Foundations of the Standard Model

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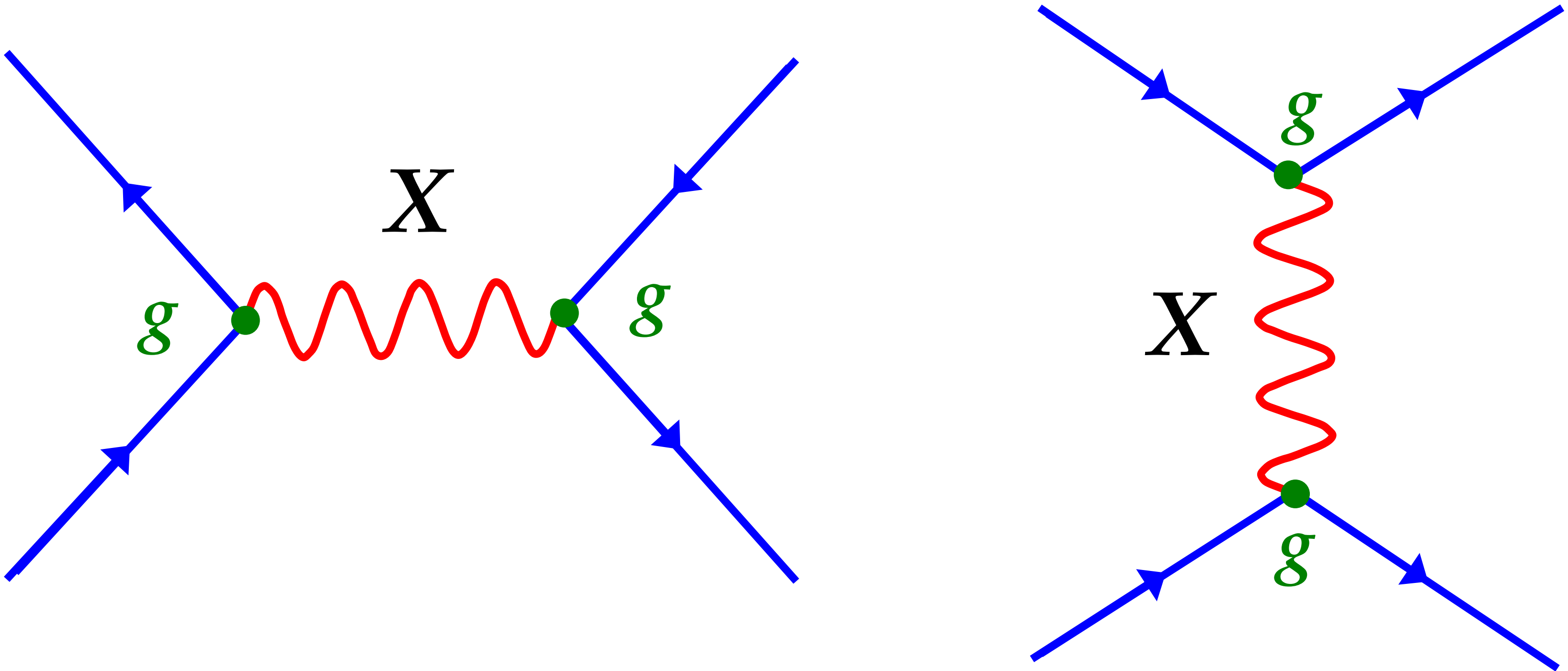


"Quantum Field Theory"

Lagrangians

- **In this short lecture, I will focus on two main elements of Quantum Field Theory as applied to particle physics:**
  - Lagrangians in field theory
  - Interaction by Particle Exchange

