Foundations of the Standard Model

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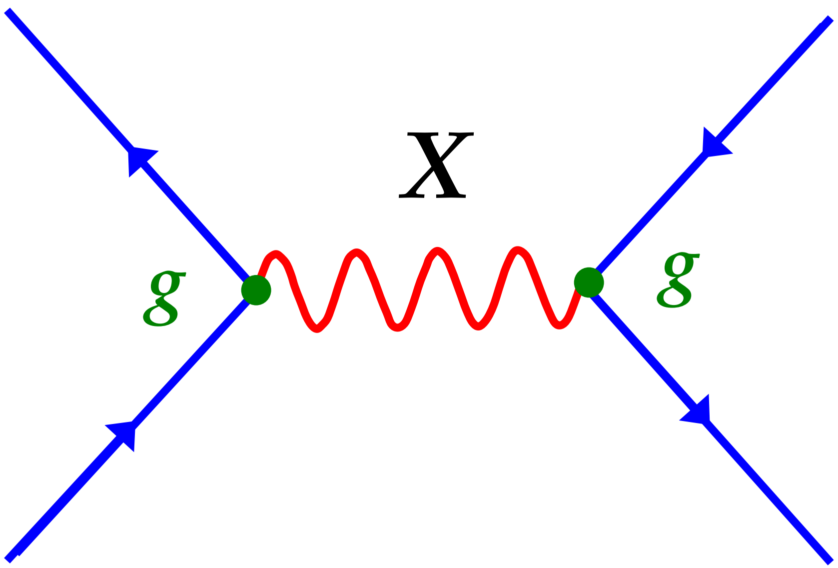
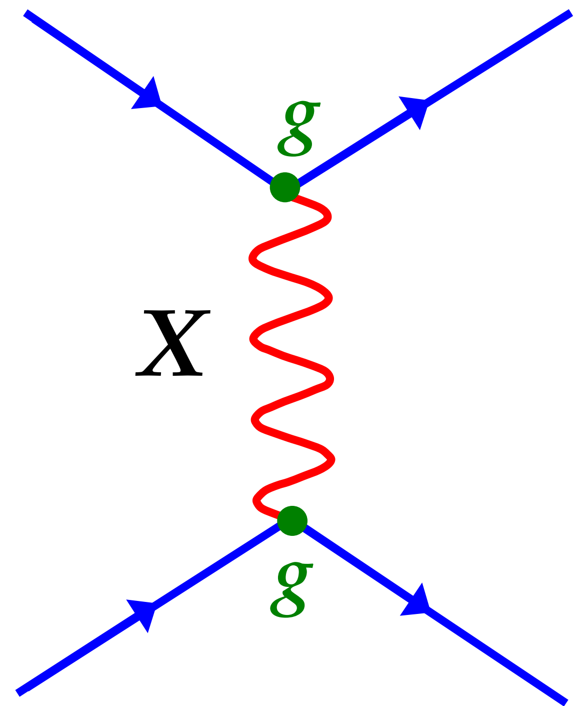




1: Introduction to Relativistic QM

These Lectures

- ★ **Impossible to cover the whole of the SM in six 40-minute lectures…**
- ★ **Will therefore focus on key concepts that form "the foundations of the Standard Model"**
  - will not have time to dig too deeply into mathematical structure
  - aim is on understanding and insight
- ★ **What are the Foundations of the SM?**
  - **Lorentz Invariance** and Relativistic Quantum Mechanics
  - **Quantum Field Theory** – a framework for relativistic calculation
  - **Local Gauge Invariance** – determines the nature of the forces
  - **Higgs Mechanism** – the glue that makes electroweak unification work and much much more

Preliminaries: Natural Units

- S.I. UNITS: **kg m s** are a natural choice for "everyday" objects

  e.g. M(Human) ~ 100 kg

- not very natural in particle physics
- instead use **Natural Units** based on the language of particle physics
  - **From Quantum Mechanics** - the unit of action : $\hbar$
  - **From relativity** - the speed of light: c
  - **From Particle Physics** - unit of energy: GeV (1 GeV ~ proton rest mass energy)

★ **Units become (i.e. with the correct dimensions):**

Energy	GeV	Time	$(\mathrm{GeV}/\hbar)^{-1}$
Momentum	GeV/c	Length	$(\mathrm{GeV}/\hbar c)^{-1}$
Mass	GeV/c^2	Area	$(\mathrm{GeV}/\hbar c)^{-2}$

★ **Simplify algebra by setting:** $\hbar = c = 1$

- Now all quantities expressed in powers of GeV

Energy	GeV	Time	GeV^{-1}
Momentum	GeV	Length	GeV^{-1}
Mass	GeV	Area	GeV^{-2}

To convert back to S.I. units, need to restore missing factors of $\hbar$ and c

Special Relativity and 4-Vector Notation

- Will use 4-vector notation with p^0 as the time-like component, e.g.

$$p^\mu = \{E, \vec{p}\} = \{E, p_x, p_y, p_z\}$$ (contravariant)

$$p_\mu = g_{\mu\nu} p^\mu = \{E, -\vec{p}\} = \{E, -p_x, -p_y, -p_z\}$$ (covariant)

with

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- In particle physics, we usually deal with relativistic particles. Require all calculations to be Lorentz Invariant. L.I. quantities formed from 4-vector scalar products, e.g.

$$p^\mu p_\mu = E^2 - p^2 = m^2$$ Invariant mass

$$x^\mu p_\mu = Et - \vec{p}.\vec{r}$$ Phase

- A few words on NOTATION

  Four vectors written as either: p^μ or p

  Four vector scalar product: $p^\mu q_\mu$ or $p.q$

  Three vectors written as: $\vec{p}$

- Using Natural Units throughout

$$\boxed{\hbar = c = 1}$$

Overview of The Standard Model

Particle Physics is the study of:

- ★ **MATTER: the fundamental constituents of the universe, i.e. the elementary particles**
- ★ **FORCE: the fundamental forces of nature, i.e. the interactions between the elementary particles**

