid angle and volume. We generally have that the intensity, for an optically thin source, is $I = \int_{\Delta R} j dr$, where ΔR is the length of the region containing the emitting particles. The emissivity transforms like $j = j'\delta^3$, namely with one power of δ less than the intensity.

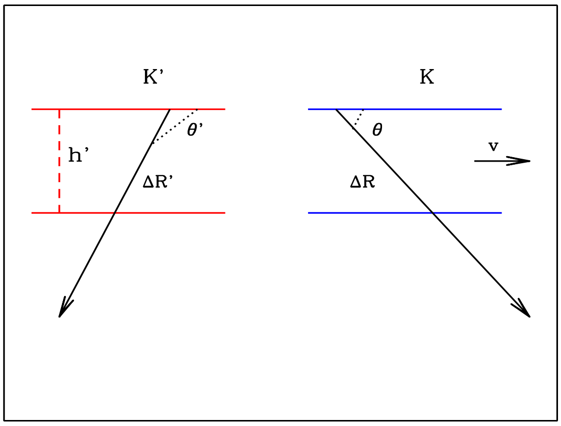

Figure 3.6: Due to aberration of light, the travel path of the a light ray is different in the two frames K and K'

To understand why, consider a slab with plasma flowing with a velocity parallel to the walls of the slab, as in Fig. 3.6. The observer in K will measure a certain ΔR which depends on her viewing angle. In K' the same path has a different length, because of the aberration of light. The height of the slab $h' = h$, since it is perpendicular to the velocity. The light ray travels a distance $\Delta R = h/\sin\theta$ in K, and the same light ray travels a distance $\Delta R' = h'/\sin\theta'$ in K'. Since $\sin\theta' = \delta\sin\theta$, then $\Delta R' = \Delta R/\delta$. Therefore the column of plasma contributing to the emission, for $\delta > 1$, is less than what the observer in K would guess by measuring ΔR. For semplicity, assume that the plasma is homogenous, allowing to simply write $I = j\Delta R$. In this case:

$$I = j\Delta R = \delta^4 I' = \delta^4 j' \Delta R' \rightarrow j = \delta^3 j' \tag{3.20}$$

And the corresponding transformation for the specific emissivity is $j(\nu) = \delta^2 j'(\nu')$.

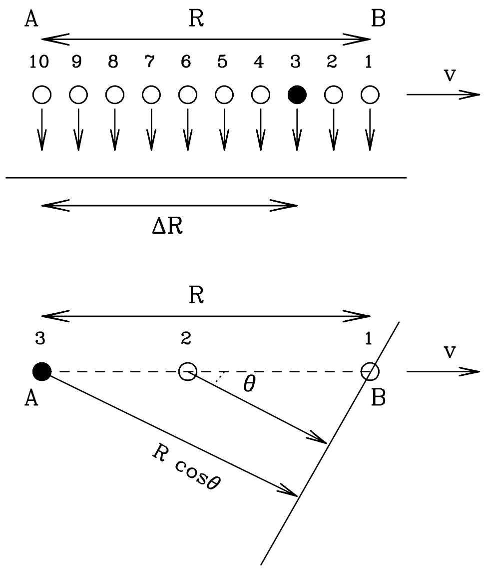

Figure 3.7: Due to the differences in light travel time, the number of blobs that can be observed simultaneously at any given time depends on the viewing angle and the velocity of the blobs. In the top panel the viewing angle is $\theta = 90°$ and all the blobs contained within a certain distance R can be seen. For smaller viewing angles, less blobs are seen. This is because the photons emitted by the rear blobs have more distance to travel, and therefore they have to be emitted before the photons produced by the front blob. Decreasing the viewing angle θ we see less blobs (3 for the case illustrated in the bottom panel).