- **In classical dynamics can work with Forces and Acceleration using Newton's second Law. Alternatively, can obtain the same dynamical equations of motion from the Lagrangian: L= T – V, where the kinetic and potential energies are expressed in terms of generalised coordinates**

$$L(q_i, \dot{q}_i)$$

- **The equations of motion are then obtained from the Euler-Lagrange eqns:**

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

- **A simple example:**

$$L = T - V = \tfrac{1}{2}m\dot{x}^2 - V(x) \quad \Rightarrow \quad m\ddot{x} = -\frac{\partial V(x)}{\partial x}$$

Continuous Field Theory

- From discrete particles to continuous systems, the Lagrangian is replaced by the Lagrangian density

$$L\left(q_i, \frac{\mathrm{d}q_i}{\mathrm{d}t}\right) \to \mathcal{L}\left(\phi_i, \partial_\mu \phi_i\right)$$

and coordinates are replaced by continuous fields and their derivatives with respect to the four space-time coordinates

$$\partial_\mu \phi_i \equiv \frac{\partial \phi_i}{\partial x^\mu}$$

- The dynamics are then obtained from

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\right) - \frac{\partial \mathcal{L}}{\partial \phi_i} = 0$$

- In QFT, single particle wavefunctions satisfying the appropriate field equations are replaced by (multi-particle) excitations of the quantum field
- The structure of the Standard Model is determined by the SM Lagrangian Density (of CERN T-shirt fame). Its quantisation determines the related Feynman rules

Spin-half Relativistic Fields

- The free-particle Dirac equation can be obtained from

$$\mathcal{L}_D = i\overline{\psi}\gamma^\mu \partial_\mu \psi - m\overline{\psi}\psi$$

- The Euler Lagrange equations could be solved using the eight independent fields in the spinor

$$\psi(x) = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} \Psi_1 + i\Phi_1 \\ \Psi_2 + i\Phi_2 \\ \Psi_3 + i\Phi_3 \\ \Psi_4 + i\Phi_4 \end{pmatrix}$$

- Alternatively, the independent components can be taken as the four-components of the spinor and the four components of the adjoint spinor. Solving the E-L equations for the components of the adjoint spinor gives

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \overline{\psi}_i)} = 0 \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial \overline{\psi}_i} = i\gamma^\mu \partial_\mu \psi - m\psi$$

- Substituting into the E-L equation $\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) - \frac{\partial \mathcal{L}}{\partial \phi_i} = 0$ gives

$$i\gamma^\mu(\partial_\mu \psi) - m\psi = 0$$

which is just the free-particle Dirac equation for the spinor field

- Not all that interesting yet… in the next lecture we'll put in interactions

Interaction by Particle Exchange

- Calculate transition rates from **Fermi's Golden Rule**

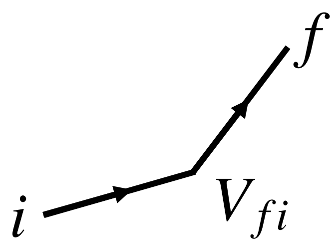
$$\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_f)$$

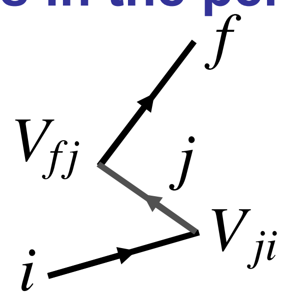
where T_{fi} is perturbation expansion for the Transition Matrix Element

$$T_{fi} = \langle f|V|i\rangle + \sum_{j\neq i} \frac{\langle f|V|j\rangle\langle j|V|i\rangle}{E_i - E_j} + \ldots$$

- For particle scattering, the first two terms in the perturbation series can be viewed as:

"scattering in a potential"