Try to categorise the PARTICLES and FORCES in as simple and fundamental manner possible

★ **Current understanding embodied in the STANDARD MODEL:**

- **Forces between particles due to exchange of particles**
- **Consistent with <u>all</u> current experimental data**
- **But it is just a "model" with many unpredicted parameters, e.g. particle masses, …**
- **As such it is not the ultimate theory (if such a thing exists), e.g. gravity is not included, grand unification, dark matter, ….**

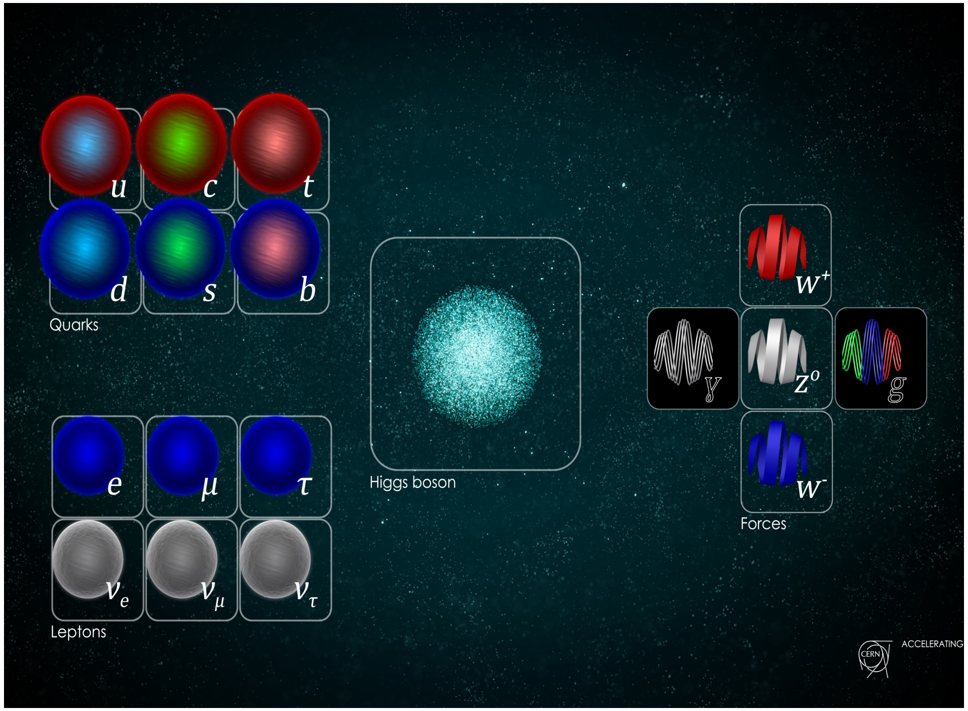
Matter in the Standard Model

★ **In the Standard Model the fundamental "matter" is described by point-like spin-1/2 fermions**

★ **The Standard Model building blocks:**
- 12 matter particles (fermions)
- 5 force carry particles (bosons)
- 1 Higgs boson (a spin-0 scalar

★ **Interactions between particles:**
- Completely defined by the Local Gauge Symmetry of the SM, namely U(1) x $SU(2)_L$ x SU(3)

★ **Fundamental particle masses**
- All from Higgs potential

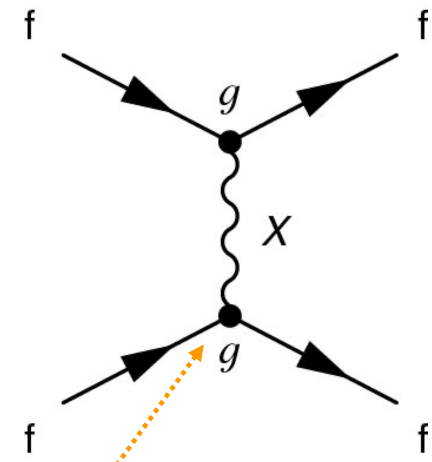


- In the SM there are three generations – the particles in each generation are copies of each other differing only in mass. (not understood why three).
- The neutrinos are much lighter than all other particles (e.g. ν_1 has m<0.5 eV) – we know know that neutrinos have non-zero mass (don't understand why they are so small)

Forces in the Standard Model

★ Forces mediated by the exchange of spin-1 Gauge Bosons

Force	Boson(s)	J^P	*m*/GeV
EM (QED)	Photon γ	1^-	0
Weak	$W^\pm$ / Z	1^-	80 / 91
Strong (QCD)	8 Gluons g	1^-	0
Gravity (?)	Graviton?	2^+	0



- Fundamental interaction strength is given by charge g.
- Related to the dimensionless coupling "constant" α

