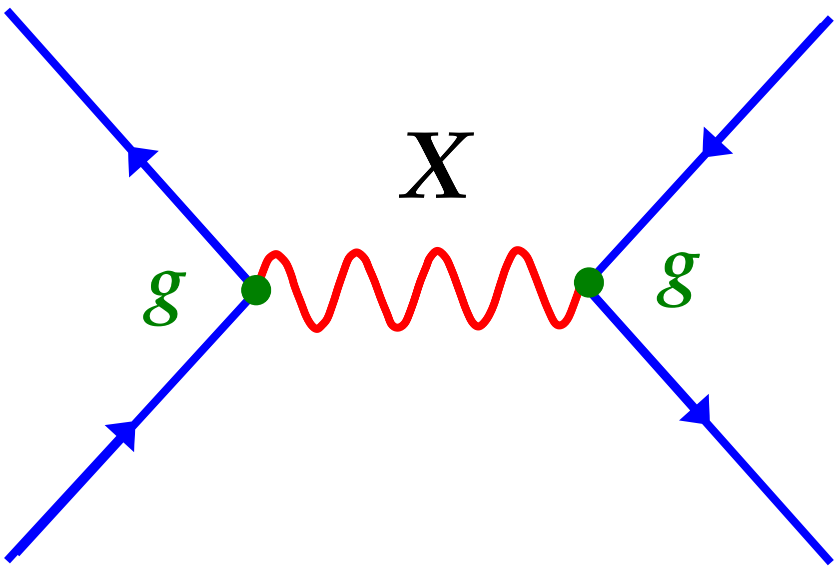
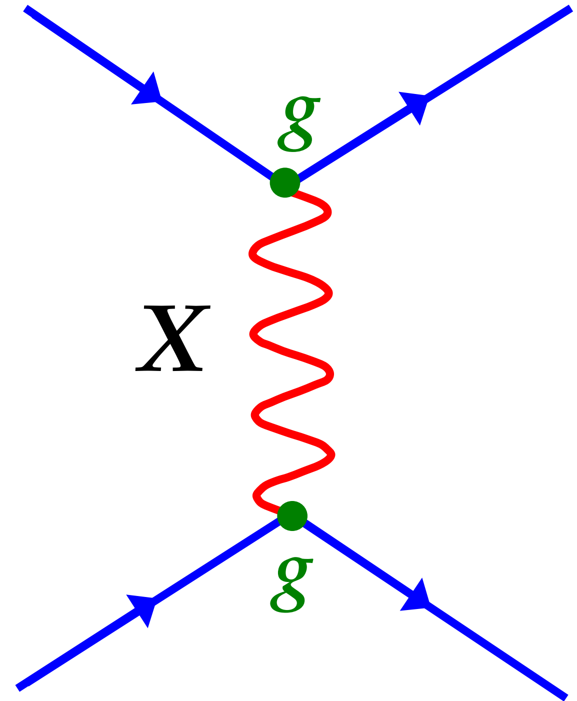
# Foundations of the Standard Model

**Prof Mark Thomson**




# 4. Quantum Chromodynamics

# Symmetries and Conservation Laws

★Suppose physics is invariant under the transformation

$$\psi \rightarrow \psi' = \hat{U}\psi$$

e.g. rotation of the coordinate axes

• To conserve probability normalisation require

$$\langle\psi|\psi\rangle = \langle\psi'|\psi'\rangle = \langle\hat{U}\psi|\hat{U}\psi\rangle = \langle\psi|\hat{U}^\dagger\hat{U}|\psi\rangle$$

$$\rightarrow \quad \boxed{\hat{U}^\dagger\hat{U} = 1}$$

i.e. $\hat{U}$ has to be **unitary**

• For physical predictions to be unchanged by the symmetry transformation, also require all QM matrix elements unchanged

$$\langle\psi|\hat{H}|\psi\rangle = \langle\psi'|\hat{H}|\psi'\rangle = \langle\psi|\hat{U}^\dagger\hat{H}\hat{U}|\psi\rangle$$

i.e. require $\quad \hat{U}^\dagger\hat{H}\hat{U} = \hat{H}$

$\times\hat{U} \quad \hat{U}\hat{U}^\dagger\hat{H}\hat{U} = \hat{U}\hat{H} \quad \rightarrow \quad \hat{H}\hat{U} = \hat{U}\hat{H}$

therefore $\quad \boxed{[\hat{H},\hat{U}] = 0}$

$\hat{U}$ **commutes with the Hamiltonian**

★Now consider the infinitesimal transformation ( $\varepsilon$ small )

$$\hat{U} = 1 + i\varepsilon\hat{G}$$

( $\hat{G}$ is called the **generator** of the transformation)

- For $\hat{U}$ to be unitary

$$\hat{U}\hat{U}^\dagger = (1 + i\varepsilon\hat{G})(1 - i\varepsilon\hat{G}^\dagger) = 1 + i\varepsilon(\hat{G} - \hat{G}^\dagger) + O(\varepsilon^2)$$

neglecting terms in $\varepsilon^2$ $\quad UU^\dagger = 1 \quad \Rightarrow \quad \boxed{\hat{G} = \hat{G}^\dagger}$

i.e. $\hat{G}$ is Hermitian and therefore corresponds to an observable quantity $G$ !

- Furthermore, $\quad [\hat{H}, \hat{U}] = 0 \quad \Rightarrow [\hat{H}, 1 + i\varepsilon\,\hat{G}] = 0 \quad \Rightarrow [\hat{H}, \hat{G}] = 0$

But from QM $\quad \frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{G}\rangle = i\langle[\hat{H}, \hat{G}]\rangle = 0$

i.e. $G$ is a conserved quantity.