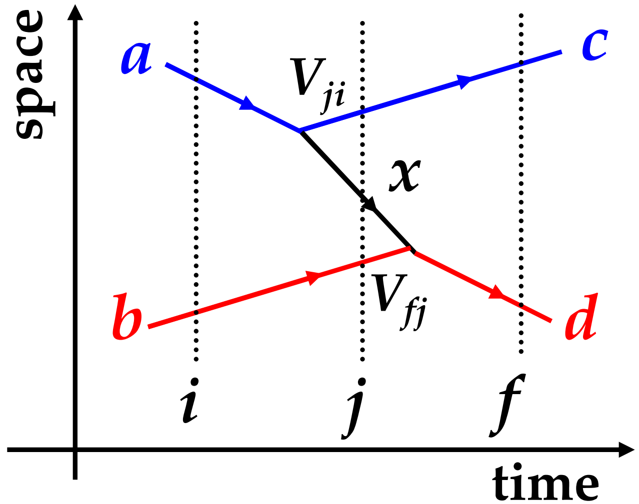




"scattering via an intermediate state"

- **"Classical picture"** – particles act as sources for fields which give rise a potential in which other particles scatter – **"action at a distance"**
- **"Quantum Field Theory picture"** – forces arise due to the exchange of virtual particles. No action at a distance + forces between particles now due to particles

- Consider the particle interaction $a+b \rightarrow c+d$ which occurs via an intermediate state corresponding to the exchange of particle x
- One possible space-time picture of this process is:



Initial state i : $a+b$
Final state f : $c+d$
Intermediate state j : $c+b+x$

- This **time-ordered** diagram corresponds to a "emitting" x and then b absorbing x

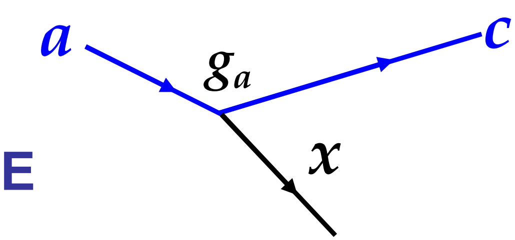
- The corresponding term in the perturbation expansion is:

$$T_{fi} = \frac{\langle f|V|j\rangle\langle j|V|i\rangle}{E_i - E_j}$$

$$T_{fi}^{ab} = \frac{\langle d|V|x+b\rangle\langle c+x|V|a\rangle}{(E_a+E_b)-(E_c+E_x+E_b)}$$

- T_{fi}^{ab} refers to the time-ordering where a emits x before b absorbs it

- Need an expression for $\langle c+x|V|a\rangle$ in non-invariant matrix element T_{fi}



- Ultimately aiming to obtain Lorentz Invariant ME
- Take it on trust that T_{fi} is related to the invariant matrix element by

$$T_{fi} = \prod_k (2E_k)^{-1/2} M_{fi}$$

where k runs over all particles in the matrix element

- Here we have

$$\langle c+x|V|a\rangle = \frac{M_{(a\to c+x)}}{(2E_a 2E_c 2E_x)^{1/2}}$$

$M_{(a\to c+x)}$ is the "Lorentz Invariant" matrix element for $a \to c + x$

★ The simplest Lorentz Invariant quantity is a scalar, in this case

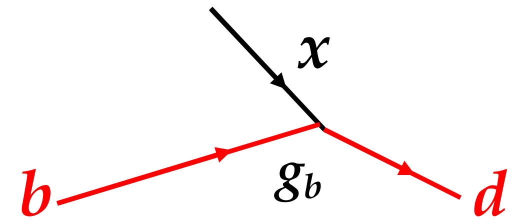
$$\langle c+x|V|a\rangle = \frac{g_a}{(2E_a 2E_c 2E_x)^{1/2}}$$

g_a is a measure of the strength of the interaction $a \to c + x$

Note : the matrix element is only LI in the sense that it is defined in terms of LI wave-function normalisations and that the form of the coupling is LI

Note : in this "illustrative" example g is not dimensionless.