Fig. 3.7 illustrates another interesting example, taken from the work of Sikora et al. (1997). Consider that within a distance R from the apex of a jet (R measured in K), at any given time there are N blobs (10 on the specific example of Fig. 3.7), moving with a velocity $v = \beta c$ along the jet. To fix the ideas, let assume that beyond R they switch off. If the viewing angle is $\theta = 90°$, the photons emitted by each blob travel the same distance to reach the observer, who will see all the 10 blobs. But if $\theta < 90°$, the photons produced by the rear blobs must travel for a longer distance in order to reach the observer, and therefore they have to be emitted before the photons produced by the front blob. The observer will then see *less* blobs. To be more quantitative, consider a viewing angle $\theta < 90°$. Photons emitted by blob numer 3 to reach blobs number 1 when it produces its last photon (before to switch off) were emitted when the blobs itself was just born (it was crossing point A). They travelled a distance $R\cos\theta$ in a time Δt. During the same time, the blob number 3 travelled a distance $\Delta R = c\beta\Delta t$ in the forward direction. The fraction f of blobs that can be seen is then

$$f = \frac{R - \Delta R}{R} = 1 - \frac{c\beta\Delta t}{R} = 1 - \beta\cos\theta \tag{3.21}$$

Where we have used the fact that $\Delta t = (R/c)\cos\theta$. This is the usual Doppler factor. We may multiply and divide by Γ to obtain

$$f = \frac{1}{\Gamma\delta} \tag{3.22}$$

The bottom line is the following: even if the flux from a single blob is boosted by δ^4, if the jet is made by many (N) equal blobs, the total flux is not just boosted by $N\delta^4$ times the intrinsic flux of a blob, because the observer will see less blobs if $\theta < 90°$.

3.2.9 Brightness Temperature

The brightness temperature is a quantity used especially in radio astronomy, and it is defined by

$$T_{\rm B} \equiv \frac{I(\nu)}{2k}\frac{c^2}{\nu^2} = \frac{F(\nu)}{2\pi k\,\theta_{\rm s}^2}\frac{c^2}{\nu^2} \tag{3.23}$$

where we have assumed that the solid angle subtended by the source is $\Delta\Omega_{\rm s} \sim \pi\theta_{\rm s}^2$, and that the received flux is $F(\nu) = \Delta\Omega_{\rm s} I(\nu)$. There are 2 ways to measure $\theta_{\rm s}$:

1. from VLBI observations, one can often resolve the source and hence directly measure the angular size. In this case the relation between the brightness temperature measured in the K and K' frames is

$$T_{\rm B} = \frac{\delta^3 F'(\nu')}{2\pi k\,\theta_{\rm s}^2}\frac{c^2}{\delta^2(\nu')^2} = \delta\, T'_{\rm B} \tag{3.24}$$

2. If the source is varying, we can estimate its size by requiring that the observed variability time–scale $\Delta t_{\rm var}$ is longer than the light travel time R/c, where R is the typical radius of the emission region. In this case

$$T_{\rm B} > \frac{\delta^3 F'(\nu')}{2\pi k}\frac{d_{\rm A}^2\delta^2}{(c\Delta t'_{\rm var})^2}\frac{c^2}{\delta^2(\nu')^2} = \delta^3\, T'_{\rm B} \tag{3.25}$$

where $d_{\rm A}$ is the angular distance, related to the luminosity distance $d_{\rm L}$ by $d_{\rm A} = d_{\rm L}/(1+z)^2$.

There is a particular class of extragalactic radio sources, called Intra–Day Variable (IDV) sources, showing variability time–scales of hours in the radio band. For them, the corresponding observed brightess temperature can exceed 10^{16} K, a value much larger than the theoretical limit for an incoherent synchrotron source, which is between 10^{11} and 10^{12} K. If the variability is indeed intrinsic, namely not produced by interstellar scintillation, then one would derive a limit on the beamig factor δ, which should be larger than about 100.

3.2.10 Moving in an homogeneous radiation field

Jets in AGNs often moves in an external radiation field, produced by, e.g. the accretion disk, or by the Broad Line Region (BLR) which intercepts a fraction of the radiation produced by the disk and re–emits it in the form of emission lines. It it therefore interesting to calculate what is the energy density seen by a an observer which is comoving with the jet plasma.

