PARAMETRIZATION OF THE PROPAGATOR AND VERTEX FOR SPIN 3/2 RESONANCES FOR THE STUDY OF PION-NUCLEON SCATTERING

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Abstract

The exact formulation of quantum field theories for fundamental particles with spin $\frac{3}{2}$ represents a significant challenge in theoretical physics due to the inherent complexities in describing these systems. In particular, elastic pion-nucleon scattering involves intermediate states with spin $\frac{3}{2}$, corresponding to the $\Delta(1232)$ resonance, which underscores the importance of studying these fields. In this work, we perform a systematic analysis of the description of Rarita-Schwinger fields and the parameterization of the propagator within an effective Lagrangian model. This model is consistently constructed to preserve the relevant symmetries, allowing for the generation of the necessary amplitudes to describe the pion-nucleon scattering process. This approach enables the accurate calculation of physical observables, such as the cross-section, which are crucial for understanding the dynamic properties of the involved hadronic resonances.

Free Field $\frac{3}{2}$

The structure of this field is obtained from the tensor product of a Dirac spinor and a vector field, which in the fundamental Lorentz representation becomes:

$$\begin{aligned}\psi_\mu \rightarrowtail V_\mu \otimes \psi : \left(\frac{1}{2},\frac{1}{2}\right) \otimes \left(\left(\frac{1}{2},0\right) \oplus \left(0,\frac{1}{2}\right)\right) = \\ = \left(1,\frac{1}{2}\right) \oplus \left(0,\frac{1}{2}\right) \oplus \left(\frac{1}{2},1\right) \oplus \left(\frac{1}{2},0\right)\end{aligned}$$

In addition to the spin-$\frac{3}{2}$, this product contains two irreducible representations of spin-$\frac{1}{2}$ given by

$$\psi_1(x) = \partial^\mu \psi_\mu \quad \text{and} \quad \psi_2(x) = \gamma^\mu \psi_\mu$$

Rarita-Schwinger-Equation

To described spin-$\frac{3}{2}$, field Rarita-Schwinger proposed the equations

$$(i\partial\!\!\!/ - m)\psi_\mu(x) = 0$$
$$\partial^\mu \psi_\mu = 0$$
$$\gamma^\mu \psi_\mu = 0$$

These equations can be obtained from the Lagrangian:

$$\mathscr{L} = \overline{\psi}_\mu (i\gamma^{\mu\sigma\nu}\partial_\sigma + m\gamma^{\mu\nu})\psi_\nu$$

where $\gamma^{\mu_1 \cdot \mu_n}$ is antisymmetric product of n γ's

$$\gamma^{\mu\nu} \equiv \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\mu\gamma^\nu)$$

$$\gamma^{\mu\rho\nu} = \frac{1}{2}(\gamma^\mu\gamma^\rho\gamma^\nu - \gamma^\mu\gamma^\rho\gamma^\nu)$$

Projectors

The Rarita-Schwinger Lagrangian is writen $\mathcal{L} = \bar{\psi}_\mu \Lambda^{\mu\nu} \psi_\nu$ with

$$\Lambda^{\mu\nu} = (\not{p} - m)(g^{\mu\nu} - \gamma^\mu \gamma^\nu) - \gamma^\mu p^\nu + p^\mu \gamma^\nu$$

It is convenient to rewrite the Lagrangian in function of some projectors generated by the vectors $P_1^\mu = \frac{1}{\sqrt{3}}(\gamma^\mu - \frac{p^\mu}{\not{p}})$ y $P_2^\mu = \frac{p^\mu}{\not{p}}$ that fulfills $P_i \cdot P_j = \delta_{ij}$, $[\not{p}, P_2^\mu] = 0$ and $\{\not{p}, P_1^\mu\} = 0$.

The proyectors are $P_{ij}^{\mu\nu} = P_i^\mu P_j^\nu$ and $P_{00}^{\mu\nu} = g^{\mu\nu} - P_{11}^{\mu\nu} - P_{22}^{\mu\nu}$, which satisfy the algebra

$$P_{ij} \cdot P_{kl} = \delta_{jk} P_{il}$$

Projectors

In function of these projectors the change of basis matrix is given:

$$\begin{pmatrix} g^{\mu\nu} \\ \gamma^{\mu}\gamma^{\nu} \\ \gamma^{\mu}p^{\nu} \\ p^{\mu}\gamma^{\nu} \\ p^{\mu}p^{\nu} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 3 & \sqrt{3} & \sqrt{3} & 1 \\ 0 & 0 & -\sqrt{3}\not{p} & 0 & \not{p} \\ 0 & 0 & 0 & \sqrt{3}\not{p} & \not{p} \\ 0 & 0 & 0 & 0 & p^2 \end{pmatrix} \begin{pmatrix} P_{00}^{\mu\nu} \\ P_{11}^{\mu\nu} \\ P_{12}^{\mu\nu} \\ P_{21}^{\mu\nu} \\ P_{22}^{\mu\nu} \end{pmatrix}$$