Similarly $\langle d|V|x+b\rangle = \frac{g_b}{(2E_b 2E_d 2E_x)^{1/2}}$



Giving
$$T_{fi}^{ab} = \frac{\langle d|V|x+b\rangle\langle c+x|V|a\rangle}{(E_a+E_b)-(E_c+E_x+E_b)}$$
$$= \frac{1}{2E_x}\cdot\frac{1}{(2E_a 2E_b 2E_c 2E_d)^{1/2}}\cdot\frac{g_a g_b}{(E_a-E_c-E_x)}$$

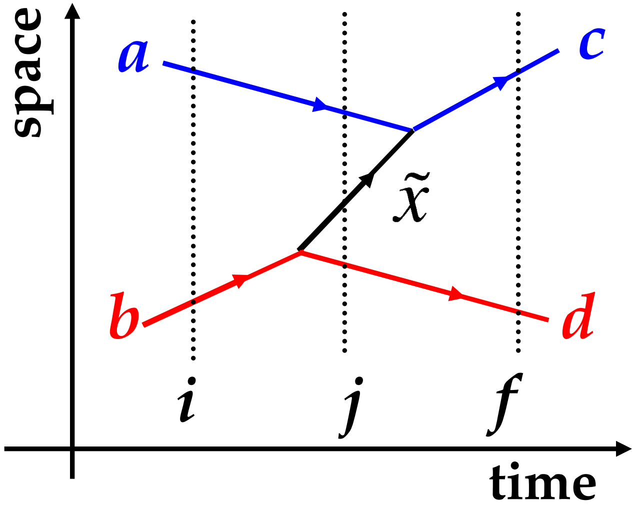
★The "Lorentz Invariant" matrix element for the entire process is

$$M_{fi}^{ab} = (2E_a 2E_b 2E_c 2E_d)^{1/2} T_{fi}^{ab}$$
$$= \frac{1}{2E_x}\cdot\frac{g_a g_b}{(E_a-E_c-E_x)}$$

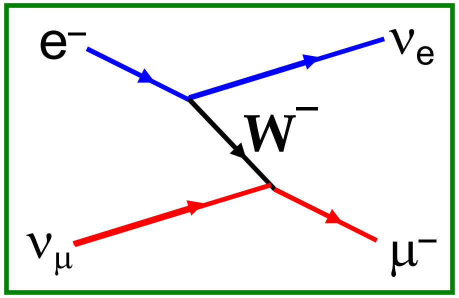
Note:

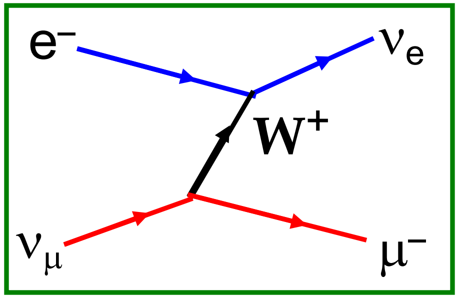
- M_{fi}^{ab} refers to the time-ordering where a emits x before b absorbs it
  It is not Lorentz invariant, order of events in time depends on frame
- Momentum is conserved at each interaction vertex but not energy
  $E_j \neq E_i$
- Particle x is "on-mass shell" i.e. $E_x^2 = \vec{p}_x^{\,2} + m^2$

★ But need to consider also the other time ordering for the process



- This **time-ordered** diagram corresponds to *b* "emitting" $\tilde{x}$ and then *a* absorbing $\tilde{x}$
- $\tilde{x}$ is the anti-particle of x e.g.





- The **Lorentz invariant matrix element** for this time ordering is:

$$M_{fi}^{ba} = \frac{1}{2E_x} \cdot \frac{g_a g_b}{(E_b - E_d - E_x)}$$

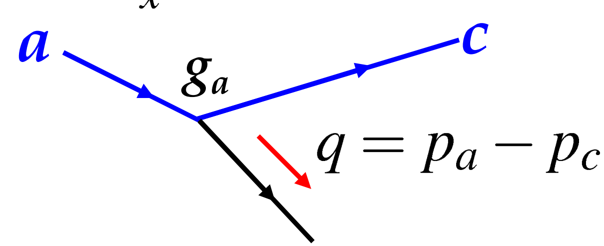
★ In QM need to sum over matrix elements corresponding to same final state:

$$\begin{aligned} M_{fi} &= M_{fi}^{ab} + M_{fi}^{ba} \\ &= \frac{g_a g_b}{2E_x} \cdot \left(\frac{1}{E_a - E_c - E_x} + \frac{1}{E_b - E_d - E_x} \right) \\ &= \frac{g_a g_b}{2E_x} \cdot \left(\frac{1}{E_a - E_c - E_x} - \frac{1}{E_a - E_c + E_x} \right) \end{aligned}$$

Energy conservation: $(E_a + E_b = E_c + E_d)$

- **Which gives** $$M_{fi} = \frac{g_a g_b}{2E_x} \cdot \frac{2E_x}{(E_a - E_c)^2 - E_x^2} = \frac{g_a g_b}{(E_a - E_c)^2 - E_x^2}$$

- **From 1st time ordering** $E_x^2 = \vec{p}_x^{\,2} + m_x^2 = (\vec{p}_a - \vec{p}_c)^2 + m_x^2$



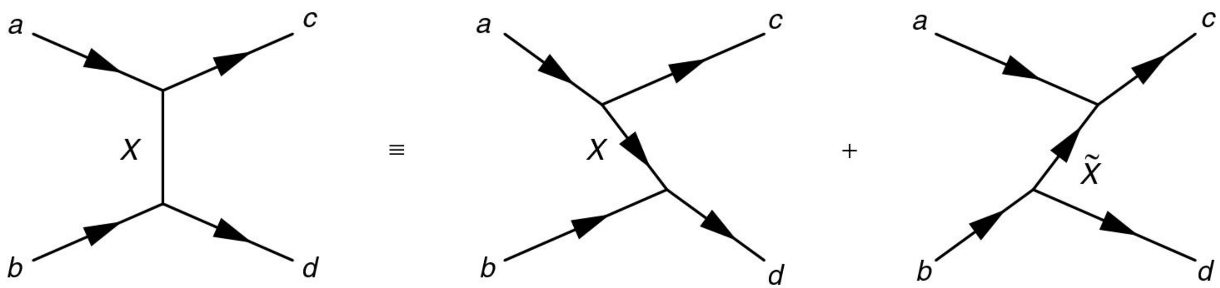
giving $$M_{fi} = \frac{g_a g_b}{(E_a - E_c)^2 - (\vec{p}_a - \vec{p}_c)^2 - m_x^2} = \frac{g_a g_b}{(p_a - p_c)^2 - m_x^2}$$

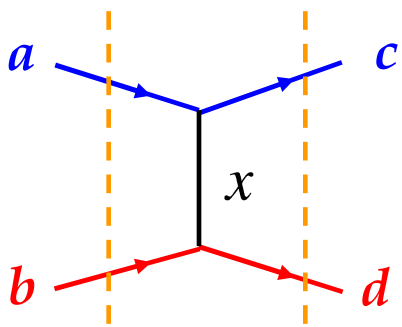
$$\Rightarrow \quad \boxed{M_{fi} = \frac{g_a g_b}{q^2 - m_x^2}}$$

- **After summing over all possible time orderings, M_{fi} is (as anticipated) Lorentz Invariant. This is remarkable result – the sum over all time orderings gives a frame independent matrix element.**
- **This simple approach is a long way from a full Quantum Field Theory derivation, but gives a sense of the interpretation of Feynman diagrams**

Feynman Diagrams

- In QFT the sum over all possible time-orderings is represented by a **FEYNMAN diagram**





In a Feynman diagram:

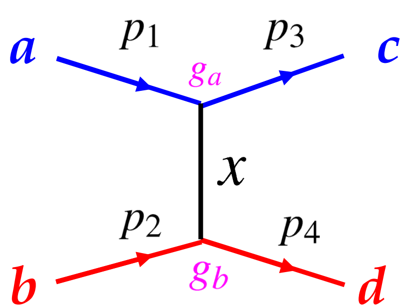
- the LHS represents the initial state
- the RHS is the final state
- everything in between is "how the interaction happened"

- It is important to remember that **energy and momentum** are conserved at each interaction vertex in the diagram.

