Short Distance Constraints on the Hadronic Light-by-Light contribution to the muon $g-2$

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Outline

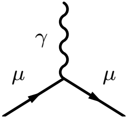
- Status of the $(g-2)_\mu$
- HLbL scattering: the basics
- OPE for the HLbL amplitude in the symmetric case

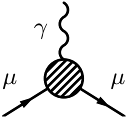
Status of the $(g-2)_\mu$

What is the anomalous magnetic moment of the muon?

Magnetic moment:
$\boldsymbol{\mu} = g\frac{q}{2m}\boldsymbol{S}$

From Dirac's equation: $g = 2$





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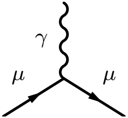
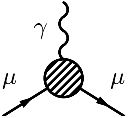
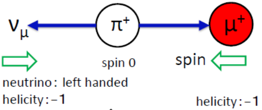




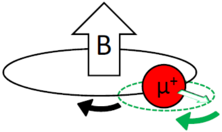
Three steps of g-2 measurement

1. Prepare a polarized muon beam.

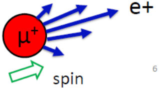


2. Store in a magnetic field (muon's spin precesses)

$$\vec{\omega} = -\frac{e}{m}\left[a_\mu \vec{B} - \left(a_\mu - \frac{1}{\gamma^2-1}\right)\frac{\vec{\beta}\times\vec{E}}{c} + \frac{\eta}{2}\left(\vec{\beta}\times\vec{B} + \frac{\vec{E}}{c}\right)\right]$$



3. Measure decay positron



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Magnetic moment:
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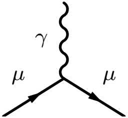
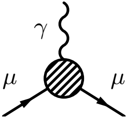
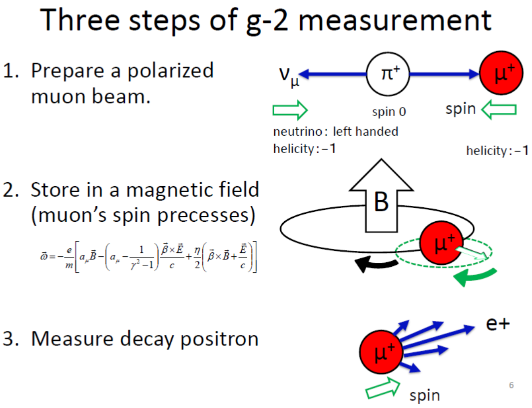


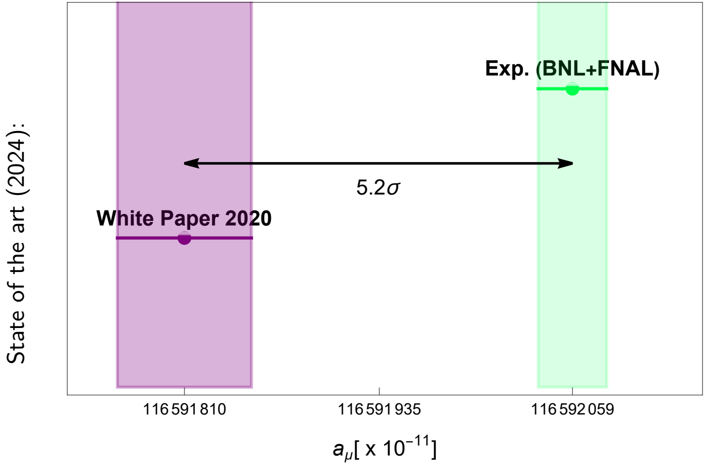




Quantum corrections produce an anomaly:

$$a_\mu = \frac{g-2}{2}$$

Why is the anomalous magnetic moment of the muon relevant?



$$\Delta a_\mu = a_\mu^{\text{exp}} - a_\mu^{\text{WP}} = 249\,(48) \times 10^{-11} \text{ or } \boxed{5.2\ \sigma}\,(?)$$

$$a_\mu^{\text{exp}} = 116\ 592\ 059\,(22) \times 10^{-11} \qquad a_\mu^{\text{WP}} = 116\ 591\ 810\,(43) \times 10^{-11}$$

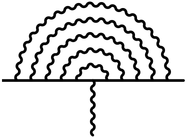
Where does a_μ^{WP} come from?

Theoretical prediction must keep up with present and coming experimental uncertainty improvements.

Where can this reduction come from?

