Chern-Simons on the Null Plane

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Null-Plane Coordinates

The null plane time x^+ and longitudinal coordinate x^- are defined, respectively, as

$$x^+ \equiv \frac{x^0 + x^3}{\sqrt{2}} \qquad x^- \equiv \frac{x^0 - x^3}{\sqrt{2}}, \tag{1}$$

with the transverse coordinate kept unchanged. In the null-plane coordinates the metric is given by,

$$\eta_{\mu\nu} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{2}$$

Explicitly, $x^+ = x_-$ and the derivatives with respect to x^+ and x^- are defined as: $\partial_+ \equiv \frac{\partial}{\partial x^+}$ and $\partial_- \equiv \frac{\partial}{\partial x^-}$, with $\partial^+ = \partial_-$. The Levi-Civita tensor has the following components,

$$\varepsilon^{+-1} = \varepsilon_{+-1} = 1. \tag{3}$$

Abelian Chern-Simons Theory

The abelian Chern-Simons theory is describe by the following Lagrangian density:

$$\mathcal{L}_{CS} = \frac{\kappa}{4}\varepsilon^{\mu\nu\alpha}F_{\mu\nu}A_{\alpha}, \tag{4}$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. The Lagrangian above is invariant by the following transformations,

$$A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \Lambda(x) \quad , \quad \mathcal{L}_{CS} \rightarrow \mathcal{L}_{CS} + \partial_\alpha\left(\frac{\kappa}{4e}\varepsilon^{\mu\nu\alpha}F_{\mu\nu}\Omega\right). \tag{5}$$

The field equations are:

$$\frac{\kappa}{2}\varepsilon^{\rho\sigma\alpha}F_{\sigma\alpha} = 0, \tag{6}$$

From (4) it is easy to write the first-order Lagrangian by introducing the momentum π^{μ} with respect to the fields A_{μ},

$$\pi^{\mu} = \frac{\partial \mathcal{L}}{\partial\left(\partial_{+} A_{\mu}\right)} = \frac{\kappa}{2}\varepsilon^{+\rho\alpha} A_{\alpha}. \tag{7}$$

The theory has the following set of primary constraints:

$$\Omega_1 \equiv \pi^{+} \approx 0 \qquad , \qquad \Omega_{2,a} \equiv \pi^{a} - \frac{\kappa}{2}\varepsilon_{ab} A_b \approx 0, \qquad a = -, 1, \tag{8}$$

where we have defined the following antisymmetric tensor,

$$\varepsilon_{-1} = -\varepsilon_{1-} = 1. \tag{9}$$

The canonical Hamiltonian density is defined by,

$$\mathcal{H}_C = \pi^{\mu}\partial_{+}A_{\mu} - \mathcal{L} = -\kappa A_{+}\varepsilon_{ab}\partial_{a}A_{b}. \tag{10}$$

The canonical Hamiltonian take the form,

$$H_C = -\kappa \int d^2x A_{+}\varepsilon_{ab}\partial_{a}A_{b}, \tag{11}$$

where $d^2x \equiv dx^{-}dx^{1}$. Thus, the dynamics is determined by the primary Hamiltonian defined by,

$$H_P = H_C + \int d^2y \left[u^{1}\left(y\right)\Omega_{1}\left(y\right) + u^{2,a}\left(y\right)\Omega_{2,a}\left(y\right)\right]. \tag{12}$$

The fundamental Poisson brackets of the theory are,

$$\left\{ A_\mu(x), \pi^\nu(y) \right\} = \delta_\mu^\nu \delta^2(x-y), \tag{13}$$

The consistence conditions on the constraints are,

$$\dot{\Omega}_1(x) = \left\{ \Omega_1(x), H_P \right\} = \kappa \varepsilon_{ab} \partial_a A_b \approx 0,$$

Then, a secondary constraint arise defined by,

$$\Omega_4 \equiv \varepsilon_{ab} \partial_a A_b \approx 0. \tag{14}$$

The consistence of Ω_4 determine,

$$\dot{\Omega}_4(x) = \left\{ \Omega_4(x), H_P \right\} = \varepsilon_{ab}\partial_a u^{2,b} \approx 0, \qquad (15)$$

thus, no more constraints are generated from Ω_4.

The consistence condition of $\Omega_{2,a}$ determine,

$$\dot{\Omega}_{2,a}(x) = \left\{ \Omega_{2,a}(x), H_P \right\} = u^{2,a} - \varepsilon_{ab}\partial_b A_+ \approx 0. \qquad (16)$$

This relations is consistente with (18).