- The factor $1/(q^2 - m_x^2)$ is the propagator; it arises naturally from the above discussion of interaction by particle exchange

★ **The matrix element:** $M_{fi} = \frac{g_a g_b}{q^2 - m_x^2}$ **depends on:**

- **The fundamental strength of the interaction at the two vertices** g_a, g_b
- **The four-momentum,** q **, carried by the (virtual) particle which is determined from energy/momentum conservation at the vertices. Note** q^2 **can be either positive or negative.**

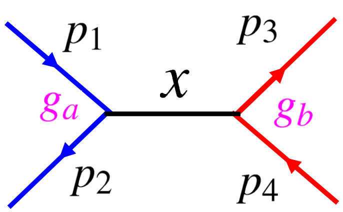


Here $q = p_1 - p_3 = p_4 - p_2 = t$ **"t-channel"**

For elastic scattering: $p_1 = (E, \vec{p}_1); \quad p_3 = (E, \vec{p}_3)$

$$q^2 = (E - E)^2 - (\vec{p}_1 - \vec{p}_3)^2$$

$q^2 < 0$ **termed "space-like"**



Here $q = p_1 + p_2 = p_3 + p_4 = s$ **"s-channel"**

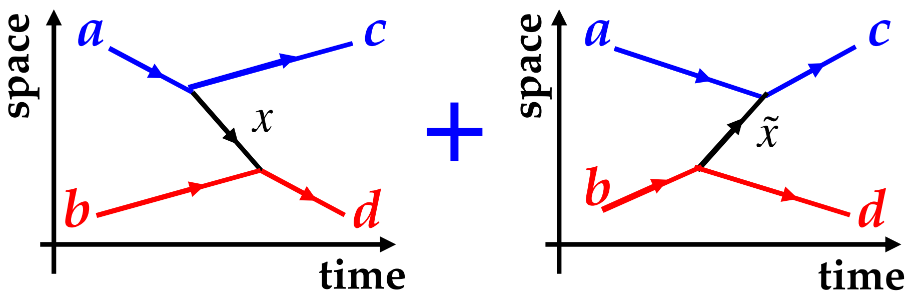
In CoM: $p_1 = (E, \vec{p}); \quad p_2 = (E, -\vec{p})$

$$q^2 = (E + E)^2 - (\vec{p} - \vec{p})^2 = 4E^2$$

$q^2 > 0$ **termed "time-like"**

Virtual Particles

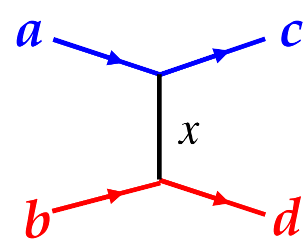
"Time-ordered QM"



- Momentum conserved at vertices
- Energy **not** conserved at vertices
- Exchanged particle **"on mass shell"**

$$E_x^2 - |\vec{p}_x|^2 = m_x^2$$

Feynman diagram



$$M_{fi} = \frac{g_a g_b}{q^2 - m_x^2}$$

- Momentum **AND** energy conserved at interaction vertices
- Exchanged particle **"off mass shell"**

$$E_x^2 - |\vec{p}_x|^2 = q^2 \neq m_x^2$$

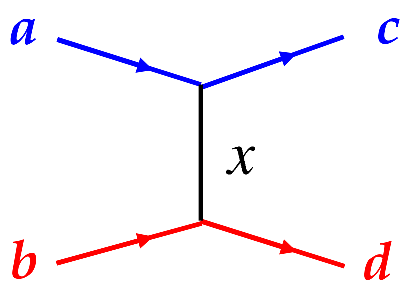
VIRTUAL PARTICLE

- Can think of observable "on mass shell" particles as propagating waves and unobservable virtual particles as normal modes between the source particles:

Aside: V(r) from Particle Exchange

★ Can view the scattering of an electron by a proton at rest in two ways:

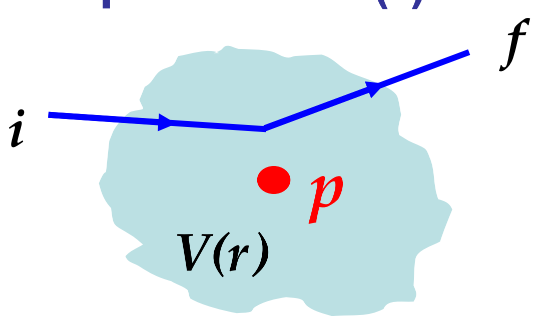
- Interaction by particle exchange in 2nd order perturbation theory.



$$M_{fi} = \frac{g_a g_b}{q^2 - m_x^2}$$

- Could also evaluate the same process in first order perturbation theory treating proton as a fixed source of a field which gives rise to a potential V(r)

$$M = \langle \psi_f | V(r) | \psi_i \rangle$$



Obtain same expression for M_{fi} using

$$V(r) = g_a g_b \frac{e^{-mr}}{r}$$

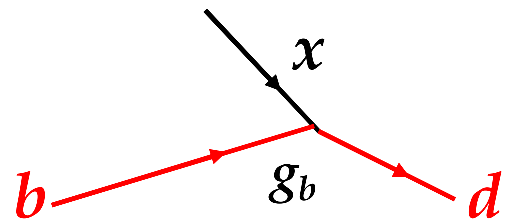
YUKAWA potential

★ In this way can relate potential and forces to the particle exchange picture

★ However, scattering from a fixed potential $V(r)$ is not a relativistic invariant view…

The Interaction Vertex

- In the simple example, simply put in a "constant" at each interaction vertex



- Gave rise to a manifestly Lorentz invariant matrix element

$$M_{fi} = \frac{g_a g_b}{q^2 - m_x^2}$$

- This was the simplest choice – but it turns out that possibilities are very limited by the requirement of Lorentz invariance
- Beyond a simple scalar, we need to construct possible operators from four-by-four matrices, since the operator is sandwiched between Dirac 4-component spinors.

Bilinear Covariants

★ The requirement of Lorentz invariance of the matrix element severely restricts the form of the interaction vertex. QED and QCD are "VECTOR" interactions:

$$j^\mu = \overline{\psi}\gamma^\mu \phi$$

★ As already described, this combination transforms as a 4-vector

★ In general, there are only 5 possible combinations of two spinors and the gamma matrices that form Lorentz invariant currents, called "bilinear covariants":

Type	Form	Components	"Boson Spin"
♦ SCALAR	$\overline{\psi}\phi$	1	0
♦ PSEUDOSCALAR	$\overline{\psi}\gamma^5\phi$	1	0
♦ VECTOR	$\overline{\psi}\gamma^\mu\phi$	4	1
♦ AXIAL VECTOR	$\overline{\psi}\gamma^\mu\gamma^5\phi$	4	1
♦ TENSOR	$\overline{\psi}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\phi$	6	2

★ Note that in total the sixteen components correspond to the 16 elements of a general 4x4 matrix: "decomposition into Lorentz invariant combinations"

Summary

★ Interaction by particle exchange naturally gives rise to **Lorentz Invariant Matrix Element** of the form

$$M_{fi} = \frac{g_a g_b}{q^2 - m_x^2}$$

★ We have seen how interaction by particle exchange gives rise to the "propagator" term and there are limited options for what happens at the interaction vertex

★ In the next lecture will explore some of the features of QED