- QED gives the largest contribution, but its uncertainty is under control:

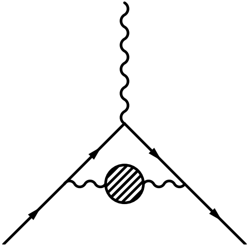
$$a_\mu^{\text{QED}} = 116\,584\,718.931\,(104) \times 10^{-11}$$
$$a_\mu^{\text{WP}} - a_\mu^{\text{QED}} = 7\,091 \times 10^{-11}$$



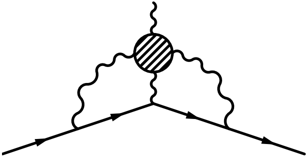
Three loops analytically
Five loops numerically.

Where does a_μ^{WP} come from?

Hadronic vacuum polarization (HVP).



Hadronic Light–by–Light scattering (HLbL).



Where does a_μ^{WP} come from?

➥ The hadronic part has the largest uncertainty:

$$a_\mu^{\text{HVP}} = 6\,845\,(40) \times 10^{-11} \quad \text{Disp., Lattice, CMD-3?}$$
$$a_\mu^{\text{HLbL}} = 92\,(18) \times 10^{-11}$$

➥ HVP is the biggest uncertainty source.

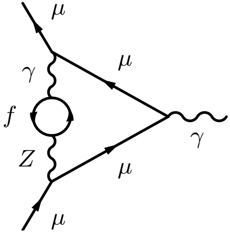
➥ HLbL has the largest relative uncertainty of any of the main contributions!

Where does a_μ^{WP} come from?

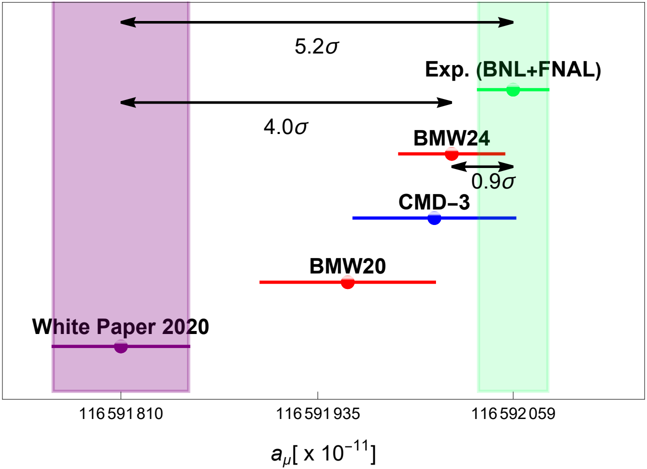
➥ Electroweak contributions are under control:

$$a_\mu^{\text{EW}} = 153.6\,(1.0) \times 10^{-11}$$

Up to two loops.



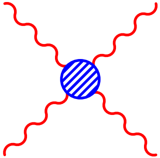
Brief interlude: the state of the HVP contribution



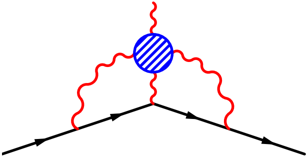
- Tension between dispersive and lattice estimates.
- Tension between experimental results for $e^+e^- \longrightarrow \pi^+\pi^-$
- Why is there tension between theoretical descriptions?

HLbL scattering: the basics

Basics of $\Pi^{\mu_1\mu_2\mu_3\mu_4}$



$$\Pi^{\mu_1\mu_2\mu_3\mu_4} = -i\int d^4x\int d^4y\int d^4z\, e^{-i(q_1x+q_2y+q_3z)} \times \langle\Omega|J^{\mu_1}(x)J^{\mu_2}(y)J^{\mu_3}(z)J^{\mu_4}(0)|\Omega\rangle$$



➥ Loops on q_1 and q_2 sweep very different kinematic regions. Alternate tools required above and below Λ_{QCD}.

Dispersive computation of $\Pi^{\mu_1\mu_2\mu_3\mu_4}$

Analyticity: (Sugawara–Kanazawa + Mandelstam hypothesis)

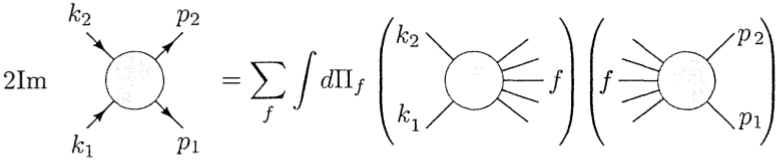
$$g(z) = \sum_i \frac{R_i}{z - x_i} + \frac{1}{\pi}\left(\int_{c_1}^{\infty} + \int_{-\infty}^{-c_2} \right)\frac{\Delta_x g(x)}{x - z}\, dx + \lim_{x\to\infty} \overline{g}(x)$$

$$\Delta_x g(x) = \frac{1}{2i}\{g(x + i\epsilon) - g(x - i\epsilon)\}$$

$$\overline{g}(x) = \frac{1}{2}\{g(x + i\epsilon) + g(x - i\epsilon)\}$$

Unitarity:

Schwarz: $\mathrm{Im}\, g \to \Delta_x g$



Dispersive computation of $\Pi^{\mu_1\mu_2\mu_3\mu_4}$

The "basis" of $\Pi^{\mu_1\mu_2\mu_3\mu_4}$ free of kinematic singularities and zeros:

$$\Pi^{\mu_1\mu_2\mu_3\mu_4} = \sum_{i}^{54} \Pi_i T_i^{\mu_1\mu_2\mu_3\mu_4}$$

Colangelo et al. (2017)

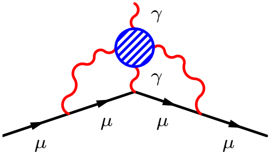
- From 138 to 43 independent tensors vs. 1 in HVP.
- To avoid kinematic zeros/singularities it is necessary to expand to 54 redundant Π_i. (Tarrach)
- Richer tensor structure made BTT basis much harder to find.

OPE for the HLbL amplitude in the symmetric case

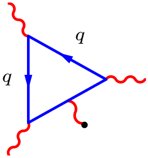
The leading contribution: a quark loop

Bijnens et al. (2019)

$Q_1^2, Q_2^2, Q_3^2 \gg \Lambda^2_{\text{QCD}}$



$\Longrightarrow$



Wilson coefficient of $\langle F_{\mu\nu} \rangle$, proportional to:

$$\int \frac{d^4p}{(2\pi)^4} \partial^{\nu_4} \sum \gamma^{\mu_3} S^0(p+q_1+q_2+q_4)\gamma^{\mu_4} S^0(p+q_1+q_2)\gamma^{\mu_1} S^0(p+q_2)\gamma^{\mu_2} S^0(p) \Big|_{q_4=0}$$

Due to operator mixing, only its massless part is the true LO of the OPE.

Quark loop computation

Bijnens et al. (2020) uses the standard approach:

- ➥ Project $\overline{\Pi}_i$ out of the quark loop.
- ➥ Expand to $O(m^2)$.
- ➥ Use Integration By Parts (IBP) to arrive at a small number of master integrals.

Quark loop computation

We tensor decompose the quark loop amplitude following Davydychev et al. (1991), for instance

$$
\begin{aligned}
I^{(2)}_{\mu_1\mu_2}(d;\nu_1,\nu_2) = &-\frac{1}{2\pi} g_{\mu_1\mu_2} I^{(2)}(d+2;\{\nu_i\}) \\
&+ \frac{1}{\pi^2} q_{1\mu_1} q_{1\mu_2} \nu_1(\nu_1+1) I^{(2)}(d+4;\nu_1+2,\nu_2) \\
&+ \frac{1}{\pi^2} q_{2\mu_1} q_{2\mu_2} \nu_2(\nu_2+1) I^{(2)}(d+4;\nu_1,\nu_2+2) \\
&+ \frac{1}{\pi^2} (q_{1\mu_1} q_{2\mu_2} + q_{1\mu_2} q_{2\mu_1}) \nu_1 \nu_2 I^{(2)}(d+4;\nu_1+2,\nu_2+1)
\end{aligned}
$$

It does not introduce kinematic singularities, but at the cost of shifting dimensions.

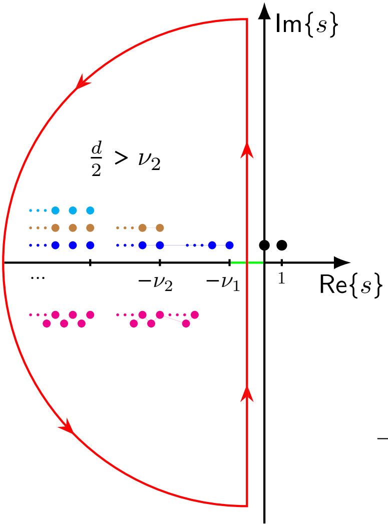
Scalar integrals in shifted dimensions

We find 133 different scalar loop integrals in shifted dimensions.