e.g. QED $$g_{em} = e = \sqrt{4\pi\alpha\varepsilon_0\hbar c}$$

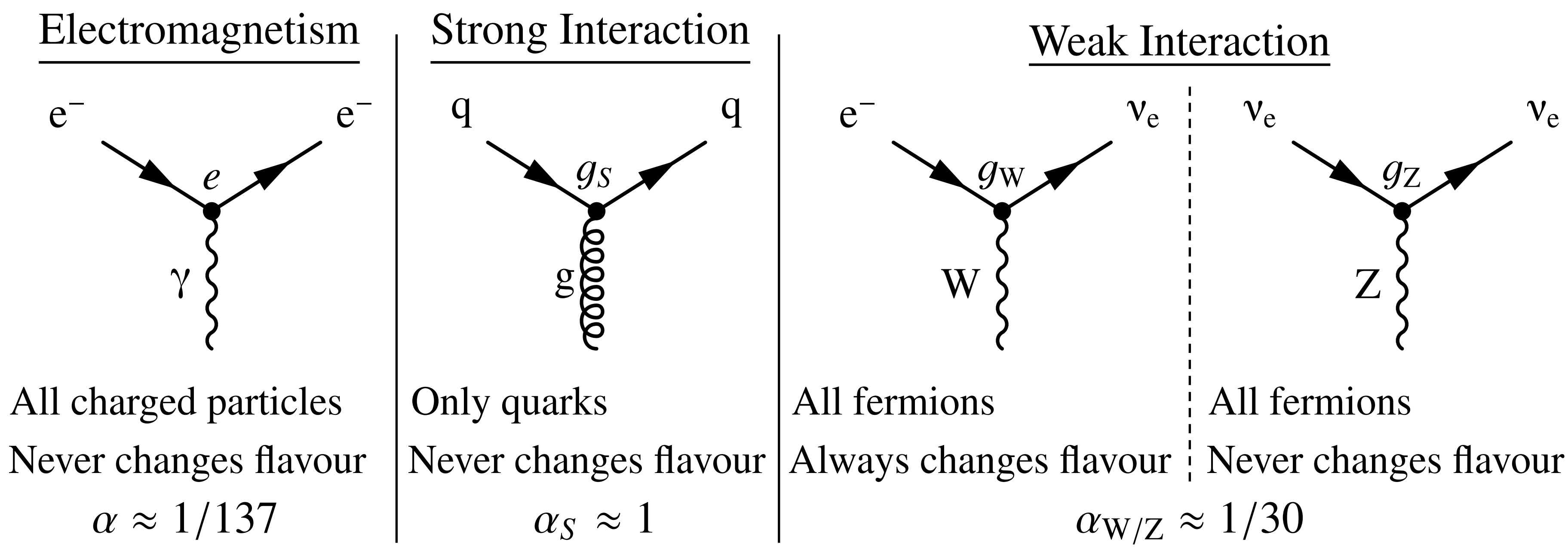
★ In Natural Units $$g = \sqrt{4\pi\alpha}$$ (both g and α are dimensionless, but g contains a "hidden" $\hbar c$)

★ Convenient to express couplings in terms of α which, being genuinely dimensionless does not depend on the system of units (this is not true for the numerical value for e)

Standard Model Vertices

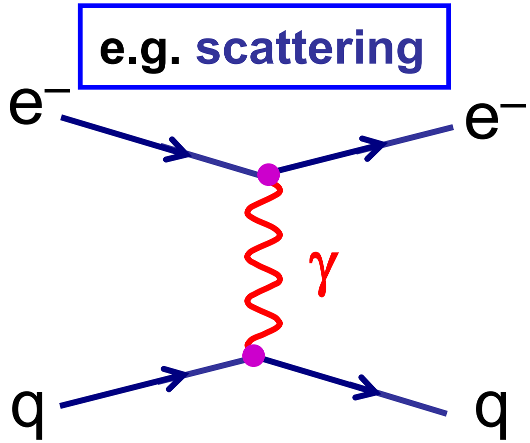
★ **Interaction of gauge bosons with fermions described by SM vertices**

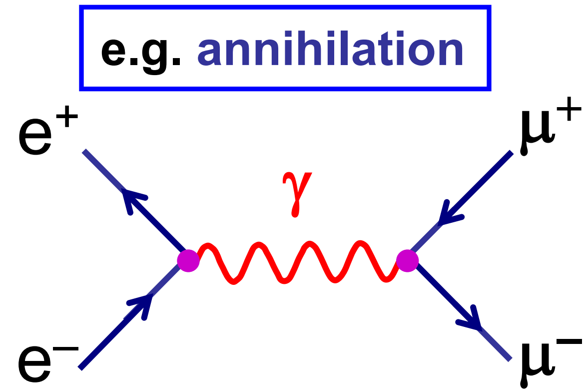
★ **Properties of the gauge bosons and nature of the interaction between the bosons and fermions determine the properties of the interaction**



Feynman Diagrams

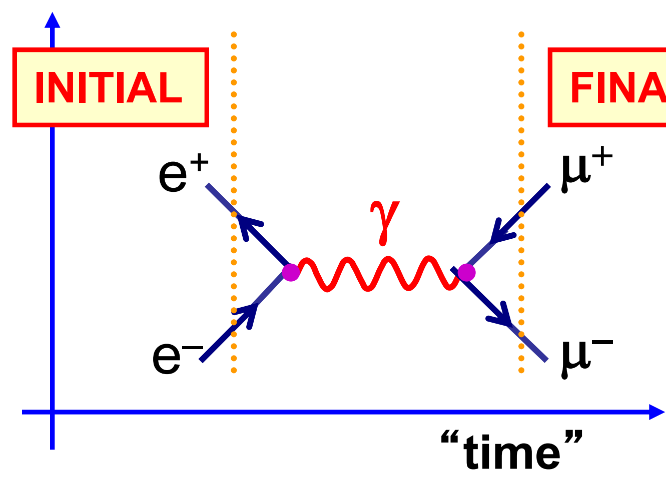
★ **Particle interactions described in terms of Feynman diagrams**





★ **IMPORTANT POINTS TO REMEMBER:**

- **"time" runs from left – right, only in sense that:**
  - LHS of diagram is initial state
  - RHS of diagram is final state
  - Middle is "how it happened"
- **anti-particle arrows in –ve "time" direction**
- **Energy, momentum, angular momentum, etc. conserved at all interaction vertices**
- **All intermediate particles are "virtual"** i.e. $E^2 \neq |\vec{p}|^2 + m^2$



Towards Relativistic QM

- For particle physics need a relativistic formulation of quantum mechanics. But first take a few moments to review the non-relativistic formulation QM
- Take as the starting point non-relativistic energy:

$$E = T + V = \frac{\vec{p}^2}{2m} + V$$

- In QM we identify the energy and momentum operators:

$$\vec{p} \to -i\vec{\nabla}, \quad E \to i\frac{\partial}{\partial t}$$

which gives the time dependent Schrödinger equation (take V=0 for simplicity)

$$-\frac{1}{2m}\vec{\nabla}^2\psi = i\frac{\partial \psi}{\partial t} \qquad \text{(S1)}$$

with plane wave solutions: $\psi = Ne^{i(\vec{p}.\vec{r}-Et)}$ where $\begin{cases} -i\nabla\psi = \vec{p}\psi \\ i\frac{\partial \psi}{\partial t} = E\psi \end{cases}$