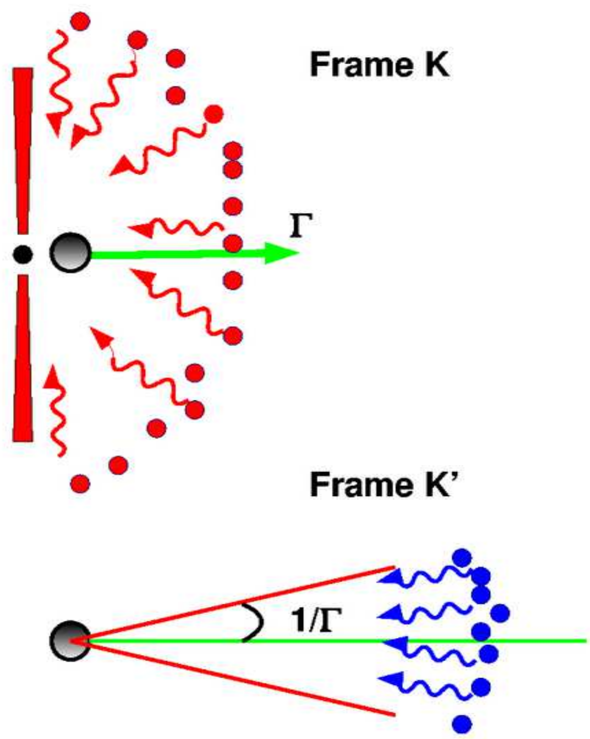

Figure 3.8: A real case: a relativistic bob is moving within the Broad Line Region of a radio loud AGN, with Lorentz factor Γ. In the rest frame K' of the blob the photons coming from 90° in frame K are seen to come at an angle $1/\Gamma$. The energy density as seen by the blob is enhanced by a factor $\sim \Gamma^2$.

To make a specific example, as illustrated by Fig. 3.8, assume that a portion of the jet is moving with a bulk Lorentz factor Γ, velocity βc and that it is surrounded by a shell of broad line clouds. For simplicity, assume that the broad line photons are produced by the surface of a sphere of radius R and that the jet is within it. Assume also that the radiation is monochromatic at some frequency ν_0 (in frame K). The comoving (in frame K') observer will see photons coming from a cone of semi–aperture $1/\Gamma$ (the other half may be hidden by the accretion disk): photons coming from the forward direction are seen blue–shifted by a factor $(1+\beta)\Gamma$, while photons that the observer in K sees as coming from the side (i.e. 90° degrees) will be observed in K' as coming from an angle given by $\sin\theta' = 1/\Gamma$ (and $\cos\theta' = \beta$) and will be blue–shifted by a factor Γ. As seen in K', each element of the BLR surface is moving in the opposite direction of the actual jet velocity, and the photons emitted by this element form an angle θ' with respect the element velocity. The Doppler factor used by K' is then

$$\delta' = \frac{1}{\Gamma(1-\beta\cos\theta')} \tag{3.26}$$

The intensity coming from each element is seen boosted as (cfr Eq. 3.2.10):

$$I' = \delta'^4 I \tag{3.27}$$

The radiation energy density is the integral over the solid angle of the intensity, divided by c:

$$\begin{aligned} U' &= \frac{2\pi}{c}\int_\beta^1 I' d\cos\theta' \\ &= \frac{2\pi}{c}\int_\beta^1 \frac{I}{\Gamma^4(1-\beta\cos\theta')^4} d\cos\theta' \\ &= \left(1+\beta+\frac{\beta^2}{3}\right)\Gamma^2\,\frac{2\pi I}{c} \\ &= \left(1+\beta+\frac{\beta^2}{3}\right)\Gamma^2\, U \end{aligned} \tag{3.28}$$

Note that the limits of the integral correspond to the angles $0'$ and 90° in frame K. The radiation energy density, in frame K', is then boosted by a factor $(7/3)\Gamma^2$ when $\beta \sim 1$. Doing the same calculation for a sphere, one would obtain $U' = \Gamma^2 U$.