The theory is characterized by the following set of constraints,

$$\begin{aligned} \Omega_1 &\equiv \pi^{+} \approx 0, \\ \Omega_{2,a} &\equiv \pi^{a} - \frac{\kappa}{2}\varepsilon_{ab}A_b \approx 0, \\ \Omega_4 &\equiv \varepsilon_{ab}\partial_a A_b \approx 0. \end{aligned} \quad (17)$$

The theory has the following set of first class constraints.

$$\Theta_1 \equiv \pi^{+} \approx 0 \qquad , \qquad \Theta_2 \equiv \partial_a \pi^{a} + \frac{\kappa}{2}\varepsilon_{ab}\partial_a A_b \approx 0. \quad (18)$$

and a set os second class constraints,

$$\Phi_a \equiv \pi^{a} - \frac{\kappa}{2}\varepsilon_{ab}A_b \approx 0, \qquad a = -, 1, \quad (19)$$

where,

$$\left\{ \Phi_a\left(x\right), \Phi_b\left(y\right) \right\} = -\kappa\varepsilon_{ab}\delta^2\left(x-y\right). \quad (20)$$

We are going to eliminate Φ_a using the following Dirac Brackets between two dynamical variables,

$$\left\{ A_m(x), B_n(y) \right\}_{D1} = \left\{ A_m(x), B_n(y) \right\} - \int d^2u d^2v \left\{ A_m(x), \Phi_a(u) \right\} C_{ab}^{-1}(u,v) \left\{ \Phi_b(v), B_n(y) \right\}, \quad (21)$$

where C_{ab}^{-1} is the inverse of the second class constraint matrix defined by,

$$C_{ab}(u,v) \equiv \left\{ \Phi_a(x), \Phi_b(y) \right\} = -\kappa \varepsilon_{ab} \delta^2(x-y), \quad (22)$$

where C_{ab}^{-1} is determined by,

$$\int d^2z C_{ac}(x,z) C_{cb}^{-1}(z,y) = \delta_{ab} \delta^2(x-y). \quad (23)$$

It is possible to show,

$$C_{ab}^{-1}(x,y) = \frac{1}{\kappa}\varepsilon_{ab}\delta^2(x-y)\,, \qquad (24)$$

Under this DB, the constraint Φ_a is a strong identity, thus, we can assume,

$$\pi^a = \frac{\kappa}{2}\varepsilon_{ab}A_b. \qquad (25)$$

then A_b are the freedom degree. Thus, it is possible determine:

$$\Big\{A_a(x)\,, A_b(y)\Big\}_{D1} = \frac{1}{\kappa}\varepsilon_{ab}\delta^2(x-y)\,. \qquad (26)$$

The first class constraints can be written,

$$\Theta_1 = \pi^+ \approx 0 \qquad , \qquad \Theta_2 = \kappa \varepsilon_{ab} \partial_a A_b. \tag{27}$$

To eliminate the first class constraints it is necessary to introduce two gauge condition,

$$\Theta_3 = A_+ \approx 0 \qquad , \qquad \Theta_4 = A_- \approx 0, \tag{28}$$

and define a matrix with the following elements,

$$D_{ab}\left(x, y\right) \equiv \left\{ \Theta_a\left(x\right), \Theta_b\left(y\right) \right\}_{D1}. \qquad a = 1, 2, 3, 4 \tag{29}$$

Now, we can define the final set of DB of the theory for two dynamical variables,

$$\left\{ A_m(x), B_n(y) \right\}_D = \left\{ A_m(x), B_n(y) \right\}_{D1} - \int d^2u d^2v \left\{ A_m(x), \Theta_a(u) \right\}_{D1} D_{ab}^{-1}(u,v) \left\{ \Theta_b(v), B_n(y) \right\}_{D1}. \quad (30)$$

where

$$D^{-1}(z,y) \equiv \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{1}{\partial_-^x} \\ -1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\partial_-^x} & 0 & 0 \end{pmatrix} \delta^2(x-y). \quad (31)$$

It is possible to show

$$\left\{ A_1(x), A_1(y) \right\}_D = \frac{1}{\kappa} \frac{\partial_1^x}{\partial_-^x} \delta^2(x-y). \quad (32)$$

Non Abelian Chern-Simons Theory

The Lagrangian density of the pure $SU(N)$ Chern-Simons theory in $(2+1)$ dimensions is given by

$$\mathcal{L}_{CS} = -\frac{\kappa}{4\pi}\varepsilon^{\mu\nu\alpha} tr\left(\partial_\mu \mathbf{A}_\nu \mathbf{A}_\alpha + \frac{2}{3}\mathbf{A}_\mu \mathbf{A}_\nu \mathbf{A}_\alpha\right), \qquad (33)$$

where $\mathbf{A}_\mu = A_\mu^a T^a$ is the gauge potential giving rise to the field strength

$$\mathbf{F}_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + [\mathbf{A}_\mu, \mathbf{A}_\nu],$$

where a, b denote group indices. The Lagrangian density can be written,

$$\mathcal{L}_{CS} = \frac{\kappa}{8\pi}\varepsilon^{\mu\nu\alpha}\left(\partial_\mu A_\nu^a A_\alpha^a + \frac{1}{3}f^{abc}A_\mu^a A_\nu^b A_\alpha^c\right). \qquad (34)$$

The field equation take the form,

$$\frac{\kappa}{8\pi}\varepsilon^{\rho\mu\nu}F^{d}_{\mu\nu}=0. \tag{35}$$

The canonical momentum associated is determined of,

$$\pi^{\rho}_{d}\equiv\frac{\partial\mathcal{L}_{CS}}{\partial\left(\partial_{+}A^{d}_{\rho}\right)}=\frac{\kappa}{8\pi}\varepsilon^{+\rho\alpha}A^{d}_{\alpha}, \tag{36}$$