- The SE is **first order** in the time derivatives and **second order** in spatial derivatives – and is manifestly not Lorentz invariant.
- In what follows we will use probability density/current extensively. For the non-relativistic case these are derived as follows

(S1)* ➡

$$-\frac{1}{2m}\vec{\nabla}^2\psi^* = -i\frac{\partial \psi^*}{\partial t} \qquad \text{(S2)}$$

$$\psi^* \times (\mathbf{S1}) - \psi \times (\mathbf{S2}): \quad -\frac{1}{2m}\left(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*\right) = i\left(\psi^*\frac{\partial\psi}{\partial t} + \psi\frac{\partial\psi^*}{\partial t}\right)$$

$$-\frac{1}{2m}\vec{\nabla}.\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right) = i\frac{\partial}{\partial t}(\psi^*\psi)$$

- **Which by comparison with the continuity equation**

$$\vec{\nabla}.\vec{j} + \frac{\partial\rho}{\partial t} = 0$$

leads to the following expressions for probability density and current:

$$\rho = \psi^*\psi = |\psi|^2 \qquad \vec{j} = \frac{1}{2mi}\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right)$$

- **For a plane wave** $\psi = Ne^{i(\vec{p}.\vec{r}-Et)}$

$$\rho = |N|^2 \quad \text{and} \quad \vec{j} = |N|^2\frac{\vec{p}}{m} = |N|^2\vec{v}$$

★ **The number of particles per unit volume is** $|N|^2$

★ **For** $|N|^2$ **particles per unit volume moving at velocity** $\vec{v}$**, have** $|N|^2|\vec{v}|$ **passing through a unit area per unit time (particle flux). Therefore** $\vec{j}$ **is a vector in the particle's direction with magnitude equal to the flux.**

The Klein-Gordon Equation

- Applying $\vec{p} \rightarrow -i\vec{\nabla}, \quad E \rightarrow i\partial/\partial t$ to the relativistic equation for energy:

$$E^2 = |\vec{p}|^2 + m^2 \qquad \text{(KG1)}$$

gives the **Klein-Gordon equation:**

$$\frac{\partial^2 \psi}{\partial t^2} = \vec{\nabla}^2 \psi - m^2 \psi \qquad \text{(KG2)}$$

- Using $\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ ➡ $\partial^\mu \partial_\mu \equiv \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$

KG can be expressed compactly as

$$\boxed{(\partial^\mu \partial_\mu + m^2)\psi = 0} \qquad \text{(KG3)}$$

- For plane wave solutions, $\psi = N e^{i(\vec{p}.\vec{r} - Et)}$ the KG equation gives:

$$-E^2 \psi = -|\vec{p}|^2 \psi - m^2 \psi$$

$$\Rightarrow \quad E = \pm\sqrt{|\vec{p}|^2 + m^2}$$

★ Not surprisingly, the KG equation has negative energy solutions – this is just what we started with in eq. KG1

♦ Historically the **–ve** energy solutions were viewed as problematic. **But** for the KG there is also a problem with the **probability density which could also be negative**

The Dirac Equation

★ **Historically, it was thought that there were two main problems with the Klein-Gordon equation:**

- Negative energy solutions
- The negative particle densities associated with these solutions

$$\rho = 2E|N|^2$$

★ **We now know that in Quantum Field Theory these problems are overcome and the KG equation is used to describe spin-0 particles (inherently single particle description → multi-particle quantum excitations of a scalar field).**

Nevertheless:

★ **These problems motivated Dirac (1928) to search for a different formulation of relativistic quantum mechanics in which all particle densities are positive.**

★ **The resulting wave equation had solutions which not only solved this problem but also fully describe the intrinsic spin and magnetic moment of the electron!**

The Dirac Equation :

- **Schrödinger eqn:** $-\frac{1}{2m}\vec{\nabla}^2\psi = i\frac{\partial\psi}{\partial t}$ **1st order in** $\partial/\partial t$ **2nd order in** $\partial/\partial x, \partial/\partial y, \partial/\partial z$
- **Klein-Gordon eqn:** $(\partial^\mu\partial_\mu + m^2)\psi = 0$ **2nd order throughout**
- **Dirac looked for an alternative which was 1st order throughout:**

$$\hat{H}\psi = (\vec{\alpha}.\vec{p} + \beta m)\psi = i\frac{\partial\psi}{\partial t} \qquad \textbf{(D1)}$$

where $\hat{H}$ **is the Hamiltonian operator and, as usual,** $\vec{p} = -i\vec{\nabla}$

- **Writing (D1) in full:**

$$\left(-i\alpha_x\frac{\partial}{\partial x} - i\alpha_y\frac{\partial}{\partial y} - i\alpha_z\frac{\partial}{\partial z} + \beta m\right)\psi = \left(i\frac{\partial}{\partial t}\right)\psi$$

"squaring" this equation gives

$$\left(-i\alpha_x\frac{\partial}{\partial x} - i\alpha_y\frac{\partial}{\partial y} - i\alpha_z\frac{\partial}{\partial z} + \beta m\right)\left(-i\alpha_x\frac{\partial}{\partial x} - i\alpha_y\frac{\partial}{\partial y} - i\alpha_z\frac{\partial}{\partial z} + \beta m\right)\psi = -\frac{\partial^2\psi}{\partial t^2}$$