Davydychev (1991) gives Mellin–Barnes (MB) representation for arbitrary scalar integrals:

$$I^{(2)}(d;\nu_1,\nu_2) = \int_{s_1} \left(-\frac{q_1^2}{m^2}\right)^{s_1} \Gamma(-s_1) \frac{\Gamma(s_1+\nu_1)\Gamma(s_1+\nu_2)}{\Gamma(2s_1+\nu_1+\nu_2)} \times \Gamma\left(s_1+\nu_1+\nu_2-\frac{d}{2}\right)$$

Scalar integrals in shifted dimensions



$$I^{(2)}(d;\nu_1,\nu_2) =$$

$$\int_{s_1}\left(-\frac{q_1^2}{m^2}\right)^{s_1}\Gamma(-s_1)\frac{\Gamma(s_1+\nu_1)\Gamma(s_1+\nu_2)}{\Gamma(2s_1+\nu_1+\nu_2)}$$

$$\times\,\Gamma\left(s_1+\nu_1+\nu_2-\frac{d}{2}\right)$$

$\Gamma\left(s+\nu_1+\nu_2-\frac{d}{2}\right)$	
$\Gamma(s+\nu_2)$	
$\Gamma(s+\nu_1)$	$\Gamma(-s)$
$\Gamma(2s+\nu_1+\nu_2)$	

Scalar integrals in shifted dimensions

Nested MB integrals require multivariate residues (*MBConicHulls (B. Ananthanarayan et al, 2021)*, *Multivariate residues (Larsen, 2017)*)

- ➥ There are several different triple hypergeometric series representation for one integral.
- ➥ Convergence of hypergeometric series studied with Horn's theorem.
- ➥ Too many integrals with too many representations: We wrote a script to check the convergence region of series representations.

Scalar integrals in shifted dimensions

Two–point example for $|q^2| > 4m^2$:

$$I^{(2)}(6;\nu,\nu+1) = \frac{-i}{(4\pi)^{5/2}}\frac{(m^4)^{1-\nu}}{\Gamma(\nu)\Gamma(\nu+1)}$$

$$\times\Bigg\{\left(-\frac{4m^2}{q^2}\right)^{\nu}\sum_{k=0}^{\nu-3}\left(\frac{4m^2}{q^2}\right)^{k}\frac{\Gamma(\nu+k)}{\Gamma(-k+\frac{1}{2})}\frac{\Gamma(-k+\nu-2)}{\Gamma(k+1)}$$

$$-\left(-\frac{16m^4}{q^4}\right)^{\nu-1}\sum_{l=0}^{\infty}\boxed{\left(-\frac{4m^2}{q^2}\right)^{l}}\frac{\Gamma(-2+2\nu+l)}{\Gamma(-\nu-l+\frac{5}{2})}\frac{1}{\Gamma(l+1)\Gamma(l+\nu-1)}$$

$$\times\left(\boxed{\ln\left\{-\frac{q^2}{4m^2}\right\}}-\psi^{(0)}(-2+2\nu+l)-\psi^{(0)}(-\nu-l+\frac{5}{2})-3\psi^{(0)}(l+1)+\cdots\right.$$

Large logs and mass corrections

Check of BTT tensor decomposition of $\Pi^{\mu_1\mu_2\mu_3\mu_4}$

$$\left.\frac{\partial \Pi^{\mu_1\mu_2\mu_3\mu_4}}{\partial q_{4_{\mu_5}}}\right|_{q_4\to 0} = \sum_i^{19} \Pi_i\, \partial^{\mu_5} T_i^{\mu_1\mu_2\mu_3\mu_4}\Big|_{q_4\to 0}$$

- All form factors receive contributions from the quark loop? (Weak)

$$\Pi_i \neq 0$$

- Are there contributions to tensor structures not considered in BTT? (Strong)

$$\left.\frac{\partial \Pi^{\mu_1\mu_2\mu_3\mu_4}}{\partial q_{4_{\mu_5}}}\right|_{q_4\to 0} - \sum_i^{19} \Pi_i\, \partial^{\mu_5} T_i^{\mu_1\mu_2\mu_3\mu_4}\Big|_{q_4\to 0} = 0$$

Conclusions

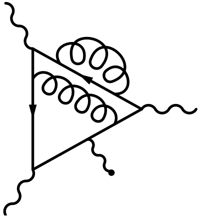
Conclusions

Main results:

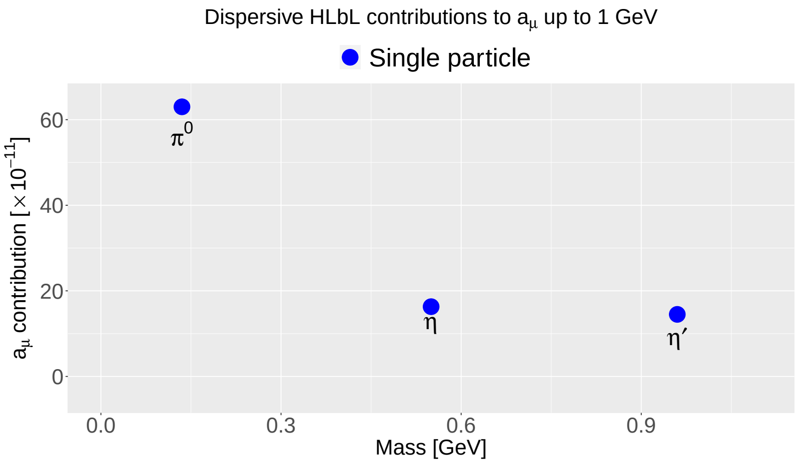
- ➥ Independent check for the quark loop computation.
- ➥ Check of the complete span of the dispersive tensor basis for the quark loop amplitude.
- ➥ Closed expression for all-order mass expansion.

Perspective:

- ➥ Three-loop perturbative result:



Low energy, data–driven a_μ^{HLbL}



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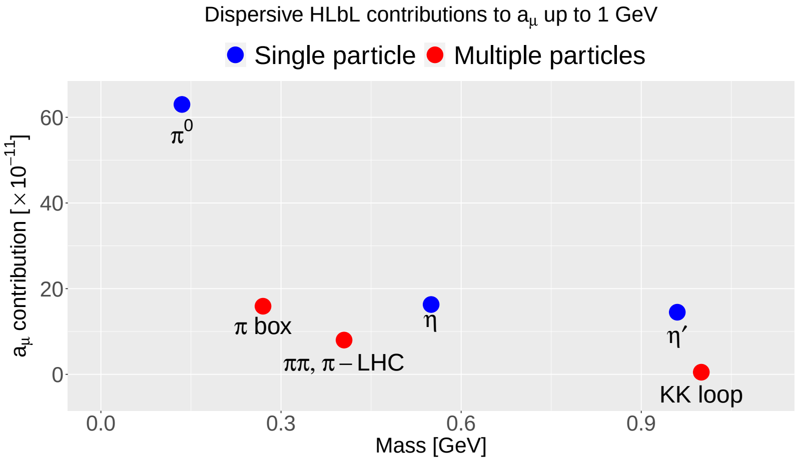


Estimation of a_μ^{HLbL} in 1 GeV – 2 GeV

Hurdles for dispersive computation in the medium–energy region:

- Lack of data for: known form factors at high energy. and new unknown form factors.
- Increasing complexity of unitarity diagrams.
- Double counting issues.

The expected small size of contributions motivates narrow width approximation, which results in:

$$a_\mu^{\text{scalars+tensor}} = -1(3) \times 10^{-11}$$
$$a_\mu^{\text{axial vectors}} = 6(6) \times 10^{-11}$$

a_μ^{HLbL} at high energy and SDC

SDC = asymptotic behaviour of amplitudes.

Useful for:

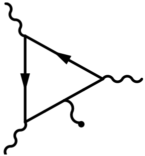
- High energy contributions from light intermediate states. (Form factors)
- Full contributions from heavy intermediate states. (HLbL amplitude)

We focus in the latter case!

Two high virtualities regimes:

$$|q_1^2| \sim |q_2^2| \sim |q_3^2| \gg \Lambda_{\text{QCD}}^2 \qquad |q_1^2| \sim |q_2^2| \gg |q_3^2|,\ \Lambda_{\text{QCD}}^2$$

The quark loop



$S_{1,\mu\nu} \to \Delta a_\mu^{HLbL} = 1.7 \times 10^{-10}$

$S_{2,\mu\nu} \to \Delta a_\mu^{HLbL} = -1.2 \times 10^{-12}$

...

Wilson coefficient of $\langle F_{\mu\nu} \rangle$, proportional to:

$$i\frac{N_c}{2} \int \frac{d^4p}{(2\pi)^4} \frac{\partial}{\partial q_{4\nu_4}} \sum_{\sigma(1,2,4)} \Big\{ \gamma^{\mu_3} S^0(p+q_1+q_2+q_4)\gamma^{\mu_4} \times S^0(p+q_1+q_2)\gamma^{\mu_1} S^0(p+q_2)\gamma^{\mu_2} S^0(p) \Big\} \Big|_{q_4=0}$$