**Symmetry ⟷ Conservation Law**

★ **For each symmetry of nature have an observable conserved quantity**

**Example:** Infinitesimal spatial translation $x \rightarrow x + \varepsilon$

i.e. expect physics to be invariant under $\psi(x) \rightarrow \psi' = \psi(x + \varepsilon)$

$$\psi'(x) = \psi(x + \varepsilon) = \psi(x) + \frac{\partial\psi}{\partial x}\varepsilon = \left(1 + \varepsilon\frac{\partial}{\partial x}\right)\psi(x)$$

but $\quad \hat{p}_x = -i\frac{\partial}{\partial x} \quad \Rightarrow \quad \psi'(x) = (1 + i\varepsilon\hat{p}_x)\psi(x)$

The generator of the symmetry transformation is $\hat{p}_x$ ➡ $p_x$ is conserved

- **Translational invariance** of physics implies **momentum conservation !**

- In general the symmetry operation may depend on more than one parameter

$$\hat{U} = 1 + i\vec{\varepsilon}.\vec{G}$$

For example for an infinitesimal 3D linear translation : $\vec{r} \rightarrow \vec{r} + \vec{\varepsilon}$

$$\longrightarrow \quad \hat{U} = 1 + i\vec{\varepsilon}.\vec{p} \qquad \vec{p} = (\hat{p}_x, \hat{p}_y, \hat{p}_z)$$

- So far have only considered an infinitesimal transformation, however a finite transformation can be expressed as a series of infinitesimal transformations

$$\hat{U}(\vec{\alpha}) = \lim_{n\rightarrow\infty}\left(1 + i\frac{\vec{\alpha}}{n}.\vec{G}\right)^n = e^{i\vec{\alpha}.\vec{G}}$$

# SU(3) Flavour Symmetry

★ As an example consider SU(3) flavour symmetry. Since $m_s > m_u/m_d$ don't have an **exact symmetry**. But $m_s$ not so very different from $m_u/m_d$ and can treat the strong interaction (and resulting hadron states) as if it were symmetric under $u \leftrightarrow d \leftrightarrow s$

- **NOTE:** any results obtained from this assumption are only approximate as the symmetry is not exact.
- The assumed uds flavour symmetry can be expressed as

$$\begin{pmatrix} u' \\ d' \\ s' \end{pmatrix} = \hat{U} \begin{pmatrix} u \\ d \\ s \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix} \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

- The 3x3 unitary matrix depends on 9 complex numbers, i.e. 18 real parameters There are 9 constraints from $\hat{U}^{\dagger}\hat{U} = 1$

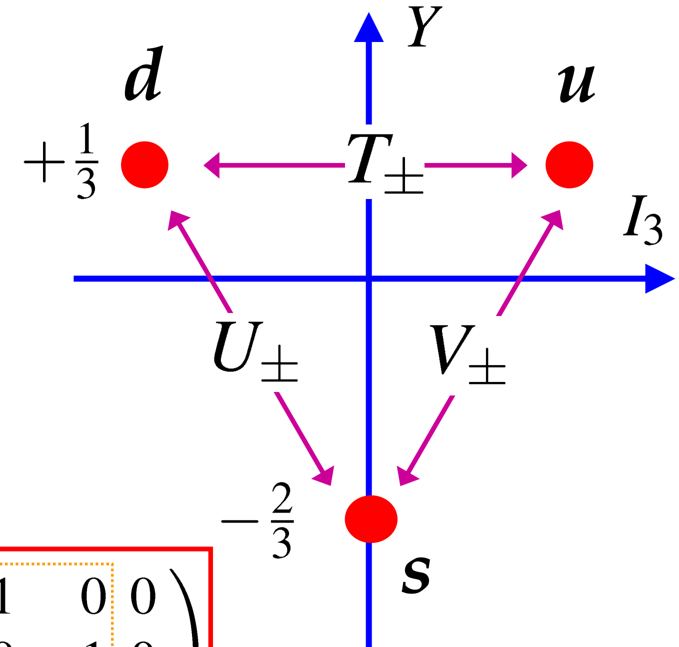
⇒ Can form 18 – 9 = 9 linearly independent matrices

**These 9 matrices form a U(3) group.**

- One matrix is simply the identity multiplied by a complex phase and is of no interest in the context of flavour symmetry
- The remaining 8 matrices have $\det U = 1$ and form an SU(3) group
- The eight matrices (the Hermitian generators) are: $\vec{T} = \frac{1}{2}\vec{\lambda}$ $\hat{U} = e^{i\vec{\alpha}.\vec{T}}$

★ **The other six matrices form six ladder operators which step between the states**

$$T_\pm = \tfrac{1}{2}(\lambda_1 \pm i\lambda_2)$$
$$V_\pm = \tfrac{1}{2}(\lambda_4 \pm i\lambda_5)$$
$$U_\pm = \tfrac{1}{2}(\lambda_6 \pm i\lambda_7)$$