Furthermore a (monochromatic) flux in K is seen, in K', at different frequencies, between $\Gamma\nu_0$ and $(1+\beta)\Gamma\nu_0$, with a slope $F'(\nu') \propto \nu'^2$. Why the slope ν'^2? This can be derived as follows: we already know that $I'(\nu') = \delta'^3 I(\nu) = (\nu'/\nu)^3 I(\nu)$. The flux at a specific frequency is

$$F'(\nu') = 2\pi \int_{\mu_1'}^{\mu_2'} d\mu' \left(\frac{\nu'}{\nu}\right)^3 I(\nu) \tag{3.29}$$

where $\mu' \equiv \cos\theta'$, and the integral is over those μ' contributing at ν'. Since

$$\frac{\nu'}{\nu} = \delta' = \frac{1}{\Gamma(1-\beta\mu')} \to \mu' = \frac{1}{\beta}\left(1 - \frac{\nu}{\Gamma\nu'}\right) \tag{3.30}$$

we have

$$d\mu' = -\frac{d\nu}{\beta\Gamma\nu'} \tag{3.31}$$

Therefore, if the intensity is monochromatic in frame K, i.e. $I(\nu) = I_0\delta(\nu - \nu_0)$, the flux density in the comoving frame is

$$\begin{aligned} F'(\nu') &= 2\pi \int_{\nu_2}^{\nu_1} \frac{d\nu}{\beta\Gamma\nu'} \left(\frac{\nu'}{\nu}\right)^3 I_0\delta(\nu-\nu_0) \\ &= \frac{2\pi}{\Gamma\beta}\frac{I_0}{\nu_0^3}\nu'^2; \qquad \Gamma\nu_0 \le \nu' \le (1+\beta)\Gamma\nu_0 \end{aligned} \tag{3.32}$$

where the frequency limits corresponds to photons produced in an emysphere in frame K, and between $0°$ and $\sin\theta' = 1/\Gamma$ in frame K'. Integrating Eq. 3.32 over frequency, one obtains

$$F' = 2\pi I_0\Gamma^2\left(1+\beta+\frac{\beta^2}{3}\right) = \Gamma^2\left(1+\beta+\frac{\beta^2}{3}\right) F \tag{3.33}$$

in agreement with Eq. 3.28.

3.3 Superluminal motion

In 1971 the Very Long Baseline Interferometry began, linking different radio–telescopes that where distant even thousands of km. The resolving power of a telescope is of the order of

$$\phi \sim \frac{\lambda}{D} \tag{3.34}$$

where λ is the wavelength to be observed, and D is the diameter of the telescope or the distance of two connected telescopes. Observing at 1 cm, with two telescopes separated by 1000 km (i.e.10^8 cm), means that we can observe details of the source down to the milli–arcsec level (m.a.s.). The first observations of the inner jet of radio–loud quasars revealed that the jet structure was not continuous, but blobby, with several radio knots. Repeating the observations allowed us to discover that the blobs were not stationary, but were moving. Comparing radio maps taken at different times one could measure the angular displacement $\Delta\theta$ between the position of the blob. Knowing the distance d, one could then transform $\Delta\theta$ is a linear size: $\Delta R = d\Delta\theta$. Dividing by the time interval $\Delta t_{\rm a}$ between the two radio maps, one obtains a velocity