On this new basis the operator Λ takes the form (omitting the Lorentz indices)

$$\Lambda = (\not{p} - m)P_{00} - 2(\not{p} - m)P_{11} + \sqrt{3}m(P_{12} + P_{21})$$

Projectors

Using the projectors algebra this operator is directly inverted

$$\Lambda^{-1} = \frac{P_{00}}{\not{p} - m} + \frac{P_{12} + P_{21}}{\sqrt{3}m} + \frac{2(\not{p} + m)}{m^2} P_{22}$$

which function of the tensor operator is written :

$$G^{\mu\nu} = \frac{(\not{p} - m)}{p^2 - m^2} \left(g^{\mu\nu} - \frac{(\gamma^\mu p^\nu + p^\mu \gamma^\nu)}{3m} + \frac{2p^\mu p^\nu}{3m^2} + -\frac{1}{3}\gamma^\mu \gamma^\nu \right)$$

The most general Lagrangian

The most general Lagrangian which satisfies the Rarita-Schwinger equation is given by:

$$\mathscr{L} = \bar{\psi}^{\mu} \Lambda_{\mu\nu} \psi^{\nu}$$

This Lagrangian is invariant under point transformation :

$$\psi^{\mu} \rightarrowtail \psi'^{\mu} = (g^{\mu\nu} + a\gamma^{\mu}\gamma^{\nu})\psi_{\nu}$$
$$A \rightarrowtail A' = \frac{A - 2a}{1 + 4a}$$

With this we find the form for the tensor given by :

$$\Lambda^{\mu\nu} \equiv (\gamma^{\mu}\not{p} - m)g^{\mu\nu} + A(\gamma^{\mu}\not{p}^{\nu} + \not{p}^{\nu}\gamma^{\mu}) + \frac{1}{2}(3A^2 + 2A + 1)\gamma^{\mu}\not{p}\gamma^{\nu} + \\ + m(3A^2 + 3A + 1)\gamma^{\mu}\gamma^{\nu}$$

The Propagator

The form of the propagator associated to this operator is :

$$G_{\mu\nu}(p) = \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}\left[g_{\mu\nu} - \frac{1}{3}\gamma_\mu\gamma_\nu - \frac{1}{3m}(\gamma_\mu p_\nu - \gamma_\nu p_\mu) - \frac{2}{3m^2}p_\mu p_\nu\right]$$
$$+ \frac{i}{3m^2}\frac{A+1}{2A+1}\left[\left(\frac{A+1}{2(2A+1)}\not{p} - \frac{A}{2A+1}m\right)\gamma_\mu\gamma_\nu + \gamma_\mu p_\nu + \frac{A}{2A+1}\gamma_\nu p_\mu\right].$$

with $A = -1$:

$$G^{\mu\nu} = \frac{(\not{p} - m)}{p^2 - m^2}\left(g^{\mu\nu} - \frac{(\gamma^\mu p^\nu + p^\mu \gamma^\nu)}{3m} + \frac{2p^\mu p^\nu}{3m^2} + -\frac{1}{3}\gamma^\mu\gamma^\nu\right)$$

Symmetries of the Strong interaction

The strong interaction is invariant under time reversal $t \to -t$ and parity $\vec{r} \to -\vec{r}$, it is also invariant under charge conjugation which transforms particles into antiparticles. Isospin symmetry is also an important concept in the physics of the strong interaction, isospin symmetry means that the strong interaction is invariant under rotations in isospin space.