- **Which can be expanded in gory details as…**

$$
\begin{aligned}
-\frac{\partial^2\psi}{\partial t^2} \quad = \quad & -\alpha_x^2\frac{\partial^2\psi}{\partial x^2} - \alpha_y^2\frac{\partial^2\psi}{\partial y^2} - \alpha_z^2\frac{\partial^2\psi}{\partial z^2} + \beta^2 m^2\psi \\
& -(\alpha_x\alpha_y + \alpha_y\alpha_x)\frac{\partial^2\psi}{\partial x\partial y} - (\alpha_y\alpha_z + \alpha_z\alpha_y)\frac{\partial^2\psi}{\partial y\partial z} - (\alpha_z\alpha_x + \alpha_x\alpha_z)\frac{\partial^2\psi}{\partial z\partial x} \\
& -(\alpha_x\beta + \beta\alpha_x)m\frac{\partial\psi}{\partial x} - (\alpha_y\beta + \beta\alpha_y)m\frac{\partial\psi}{\partial y} - (\alpha_z\beta + \beta\alpha_z)m\frac{\partial\psi}{\partial z}
\end{aligned}
$$

- **For this to be a reasonable formulation of relativistic QM, a free particle must also obey $E^2 = \vec{p}^2 + m^2$, i.e. it must satisfy the Klein-Gordon equation:**

$$
-\frac{\partial^2\psi}{\partial t^2} = -\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\psi}{\partial z^2} + m^2\psi
$$

- **Hence for the Dirac Equation to be consistent with the KG equation require:**

$$
\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = 1 \qquad \textbf{(D2)}
$$

$$
\alpha_j\beta + \beta\alpha_j = 0 \qquad \textbf{(D3)}
$$

$$
\alpha_j\alpha_k + \alpha_k\alpha_j = 0 \quad (j \neq k) \qquad \textbf{(D4)}
$$

★ **Immediately we see that the α_j and β cannot be numbers. Require 4 mutually anti-commuting matrices**

★ **Must be even dimensionality and simplest representation are 4x4 matrices**

- Consequently, the wave-function must be a <u>four-component</u> **Dirac Spinor**

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

A consequence of introducing an equation that is 1st order in time/space derivatives is that the wave-function has new degrees of freedom !

- For the Hamiltonian $\hat{H}\psi = (\vec{\alpha}.\vec{p} + \beta m)\psi = i\partial\psi/\partial t$ to be Hermitian requires

$$\alpha_x = \alpha_x^\dagger; \quad \alpha_y = \alpha_y^\dagger; \quad \alpha_z = \alpha_z^\dagger; \quad \beta = \beta^\dagger; \qquad \text{(D5)}$$

i.e. require four anti-commuting Hermitian 4x4 matrices.

- At this point it is convenient to introduce an explicit representation for $\vec{\alpha}, \beta$ It should be noted that physical results do not depend on the particular representation – everything is in the commutation relations.
- A convenient choice is based on the Pauli spin matrices:

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}$$

with $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

- The matrices are Hermitian and anti-commute with each other as required

Dirac Equation: Probability Density and Current

The continuity equation is now
$$\vec{\nabla}.(\psi^\dagger\vec{\alpha}\psi)+\frac{\partial(\psi^\dagger\psi)}{\partial t}=0$$

where $\psi^\dagger=(\psi_1^*,\psi_2^*,\psi_3^*,\psi_4^*)$

- The probability density and current can be identified as:

$$\rho=\psi^\dagger\psi \quad \text{and} \quad \vec{j}=\psi^\dagger\vec{\alpha}\psi$$

where $\rho=\psi^\dagger\psi=|\psi_1|^2+|\psi_2|^2+|\psi_3|^2+|\psi_4|^2>0$

- Unlike the KG equation, the Dirac equation has probability densities which are **always positive.**
- In addition, the solutions to the Dirac equation are **the four component Dirac Spinors**. A great success of the Dirac equation is that these components naturally give rise to the property of intrinsic spin.
- It can be shown that Dirac spinors represent spin-half particles with an intrinsic magnetic moment of

$$\vec{\mu}=\frac{q}{m}\vec{S} \quad \text{i.e.} \quad \vec{\mu}=g\frac{q}{2m}\vec{S} \quad \text{with } g=2$$

Covariant Notation: the Dirac γ Matrices

- The Dirac equation can be written more elegantly by introducing the four Dirac gamma matrices:

$$\gamma^0 \equiv \beta; \quad \gamma^1 \equiv \beta\alpha_x; \quad \gamma^2 \equiv \beta\alpha_y; \quad \gamma^3 \equiv \beta\alpha_z$$

Premultiply the Dirac equation (D6) by β

$$i\beta\alpha_x \frac{\partial\psi}{\partial x} + i\beta\alpha_y \frac{\partial\psi}{\partial y} + i\beta\alpha_z \frac{\partial\psi}{\partial z} - \beta^2 m\psi = -i\beta\frac{\partial\psi}{\partial t}$$

➡ $$i\gamma^1 \frac{\partial\psi}{\partial x} + i\gamma^2 \frac{\partial\psi}{\partial y} + i\gamma^3 \frac{\partial\psi}{\partial z} - m\psi = -i\gamma^0\frac{\partial\psi}{\partial t}$$

using $\partial_\mu = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ this can be written compactly as:

$$\boxed{(i\gamma^\mu \partial_\mu - m)\psi = 0} \qquad \text{(D9)}$$

★ **NOTE:** it is important to realise that the Dirac gamma matrices are <u>not</u> four-vectors - they are constant matrices which remain invariant under a Lorentz transformation. However it can be shown that the Dirac equation is itself Lorentz covariant

Properties of the γ matrices

- From the properties of the α and β matrices (D2)-(D4) immediately obtain:

$$(\gamma^0)^2 = \beta^2 = 1 \quad \text{and} \quad (\gamma^1)^2 = \beta\alpha_x\beta\alpha_x = -\alpha_x\beta\beta\alpha_x = -\alpha_x^2 = -1$$

- The full set of relations is

$$\begin{aligned} (\gamma^0)^2 &= 1 \\ (\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 &= -1 \\ \gamma^0\gamma^j + \gamma^j\gamma^0 &= 0 \\ \gamma^j\gamma^k + \gamma^k\gamma^j &= 0 \quad (j \neq k) \end{aligned}$$

which can be expressed as:

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$ (defines the algebra)