**with** $I_3 = \tfrac{1}{2}\lambda_3$ $\quad Y = \frac{1}{\sqrt{3}}\lambda_8$

**and the eight Gell-Mann matrices**

**u ↔ d**

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

**u ↔ s**

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

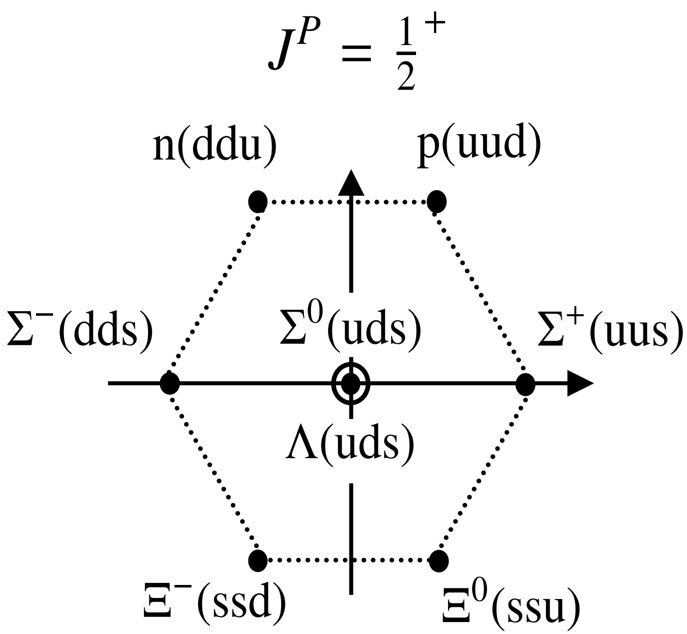
**d ↔ s**

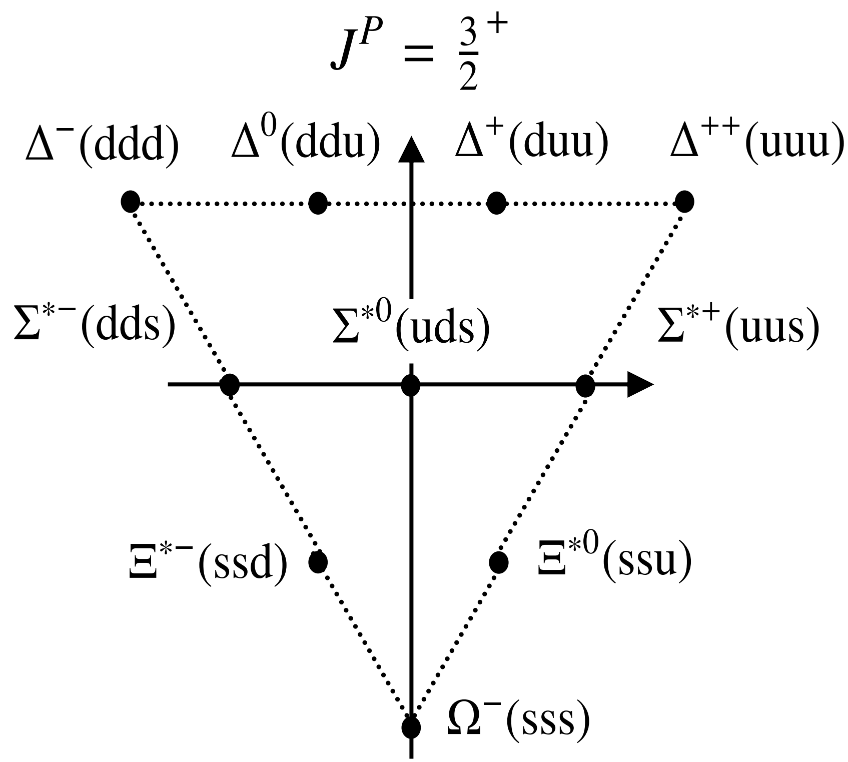
$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

# SU(3) Flavour Symmetry

★ **These ideas are incredibly powerful, e.g. with relatively little effort can explain the patterns of light mesons and light baryons**




★ **Quark flavour symmetry is not exact because the quark masses are not the same – if they were, the 10 decuplet states would all have the same mass – nevertheless there would be 10 states…**

# From QED to QCD

★ The fundamental symmetry of QCD is the exact SU(3) local gauge symmetry

"invariance under SU(3) local phase transformations"

- i.e. require invariance under $\psi \rightarrow \psi' = \psi e^{i\vec{\lambda}.\vec{\theta}(x)}$ where

$\vec{\lambda}$ are the eight 3x3 Gell-Mann matrices

$\vec{\theta}(x)$ are 8 functions taking different values at each point in space-time

⇒ 8 spin-1 gauge bosons

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$$

wave function is now a vector in COLOUR SPACE

⇒ QCD !

★ QCD is fully specified by require invariance under SU(3) **local** phase transformations

Corresponds to rotating states in colour space about an axis whose direction is different at every space-time point

⇒ interaction vertex: $-\frac{1}{2} i g_s \lambda^a_{ji} \gamma^\mu$

★ Predicts 8 massless gauge bosons – the gluons (one for each $\lambda$ )

# Gauge Invariance revisited

★ **Recall QED: invariance under U(1) local phase transformations**

- **i.e. require invariance under**

$$\psi(x) \to \psi'(x) = \hat{U}(x)\psi(x) = e^{iq\chi(x)}\psi(x)$$

- **the free particle Dirac Equation**

$$i\gamma^\mu \partial_\mu \psi = m\psi$$

**becomes**

$$i\gamma^\mu \partial_\mu (e^{iq\chi(x)}\psi) = m e^{iq\chi(x)}\psi$$

$$e^{iq\chi}\, i\gamma^\mu \left[\partial_\mu \psi + iq(\partial_\mu \chi)\psi\right] = e^{iq\chi} m\psi$$

**which shows that the Dirac equation is not invariant under this transformation**

★ **"Sanity" can be restored by modifying the Dirac equation by introducing a new field (the photon) with the following properties**

$$i\gamma^\mu(\partial_\mu + iqA_\mu)\psi - m\psi = 0$$

$$A_\mu \to A'_\mu = A_\mu - \partial_\mu \chi$$

# From U(1) to SU(3)

★ **Repeat for SU(3) local phase transformations**

- **i.e. require invariance under**

$$\psi(x) \rightarrow \psi'(x) = \exp\left[ig_S\boldsymbol{\alpha}(x)\cdot\hat{\mathbf{T}}\right]\psi(x)$$