$$v_{\rm app} = \frac{d\Delta\theta}{\Delta t_{\rm A}} \tag{3.35}$$

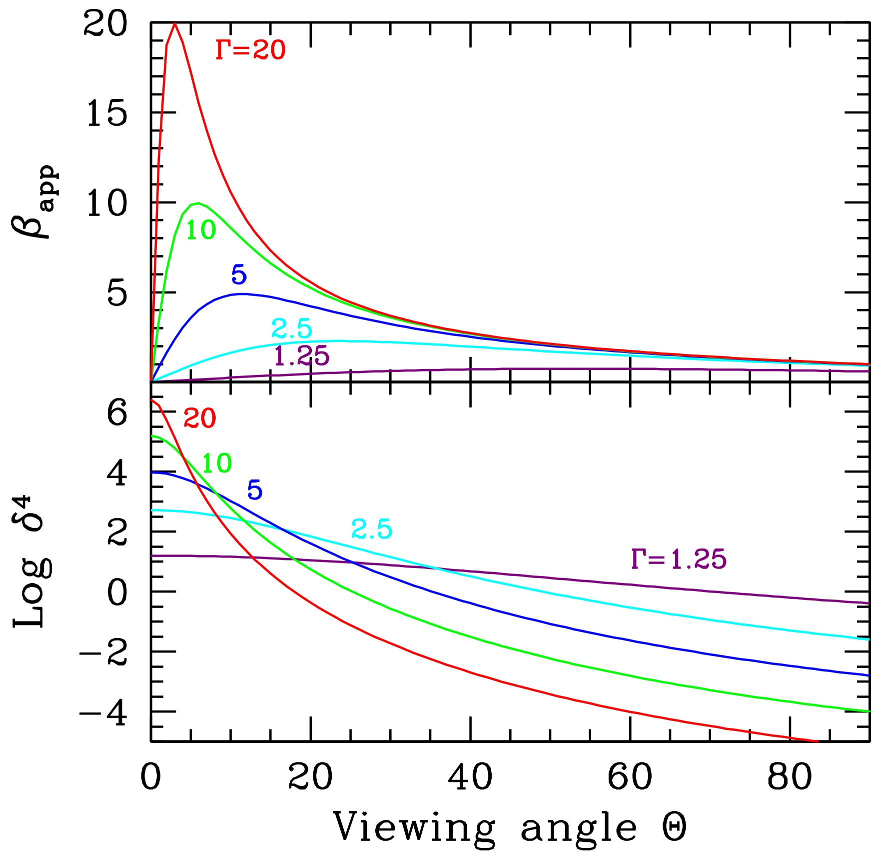

Figure 3.9: Top: The apparent velocity $\beta_{\rm app}$ as a function of the viewing angle θ for different values of Γ, as labelled. Bottom: the amplification δ^4 as a function of the viewing angle, for the same Γ as in the top panel.

With some surprise, in several objects this turned out to be larger than the light speed c. Therefore these sources were called *superluminal*. The explanation of this apparent violation of special relativity is in Fig. 3.4: the time interval $\Delta t_{\rm a}$ can be much shorter than the emission time $\Delta t_{\rm e}$. With reference to Fig. 3.4, what the observer measures in the two radio maps is the position of the blob in point A (first map) and B (second map), projected in the plane of the sky. The observed displacement is then:

$$CB \; = \; \beta c \Delta t_{\rm e} \sin\theta \tag{3.36}$$

Dividing by $\Delta t_{\rm a} = \Delta t_{\rm e}(1-\beta\cos\theta)$ we have the measured apparent velocity as

$$v_{\rm app} \; = \; \frac{\beta c \Delta t_{\rm e} \sin\theta}{\Delta t_{\rm a}} \; \longrightarrow \; \beta_{\rm app} \; = \; \frac{\beta \sin\theta}{1-\beta\cos\theta} \tag{3.37}$$

Ask yourself: Γ does not appear. Is it ok? At 0° the apparent velocity is zero. Is is ok? At what angle $\beta_{\rm app}$ is maximized? What is its maximum value? Fig. 3.9 shows $\beta_{\rm app}$ as a function of the viewing angle (angle between the line of sight and the velocity) for different Γ. Is the apparent superluminal speed given by a real motion of the emitting material? Can it be something else? If there are other possibilities, how to discriminate among them?

$\nu = \nu'\delta$	frequency
$t = t'/\delta$	time
$V = V'\delta$	volume
$\sin\theta = \sin\theta'/\delta$	sine
$\cos\theta = (\cos\theta' + \beta)/(1 + \beta\cos\theta')$	cosine
$I(\nu) = \delta^3 I'(\nu')$	specific intensity
$I = \delta^4 I'$	total intensity
$j(\nu) = j'(\nu')\delta^2$	specific emissivity
$\kappa(\nu) = \kappa'(\nu')/\delta$	absorption coefficient
$T_B = T'_B\delta$	brightn. temp. (size directly measured)
$T_B = T'_B\delta^3$	brightn. temp. (size from variability)
$U' = (1 + \beta + \beta^2/3)\Gamma^2 U$	Radiation energy density within an emisphere

Table 3.1: Useful relativistic transformations

3.4 A question

Suppose that some optically thin plasma of mass m is falling onto a central object with a velocity v and bulk Lorentz factor Γ. The central object has mass M and produces a luminosity L. Assume that the interaction is through Thomson scattering and that there are no electron–positron pairs.

a) What is the radiation force acting on the electrons?

b) What is the gravity force acting on the protons?

c) What definition of limiting ("Eddington") luminosity would you give in this case?

d) What happens if the plasma is instead going outward?

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