Hadron properties and Isospin factors

Hadron	Isospin	Spin$^{(\text{parity})}$	Mass (MeV)	Γ_{total} (MeV)
Pion	1	0^-	$m_{\pi^\pm} = 139.6$ $m_{\pi^0} = 135.0$	"stable" 8.02×10^{-6}
Nucleon	$\frac{1}{2}$	$\frac{1}{2}^+$	$M_{\text{p}} = 938.3$ $M_{\text{n}} = 939.6$	stable "stable"
$P_{33}(1232)$	$\frac{3}{2}$	$\frac{3}{2}^+$	$M_{\Delta^+} = 1206 - 1213$ $M_{\Delta^0} = 1212 - 1214$	$97 - 119$ $103 - 105$
$P_{11}(1440)$	$\frac{1}{2}$	$\frac{1}{2}^+$	M$_R$ =1360-1380	$160 - 190$

Process	$I_{N,\,R}$	I^s_Δ	I^u_Δ
$p\pi^+ \to p\pi^+$	2	1	1/3
$p\pi^- \to p\pi^-$	2	1/3	1

Interaction Lagrangians

1. πNN

$$\mathscr{L}_{\pi NN} = -\frac{f_{\pi NN}}{m_\pi}\left(\overline{\Psi}_N \gamma_5 \gamma_\mu \vec{\tau}\, \Psi_N\right) \cdot \partial^\mu \vec{\Phi}_\pi$$

2. $\pi NR^{\pm}$

$$\mathscr{L}_{\pi NR^{\pm}} = -\frac{f_{\pi NR^{\pm}}}{m_\pi}\left(\overline{\Psi}_N \Gamma_\mu \vec{\tau}\, \Psi_R\right) \cdot \partial^\mu \vec{\Phi}_\pi + \text{h.c.},\ \Gamma_\mu = \begin{cases} \gamma_\mu \gamma_5 & (+) \\ \gamma_\mu & (-) \end{cases}$$

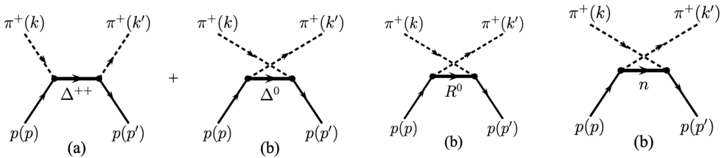
3. $\pi N\Delta$

$$\mathscr{L}_{\pi N\Delta} = \frac{f_{\pi N\Delta}}{m_\pi}\left(\overline{\Psi}^\mu_\Delta \vec{T}\, \Theta_{\mu\nu} \Psi_N\right) \cdot \partial^\nu \vec{\Phi}_\pi + \text{h.c.} \begin{cases} \Psi^\mu_X \rightarrow \Psi^\mu_X + a\gamma^\mu \gamma_\nu \Psi^\nu_X, \\ A \rightarrow \frac{A-2a}{1+4a} \end{cases}$$

where $\Theta_{\mu\nu} \equiv g_{\mu\nu} + \left[\frac{1}{2}(1+4Z)A + Z\right]\gamma_\mu\gamma_\nu \xrightarrow{A=-1} g_{\mu\nu} - \left(\frac{1}{2}+Z\right)\gamma_\mu\gamma_\nu$

Process $\pi N \to \pi N$ (charged)

1. $\pi^+ p \to \pi^+ p$



2. $\pi^- p \to \pi^- p$

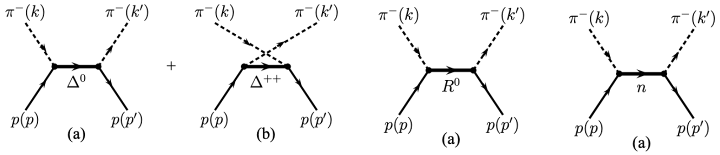


Figure: Feynman Diagrams : (a) direct or s-channel, and (b) crossed or u-chan

Amplitudes

1. N(938), R(1440) (s-channel, u-channel)

$$\mathcal{M}^{s}_{N,\,R} = \frac{f_{\pi N,\,R}}{m_\pi} \mathbf{I}_{N,\,R}\, \overline{u}(p') \not{k}' \gamma^5 S_{N,\,R}(p+k) \not{k} \gamma^5 u(p),$$

$$\mathcal{M}^{u}_{N,\,R} = \frac{f_{\pi N,\,R}}{m_\pi} \mathbf{I}_{N,\,R}\, \overline{u}(p') \not{k} \gamma^5 S_{N,\,R}(p-k) \not{k}' \gamma^5 u(p),$$