- **worth some reflection…**
  - $\hat{\mathbf{T}} = \{T^a\}$ **are the eight generators of the SU(3) symmetry group,** $T^a = \frac{1}{2}\lambda^a$
  - $\alpha^a(x)$ **are eight functions of the space time coordinate (local)**
  - **the wavefunction must have a new 3-component degree of freedom (colour)**

- **Under the SU(3) LGT, the free particle Dirac Equation becomes**

$$i\gamma^{\mu}\left[\partial_\mu + ig_S(\partial_\mu\boldsymbol{\alpha})\cdot\hat{\mathbf{T}}\right]\psi = m\psi$$

★ **Just as in QED, invariance can be restored by modifying the Dirac equation by introducing eight new fields (the gluons) associated with the 8 generators**

$$i\gamma^{\mu}\left[\partial_\mu + ig_S G^a_\mu T^a\right]\psi - m\psi = 0$$

- **with fields transforming as**

$$G^k_\mu \rightarrow G^{k\,\prime}_\mu = G^k_\mu - \partial_\mu\alpha_k - \boxed{g_S f_{ijk}\alpha_i G^j_\mu}$$

**the extra term arises because the generators of SU(3) do not commute**

# From U(1) to SU(3)

★ **The fundamental strong interaction vertex can be identified from**

$$i\gamma^\mu \left[\partial_\mu + i g_S G^a_\mu T^a\right]\psi - m\psi = 0$$

**and is…**

$$g_S T^a \gamma^\mu G^a_\mu \psi = g_S \tfrac{1}{2}\lambda^a \gamma^\mu G^a_\mu \psi$$

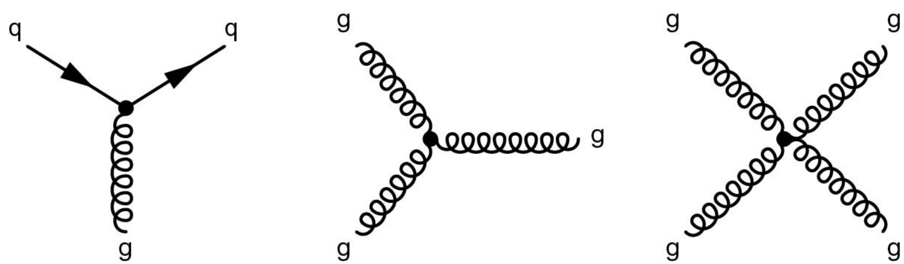
★ **Similar to QED, except that there are eight gluon fields, and the charge of QCD is a new degree of freedom of the SU(3) symmetry termed "colour"**

$$r = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad g = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

- **The new term in the field transformations**

$$G^k_\mu \to G^{k\,\prime}_\mu = G^k_\mu - \partial_\mu \alpha_k - g_S f_{ijk} \alpha_i G^j_\mu$$

**gives rise to gluon self-interactions in addition to the gluon-quark vertex**

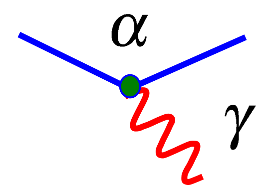


# Colour in QCD

★ The theory of the strong interaction, Quantum Chromodynamics (QCD), is very similar to QED but with 3 conserved "colour" charges

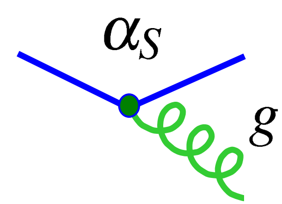
**In QED:**

- the electron carries one unit of charge $-e$
- the anti-electron carries one unit of anti-charge $+e$
- the force is mediated by a massless "gauge boson" – the photon




**In QCD:**

- quarks carry colour charge: $r, g, b$
- anti-quarks carry anti-charge: $\bar{r}, \bar{g}, \bar{b}$
- The force is mediated by massless gluons




★ In QCD, the strong interaction is invariant under rotations in colour space

$$r \leftrightarrow b;\ r \leftrightarrow g;\ b \leftrightarrow g$$

i.e. the same for all three colours

⇒ **SU(3) colour symmetry**

- This is an **exact** symmetry, unlike the approximate uds flavour symmetry touched on previously

# Colour Confinement

★ **All observed free particles are "colourless"**

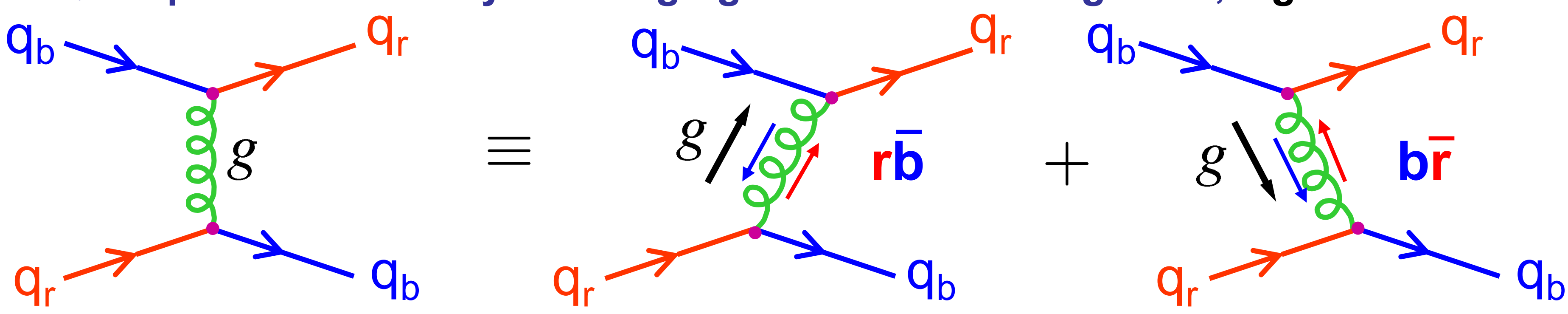
- **i.e. never observe a free quark (which would carry colour charge)**
- **consequently, quarks are always found in bound states colourless hadrons**