Thus, we have the following set of primary constraints,

$$\begin{aligned}\Omega^{d}_{1} &\equiv \pi^{+}_{d}\approx 0,\\ \Omega^{d}_{2,\mathbf{a}} &\equiv \pi^{\mathbf{a}}_{d}-\frac{\kappa}{8\pi}\varepsilon_{\mathbf{ab}}A^{d}_{\mathbf{b}}\approx 0, \qquad \mathbf{a},\mathbf{b}=-,1.\end{aligned} \tag{37}$$

The canonical Hamiltonian take the form,

$$H_{C}=-\frac{\kappa}{8\pi}\int d^{2}x A^{c}_{+}\varepsilon_{\mathbf{ab}}F^{c}_{\mathbf{ab}} \tag{38}$$

The dynamics is determined by the primary Hamiltonian defined by,

$$H_P = H_C + \int d^2y \left[u_d^1 (y) \, \Omega_1^d (y) + u_d^{2,\mathbf{a}} (y) \, \Omega_{2,\mathbf{a}}^d (y) \right], \qquad (39)$$

The fundamental Poisson brackets of the theory are,

$$\left\{ A_\mu^a (x) \, , \pi_b^\nu (y) \right\} = \delta_b^a \delta_\mu^\nu \delta^2 (x - y) \, . \qquad (40)$$

The consistence condition of $\Omega_1^g (x) = \pi_g^+ (x)$ determine a secondary constraint defined by,

$$\Omega_3^g (x) \equiv \varepsilon_{\mathbf{ab}} F_{\mathbf{ab}}^g (x) \approx 0. \qquad (41)$$

Thus, the consistence of Ω_3^d determine,

$$\dot{\Omega}_3^d(x) = \varepsilon_{\mathbf{ca}} D_{\mathbf{c}}^{de}(x)\, u_e^{2,\mathbf{a}}(x) \approx 0. \tag{42}$$

Now, if we study the consistence of $\Omega_{2,\mathbf{a}}^d$ result,

$$\dot{\Omega}_{2,\mathbf{a}}^d(x) = \frac{\kappa}{4\pi}\varepsilon_{\mathbf{ab}}\left(D_{\mathbf{b}}^{da}(x)\, A_+^a(x) - u_d^{2,\mathbf{b}}(x)\right) \approx 0. \tag{43}$$

Combining the relations (42) and (43) result:

$$f^{ead}\Omega_3^e(x)\, A_+^a(x) \approx 0. \tag{44}$$

which is consistent.

The theory is characterized by the following set of constraints:

$$\begin{aligned}
\Omega_1^d(x) &= \pi_d^+(x), \\
\Omega_{2,\mathbf{a}}^d(x) &= \pi_d^{\mathbf{a}}(x) - \frac{\kappa}{8\pi}\varepsilon_{\mathbf{ab}}A_{\mathbf{b}}^d(x), \qquad \mathbf{a} = -, 1. \\
\Omega_3^d(x) &\equiv \varepsilon_{\mathbf{ab}}F_{\mathbf{ab}}^d(x).
\end{aligned}$$

The theory has the following set of first class constraints.

$$\Theta_1^a \equiv \pi_a^+ \approx 0 \quad , \quad \Theta_2^a \equiv \left[\delta^{ab}\partial_{\mathbf{a}}^x - f^{abc}A_{\mathbf{a}}^c\right]\pi_b^{\mathbf{a}} + \frac{\kappa}{8\pi}\varepsilon_{\mathbf{ab}}\partial_{\mathbf{a}}^x A_{\mathbf{b}}^a \approx 0. \tag{45}$$

and a set of second class constraints,

$$\Phi_{\mathbf{a}}^a \equiv \Omega_{2,\mathbf{a}}^a \equiv \pi_a^{\mathbf{a}} - \frac{\kappa}{8\pi}\varepsilon_{\mathbf{ab}}A_{\mathbf{b}}^a \approx 0, \qquad \mathbf{a}, \mathbf{b} = -, 1, \tag{46}$$

where,

$$\left\{\Phi_{\mathbf{a}}^a(x), \Phi_{\mathbf{b}}^b(y)\right\} = -\frac{\kappa}{4\pi}\delta_b^a\varepsilon_{\mathbf{ab}}\delta^2(x-y). \tag{47}$$

Now, we need two gauge conditions:

$$A_{+}^{a} \approx 0 \quad , \quad A_{-}^{a} \approx 0 \tag{48}$$

It is possible to show

$$\left\{ A_{1}^{a}(x), A_{1}^{b}(y) \right\}_{D} = \frac{1}{\kappa} \delta_{b}^{a} \frac{\partial_{1}^{x}}{\partial_{-}^{x}} \delta^{2}(x-y) . \tag{49}$$

Conclusions

- We have calculated the complete set of constraints of the theory.
- The constraints have been classified and it was shown that one of the first class constraints result of a combination of constraints.
- The degrees of freedom have been determined and their corresponding DB calculated.