- Are the gamma matrices Hermitian?
  - β is Hermitian so γ^0 is Hermitian.
  - The α matrices are also Hermitian, giving

$$\gamma^{1\dagger} = (\beta\alpha_x)^\dagger = \alpha_x^\dagger\beta^\dagger = \alpha_x\beta = -\beta\alpha_x = -\gamma^1$$

  - Hence $\gamma^1,\ \gamma^2,\ \gamma^3$ are anti-Hermitian

$$\boxed{\gamma^{0\dagger} = \gamma^0,\ \ \gamma^{1\dagger} = -\gamma^1,\ \ \gamma^{2\dagger} = -\gamma^2,\ \ \gamma^{3\dagger} = -\gamma^3}$$

Pauli-Dirac Representation

- From now on we will use the Pauli-Dirac representation of the gamma matrices:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}; \quad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}$$

which when written in full are

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}; \; \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}; \; \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}; \; \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

- Using the gamma matrices $\rho = \psi^\dagger \psi$ and $\vec{j} = \psi^\dagger \vec{\alpha} \psi$ can be written as:

$$j^\mu = (\rho, \vec{j}) = \psi^\dagger \gamma^0 \gamma^\mu \psi$$

where j^μ is the four-vector current.

(The proof that j^μ is a four vector can be found in all standard text books)

- In terms of the four-vector current the continuity equation becomes

$$\partial_\mu j^\mu = 0$$

- Finally the expression for the four-vector current

$$j^\mu = \psi^\dagger \gamma^0 \gamma^\mu \psi$$

can be simplified by introducing the adjoint spinor

The Adjoint Spinor

- The adjoint spinor is defined as

$$\overline{\psi} = \psi^{\dagger}\gamma^{0}$$

i.e. $$\overline{\psi} = \psi^{\dagger}\gamma^{0} \quad = \quad (\psi^{*})^{T}\gamma^{0} = (\psi_1^{*}, \psi_2^{*}, \psi_3^{*}, \psi_4^{*}) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\overline{\psi} \quad = \quad (\psi_1^{*}, \psi_2^{*}, -\psi_3^{*}, -\psi_4^{*})$$

- In terms the adjoint spinor the four vector current can be written:

$$j^{\mu} = \overline{\psi}\gamma^{\mu}\psi$$

★ We will use this expression in the Feynman rules for the Lorentz invariant matrix element for the fundamental interactions.

★ That's enough on notation, start to investigate the free particle solutions of the Dirac equation...

Dirac Equation: Free Particle at Rest

- Look for **free particle** solutions to the Dirac equation of form:

$$\boxed{\psi = u(E,\vec{p})e^{i(\vec{p}.\vec{r}-Et)}}$$

where $u(\vec{p},E)$, which is a constant four-component spinor which must satisfy the Dirac equation

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi = 0$$

- Consider the derivatives of the free particle solution

$$\partial_0\psi = \frac{\partial\psi}{\partial t} = -iE\psi; \quad \partial_1\psi = \frac{\partial\psi}{\partial x} = ip_x\psi, \quad \ldots$$

substituting these into the Dirac equation gives:

$$(\gamma^0 E - \gamma^1 p_x - \gamma^2 p_y - \gamma^3 p_z - m)u = 0$$

which can be written: $$\boxed{(\gamma^{\mu}p_{\mu} - m)u = 0} \qquad \text{(D10)}$$

- This is the Dirac equation in "momentum" – note it contains no derivatives.

- For a **particle at rest** $\vec{p} = 0$

and $\psi = u(E,0)e^{-iEt}$

eq. (D10) → $$E\gamma^0 u - mu = 0$$

$$E\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\begin{pmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{pmatrix}=m\begin{pmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{pmatrix} \qquad \text{(D11)}$$

- **This equation has four orthogonal solutions:**

$$u_1(m,0)=\begin{pmatrix}1\\0\\0\\0\end{pmatrix};\quad u_2(m,0)=\begin{pmatrix}0\\1\\0\\0\end{pmatrix};\quad u_3(m,0)=\begin{pmatrix}0\\0\\1\\0\end{pmatrix};\quad u_4(m,0)=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

(D11) → $E = m$ (for u_1, u_2)

(D11) → $E = -m$ (for u_3, u_4)

still have NEGATIVE ENERGY SOLUTIONS

- **Including the time dependence from** $\psi = u(E,0)e^{-iEt}$ **gives**

$$\psi_1=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}e^{-imt};\quad \psi_2=\begin{pmatrix}0\\1\\0\\0\end{pmatrix}e^{-imt};\quad \psi_3=\begin{pmatrix}0\\0\\1\\0\end{pmatrix}e^{+imt};\ \text{and}\ \ \psi_4=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}e^{+imt}$$

Two spin states with E>0 (ψ_1, ψ_2)

Two spin states with E<0 (ψ_3, ψ_4)

★ **In QM mechanics can't just discard the E<0 solutions as unphysical as we require a complete set of states - i.e. 4 SOLUTIONS**

Dirac Equation: Plane Wave Solutions

★ **The four plane wave solutions are:** $\psi_i = u_i(E,\vec{p})e^{i(\vec{p}.\vec{r}-Et)}$

$$u_1 = N_1 \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}; \quad u_2 = N_2 \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}; \quad u_3 = N_3 \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x+ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix}; \quad u_4 = N_4 \begin{pmatrix} \frac{p_x-ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix}$$

• **If any of these solutions is put back into the Dirac equation, we obtain**

$$E^2 = \vec{p}^2 + m^2$$

which doesn't in itself identify the negative energy solutions.

• **One rather subtle point:** One could ask the question whether we can interpret **all four** solutions as positive energy solutions. The answer is no. If we take all solutions to have the same value of E, i.e. E = +|E|, only two of the solutions are found to be independent.