★ **Colour Confinement Hypothesis:**

> **only colour singlet states can exist as free particles**

# Gluons

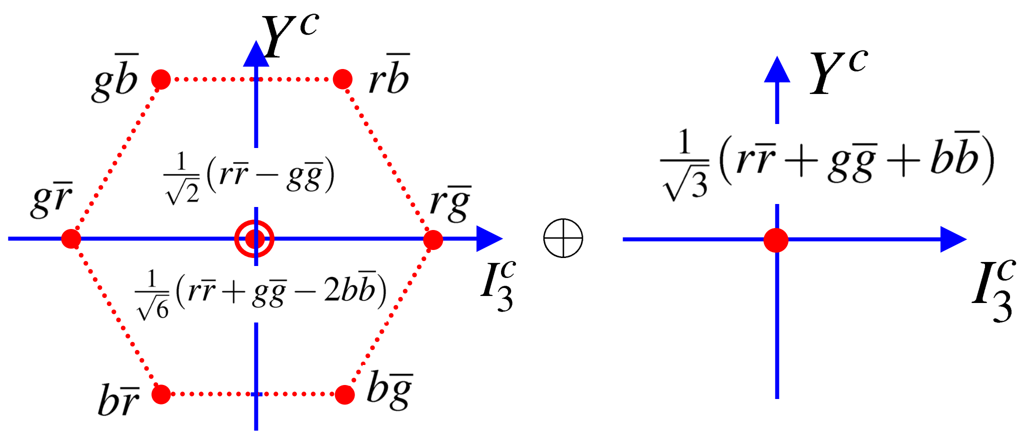
★ In QCD quarks interact by exchanging virtual massless gluons, e.g.




★ Gluons carry colour and anti-colour, e.g.


★ Gluon colour wave-functions (colour + anti-colour) are the same as those obtained for mesons (also colour + anti-colour)

⇨ OCTET + "COLOURLESS" SINGLET



# Gluon-Gluon Interactions

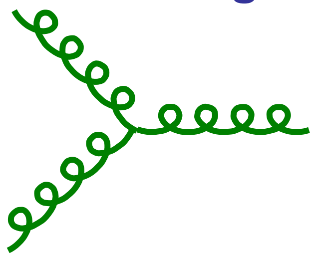
★ In **QED** the **photon** does not carry the charge of the EM interaction (photons are electrically neutral)

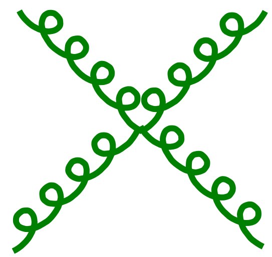
★ In contrast, in **QCD** the **gluons** do carry **colour charge**

⇨ **Gluon Self-Interactions**

★ Two new vertices (no QED analogues)

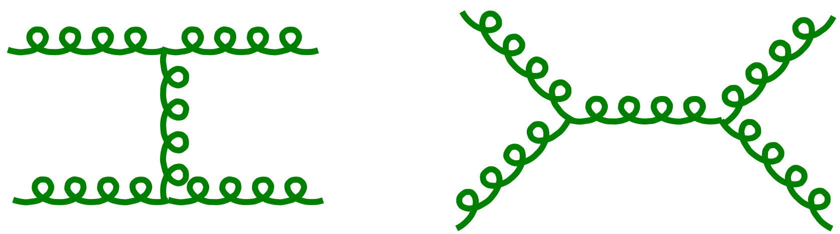
**triple-gluon vertex**



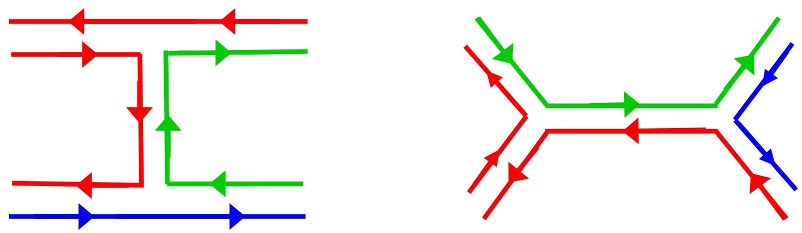


**quartic-gluon vertex**

★ In addition to quark-quark scattering, therefore can have gluon-gluon scattering

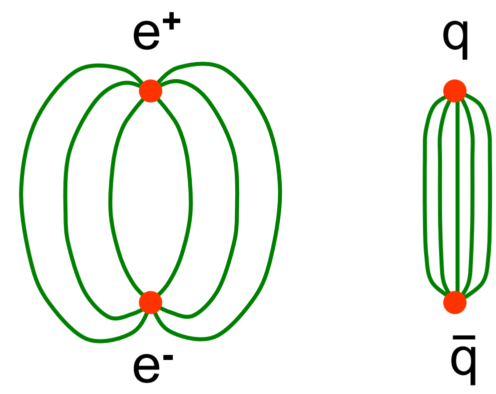


e.g. possible way of arranging the colour flow

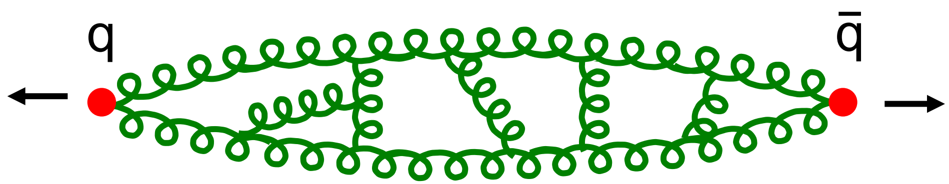


# Gluon self-Interactions and Confinement

- ★ Gluon self-interactions are believed to give rise to colour confinement
- ★ Qualitative picture:
  - Compare QED with QCD
  - In QCD "gluon self-interactions squeeze lines of force into a flux tube"