2. Δ (1232) (s-channel, u-channel)

$$\mathcal{M}^{s}_{\Delta} = \frac{f_{\pi N \Delta}}{m_\pi} \mathbf{I}^{s}_{\Delta}\, \overline{u}^{\mu}(p') k'^{\nu} \Theta_{\nu\mu} G^{\nu\alpha}(p+k) \Theta_{\alpha\beta} k^{\beta} u(p),$$

$$\mathcal{M}^{u}_{\Delta} = \frac{f_{\pi N \Delta}}{m_\pi} \mathbf{I}^{u}_{\Delta}\, \overline{u}^{\mu}(p') k^{\nu} \Theta_{\nu\mu} G^{\nu\alpha}(p-k) \Theta_{\alpha\beta} k'^{\beta} u(p),$$

Physical Observables → Total Cross-Section

1. Averaged Differential Cross-Section (center of mass)

$$\frac{d\sigma}{d\Omega^*} = \frac{|\vec{k}'|}{2|\vec{k}|} \frac{M_{\mathsf{N}}^2}{16\pi^2 s} \frac{1}{2} \sum_{s_{\mathsf{i}}} \sum_{s_{\mathsf{f}}} |\overline{u}(p')\mathcal{M}u(p)|^2$$

2. Laboratory Kinetic Energy

$$E_\pi = \frac{s^2 - (m_\pi + M)^2}{2M}$$

Results

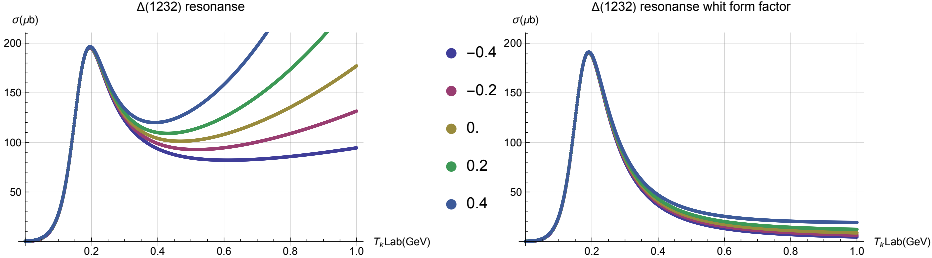


Figure: Δ contribution to Cross Section: (left) without form factor, and (right) With form factor

$$F_1(s) = \frac{\Lambda^4}{\Lambda^4 + (s - m^2)^2} \quad F_2(u) = \frac{\Lambda^4}{\Lambda^4 + (u - m^2)^2}$$

Total cross section

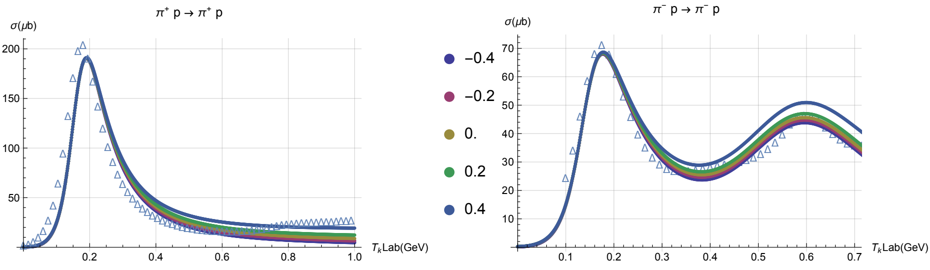


Figure: Total Cross Section: (left) $\pi^+ p \to \pi^+ p$, and (right) $\pi^- p \to \pi^- p$

Conclusions

1. We employ the *Effective Lagrangian Approach* (*ELA*) to analyze pion-nucleon elastic scattering. This method provides a consistent framework to describe the interaction vertices and propagators, incorporating the symmetries of quantum field theory. The *ELA* allows for a systematic inclusion of resonances, such as the $\Delta(1232)$ and $N(1240)$, and facilitates the study of the dynamics of pion-nucleon interactions across different energy regimes.
2. The model adjustment strongly depends on both the coupling constant and the Z-parameter of the vertex, leading to an intrinsic correlation between their values during the fitting process. This interdependence ensures consistency with the well-established experimental data for pion-nucleon scattering, highlighting the importance of these parameters in accurately describing the dynamics of the interaction.

Conclusions

1. The effects of the Z-parameter become noticeable for energies around 0.25 GeV. As momentum increases, its influence on the system's dynamics grows, emphasizing its critical role in describing interactions at higher energies.
2. To capture the experimentally observed behavior of hadrons, appropriate form factors must be included at vertices containing only hadrons, ensuring realistic interaction dynamics and consistency with data.

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