• **There are only four independent solutions when two are taken to have E<0.**

★ **To identify which solutions have E<0, refer back to particle at rest (eq. D11).**

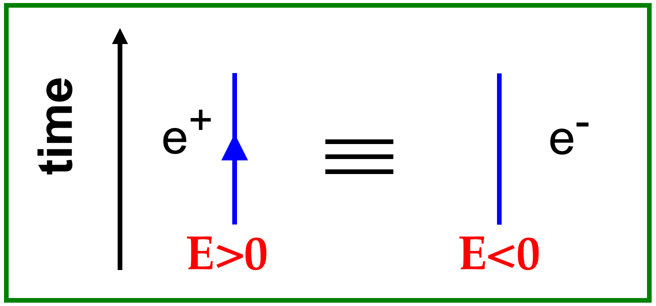
• **For** $\vec{p} = 0$ u_1, u_2 **correspond to the E>0 particle at rest solutions**

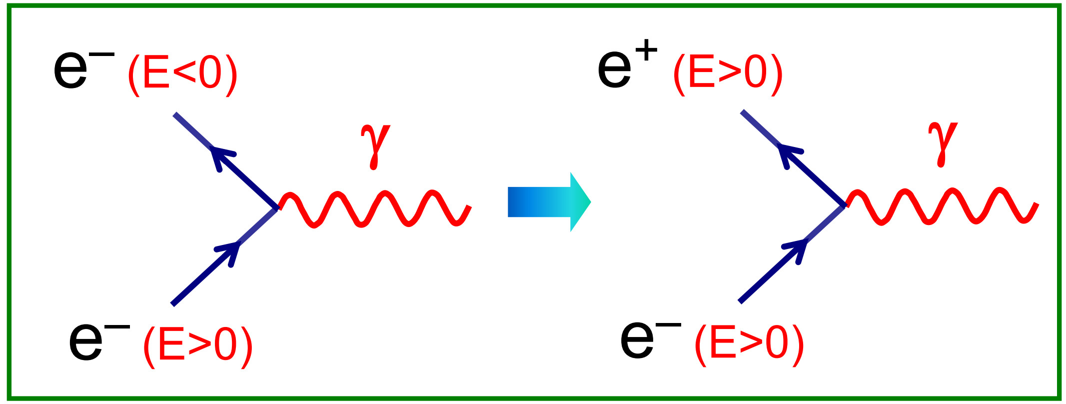
u_3, u_4 **correspond to the E<0 particle at rest solutions**

★ **So u_1, u_2 are the +ve energy solutions and u_3, u_4 are the -ve energy solutions**

Feynman-Stückelberg Interpretation

★ Interpret a negative energy solution as a negative energy particle which propagates backwards in time or equivalently a positive energy anti-particle which propagates forwards in time





$$e^{-i(-E)(-t)} \rightarrow e^{-iEt}$$

NOTE: in the Feynman diagram the arrow on the anti-particle remains in the backwards in time direction to label it an anti-particle solution.

★ At this point it become more convenient to work with anti-particle wave-functions with $E = \sqrt{|\vec{p}|^2 + m^2}$ motivated by this interpretation

Summary of Solutions to the Dirac Equation

- The normalised free **PARTICLE** solutions to the Dirac equation:

$$\psi = u(E,\vec{p})e^{+i(\vec{p}.\vec{r}-Et)} \quad \text{satisfy} \quad (\gamma^\mu p_\mu - m)u = 0$$

with

$$u_1 = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}; \qquad u_2 = \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

- The **ANTI-PARTICLE** solutions in terms of the physical energy and momentum:

$$\psi = v(E,\vec{p})e^{-i(\vec{p}.\vec{r}-Et)} \quad \text{satisfy} \quad (\gamma^\mu p_\mu + m)v = 0$$

with

$$v_1 = \sqrt{E+m}\begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix}; \qquad v_2 = \sqrt{E+m}\begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}$$

For these states the spin is given by $\hat{S}^{(v)} = -\hat{S}$

- For both particle and anti-particle solutions: $E = \sqrt{|\vec{p}|^2 + m^2}$

Spin States

- In general the spinors u_1, u_2, v_1, v_2 are **not** Eigenstates of $\hat{S}_z$

$$\hat{S}_z = \tfrac{1}{2}\Sigma_z = \tfrac{1}{2}\begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- However, particles/anti-particles travelling in the z-direction: $p_z = \pm|\vec{p}|$

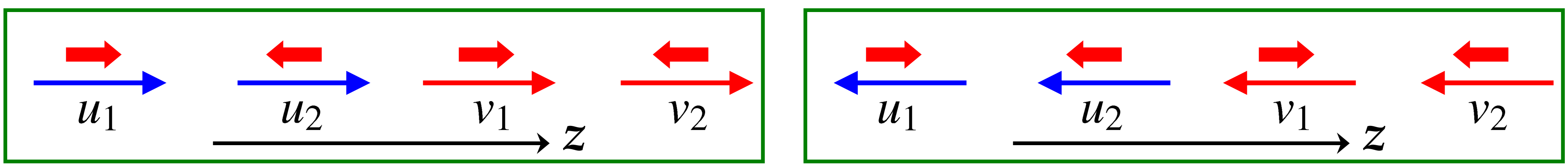
$$u_1 = N\begin{pmatrix} 1 \\ 0 \\ \frac{\pm|\vec{p}|}{E+m} \\ 0 \end{pmatrix}; \quad u_2 = N\begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{\mp|\vec{p}|}{E+m} \end{pmatrix}; \quad v_1 = N\begin{pmatrix} 0 \\ \frac{\mp|\vec{p}|}{E+m} \\ 0 \\ 1 \end{pmatrix}; \quad v_2 = N\begin{pmatrix} \frac{\pm|\vec{p}|}{E+m} \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

are Eigenstates of $\hat{S}_z$

$$\hat{S}_z u_1 = +\tfrac{1}{2}u_1 \qquad \hat{S}_z^{(v)} v_1 = -\hat{S}_z v_1 = +\tfrac{1}{2}v_1$$

$$\hat{S}_z u_2 = -\tfrac{1}{2}u_2 \qquad \hat{S}_z^{(v)} v_2 = -\hat{S}_z v_2 = -\tfrac{1}{2}v_2$$