- ★ What happens when try to separate two coloured objects e.g. $q\bar{q}$




- Form a flux tube of interacting gluons of approximately constant energy density $\sim 1\,\mathrm{GeV/fm}$

$$\Rightarrow \quad V(r) \sim \lambda r$$

- Require infinite energy to separate coloured objects to infinity
- **Coloured** quarks and gluons are always **confined** within colourless states
- In this way QCD provides a plausible explanation of confinement – but **not yet proven** (although there has been recent progress with Lattice QCD)

# Hadronisation and Jets

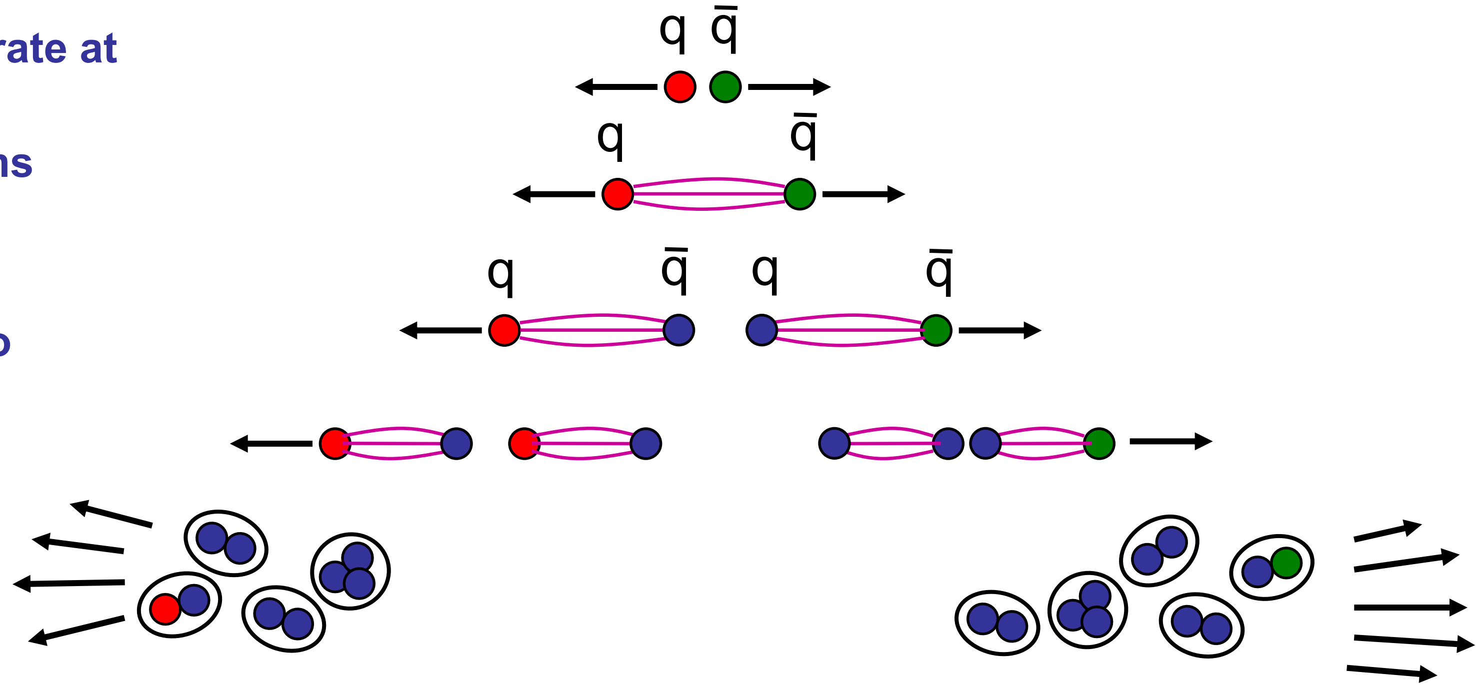
★ Consider a quark and anti-quark produced in electron positron annihilation

i) Initially Quarks separate at high velocity

ii) Colour flux tube forms between quarks

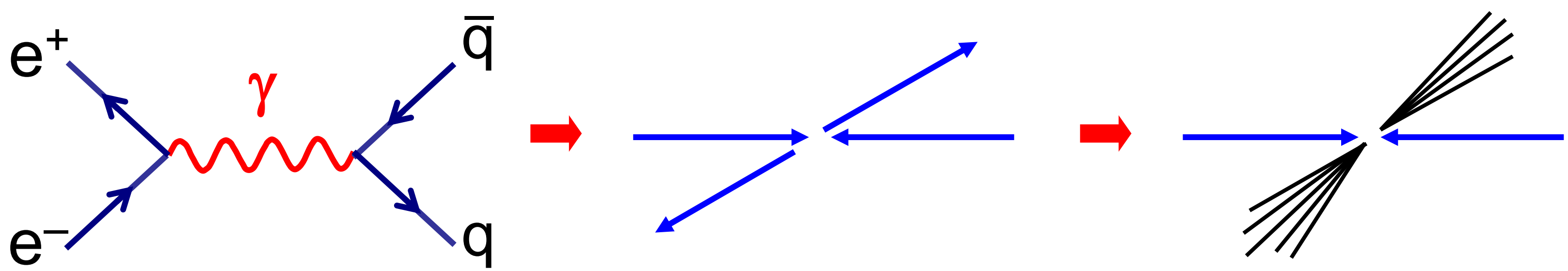
iii) Energy stored in the flux tube sufficient to produce $q\bar{q}$ pairs

iv) Process continues until quarks pair up into jets of colourless hadrons




★ This process is called **hadronisation**. It is not (yet) calculable.

★ The main consequence is that at collider experiments **quarks** and **gluons** observed as jets of particles



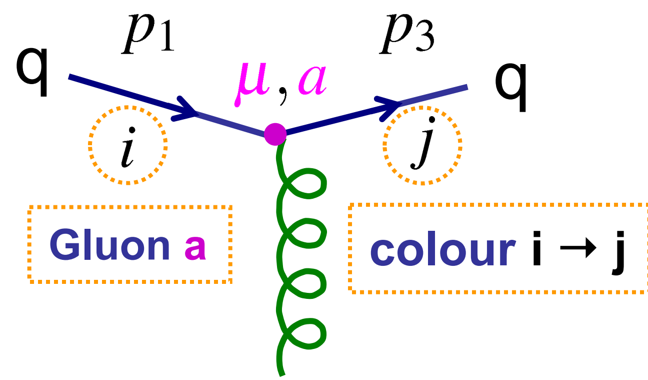
# The Quark – Gluon Interaction

- Representing the colour part of the quark wave-functions by:

$$r = c_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad g = c_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad b = c_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

- Particle wave-functions $u(p) \longrightarrow c_i u(p)$
- The QCD qqg vertex is written:

$$\overline{u}(p_3)c_j^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1)$$




- Only difference w.r.t. QED is the insertion of the 3x3 SU(3) Gell-Mann matrices
- Isolating the colour part:

$$c_j^\dagger \lambda^a c_i = c_j^\dagger \begin{pmatrix} \lambda^a_{1i} \\ \lambda^a_{2i} \\ \lambda^a_{3i} \end{pmatrix} = \lambda^a_{ji}$$

- Hence the fundamental quark - gluon QCD interaction can be written

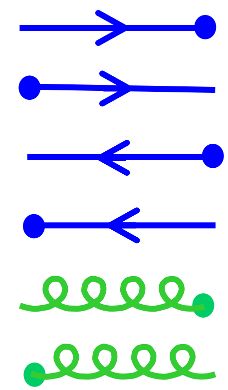
$$\overline{u}(p_3)c_j^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1) \equiv \overline{u}(p_3)\{-\tfrac{1}{2}ig_s\lambda^a_{ji}\gamma^\mu\}u(p_1)$$

# Feynman Rules for QCD

- **External Lines**

spin 1/2
- incoming quark $u(p)$
- outgoing quark $\bar{u}(p)$
- incoming anti-quark $\bar{v}(p)$
- outgoing anti-quark $v(p)$

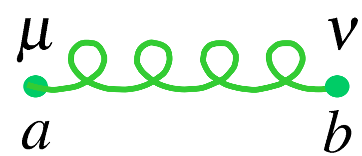
spin 1
- incoming gluon $\varepsilon^{\mu}(p)$
- outgoing gluon $\varepsilon^{\mu}(p)^{*}$



- **Internal Lines (propagators)**

spin 1 gluon

$$\frac{-ig_{\mu\nu}}{q^2}\delta^{ab}$$

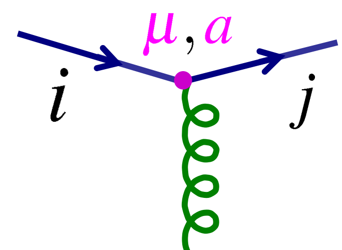



a, b = 1,2,…,8 are gluon colour indices

- **Vertex Factors**

spin 1/2 quark

$$-ig_s\tfrac{1}{2}\lambda^a_{ji}\gamma^{\mu}$$



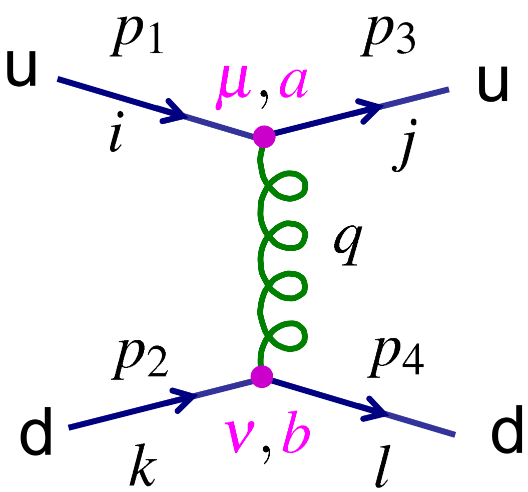

i, j = 1,2,3 are quark colours,

$\lambda^a$ a = 1,2,..8 are the Gell-Mann SU(3) matrices

- **+ 3 gluon and 4 gluon interaction vertices**
- **Matrix Element** $-iM$ = **product of all factors**

# Matrix Element for quark-quark scattering

★ Consider QCD scattering of an up and a down quark




- The incoming and out-going quark colours are labelled by $i, j, k, l = \{1,2,3\}$ (or $\{r,g,b\}$)
- In terms of colour this scattering is $ik \rightarrow jl$
- The 8 different gluons are accounted for by the colour indices $a, b = 1, 2, \ldots, 8$
- NOTE: the δ-function in the propagator ensures $a = b$, i.e. the gluon "emitted" at $a$ is the same as that "absorbed" at $b$

★ Applying the Feynman rules:

$$-iM = \left[\overline{u}_u(p_3)\{-\tfrac{1}{2}ig_s\lambda^a_{ji}\gamma^\mu\}u_u(p_1)\right]\frac{-ig_{\mu\nu}}{q^2}\delta^{ab}\left[\overline{u}_d(p_4)\{-\tfrac{1}{2}ig_s\lambda^b_{lk}\gamma^\nu\}u_d(p_2)\right]$$

where summation over $a$ and $b$ (and $\mu$ and $\nu$) is implied.