Note the change of sign of $\hat{S}$ when dealing with antiparticle spinors



★ Spinors u_1, u_2, v_1, v_2 are only eigenstates of $\hat{S}_z$ for $p_z = \pm|\vec{p}|$

Pause for Breath…

- Have found solutions to the Dirac equation which are also eigenstates $\hat{S}_z$ but only for particles travelling along the z axis.
- Not a particularly useful basis
- More generally, want to label our states in terms of "good quantum numbers", i.e. a set of commuting observables.
- Can't use z component of spin: $[\hat{H}, \hat{S}_z] \neq 0$
- Introduce a new concept **"HELICITY"**

Helicity plays an important role in much that follows

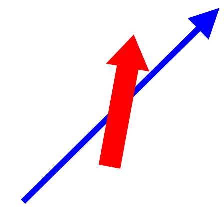
Helicity

★ The component of a particles spin along its direction of flight is a good quantum number:

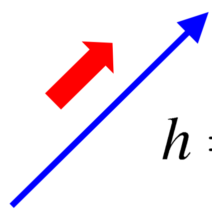
$$[\hat{H},\hat{S}.\hat{p}] = 0$$

★ Define the component of a particles spin along its direction of flight as **HELICITY**:

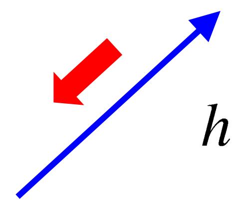
$$h \equiv \frac{\vec{S}.\vec{p}}{|\vec{S}||\vec{p}|} = \frac{2\vec{S}.\vec{p}}{|\vec{p}|} = \frac{\vec{\Sigma}.\vec{p}}{|\vec{p}|}$$



- If we make a measurement of the component of spin of a spin-half particle along any axis it can take two values $\pm 1/2$, consequently the eigenvalues of the helicity operator for a spin-half particle are: ± 1



$h = +1$



$h = -1$

Often termed: "right-handed" "left-handed"

★ **NOTE:** these are "RIGHT-HANDED" and LEFT-HANDED <u>**HELICITY**</u> eigenstates
★ Later we will discuss RH and LH <u>**CHIRAL**</u> eigenstates. **Only** in the limit $v \approx c$ are the **HELICITY** eigenstates the same as the **CHIRAL** eigenstates

★ **The particle and anti-particle helicity eigenstates states are:**

$$u_\uparrow = N\begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}\cos\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix} \qquad u_\downarrow = N\begin{pmatrix} -\sin\left(\frac{\theta}{2}\right) \\ e^{i\phi}\cos\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}\sin\left(\frac{\theta}{2}\right) \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}\cos\left(\frac{\theta}{2}\right) \end{pmatrix}$$

$$v_\uparrow = N\begin{pmatrix} \frac{|\vec{p}|}{E+m}\sin\left(\frac{\theta}{2}\right) \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}\cos\left(\frac{\theta}{2}\right) \\ -\sin\left(\frac{\theta}{2}\right) \\ e^{i\phi}\cos\left(\frac{\theta}{2}\right) \end{pmatrix} \qquad v_\downarrow = N\begin{pmatrix} \frac{|\vec{p}|}{E+m}\cos\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}e^{i\phi}\sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

particles | antiparticles

★ **For all four states, normalising to $2E$ particles/Volume again gives** $N = \sqrt{E+m}$

★ **The helicity basis is a useful tool in the ultra-relativistic limit**

Summary

★ The formulation of relativistic quantum mechanics starting from the linear Dirac equation

$$\hat{H}\psi = (\vec{\alpha}.\vec{p} + \beta m)\psi = i\frac{\partial \psi}{\partial t}$$

⇒ New degrees of freedom : found to describe Spin ½ particles

★ In terms of 4x4 gamma matrices the Dirac Equation can be written:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

★ Introduces the 4-vector current and adjoint spinor:

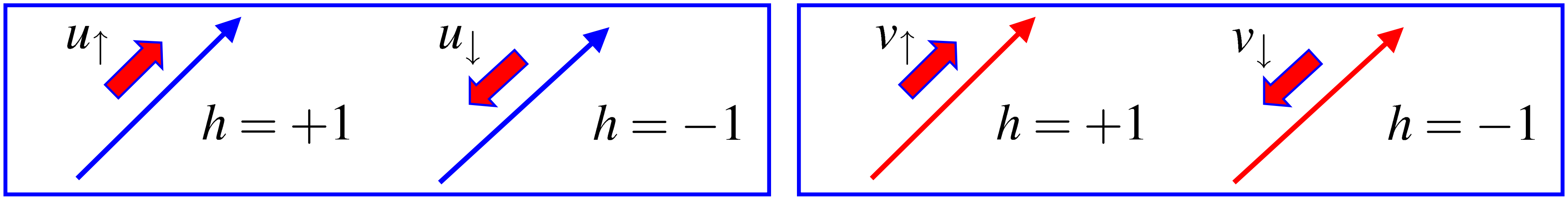
$$j^\mu = \psi^\dagger \gamma^0 \gamma^\mu \psi = \overline{\psi}\gamma^\mu \psi$$

★ With the Dirac equation: forced to have two positive energy and two negative energy solutions

★ Feynman-Stückelberg interpretation: -ve energy particle solutions propagating backwards in time correspond to physical +ve energy anti-particles propagating forwards in time

$$u_1,\ u_2,\ v_1,\ v_2$$

★ **A useful basis: particle and anti-particle helicity eigenstates**



★ **As an aside, in terms of 4-component spinors, the charge conjugation and parity operations are:**

$$\psi \rightarrow \hat{C}\psi = i\gamma^2\psi^\dagger \qquad\qquad \psi \rightarrow \hat{P}\psi = \gamma^0\psi$$

★ **Now have a relativistic description of particles… now we can discuss particle interactions and then QED.**