★ Summing over $a$ and $b$ using the δ-function gives:

$$M = -\frac{g_s^2}{4}\lambda^a_{ji}\lambda^a_{lk}\frac{1}{q^2}g_{\mu\nu}[\overline{u}_u(p_3)\gamma^\mu u_u(p_1)][\overline{u}_d(p_4)\gamma^\nu u_d(p_2)]$$

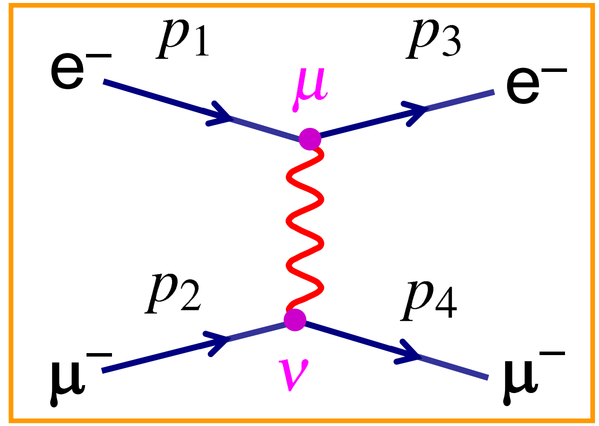
Sum over all 8 gluons (repeated indices)

# QCD vs QED

**QED**

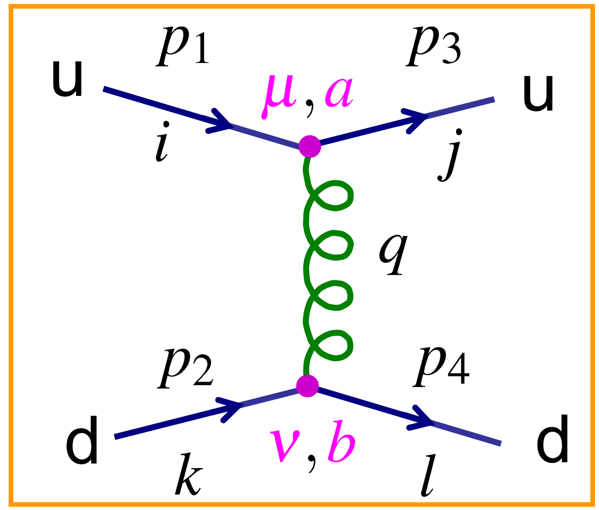
$$-iM = [\overline{u}(p_3)ie\gamma^\mu u(p_1)]\frac{-ig_{\mu\nu}}{q^2}[\overline{u}(p_4)ie\gamma^\nu u(p_2)]$$

$$M = -e^2\frac{1}{q^2}g_{\mu\nu}[\overline{u}(p_3)\gamma^\mu u(p_1)][\overline{u}(p_4)\gamma^\nu u(p_2)]$$




**QCD**

$$M = -\frac{g_s^2}{4}\lambda^a_{ji}\lambda^a_{lk}\frac{1}{q^2}g_{\mu\nu}[\overline{u}_u(p_3)\gamma^\mu u_u(p_1)][\overline{u}_d(p_4)\gamma^\nu u_d(p_2)]$$




★ **QCD Matrix Element = QED Matrix Element with:**

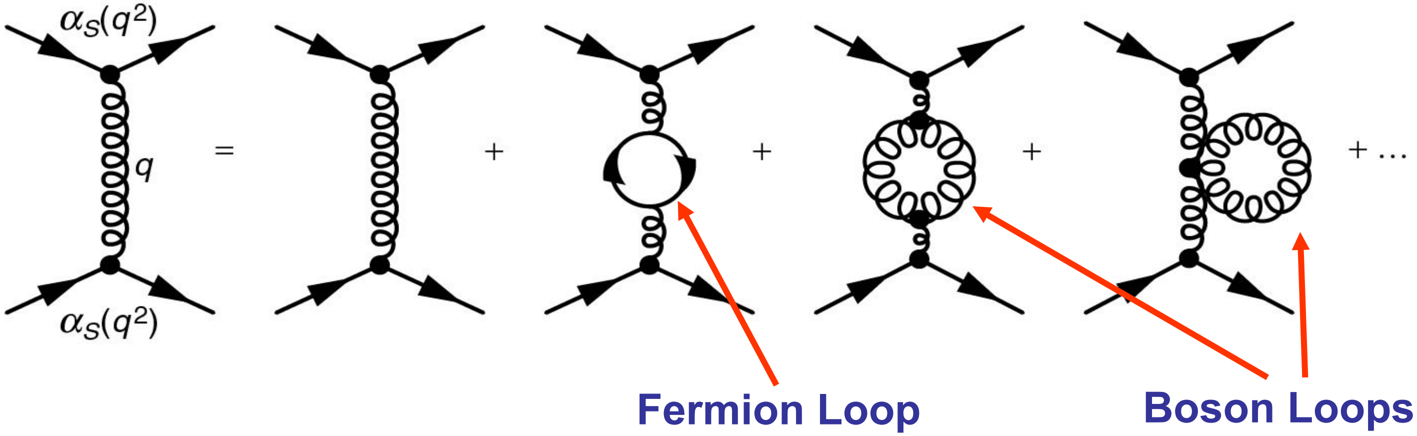
- $e^2 \rightarrow g_s^2$ **or equivalently** $\alpha = \frac{e^2}{4\pi} \rightarrow \alpha_s = \frac{g_s^2}{4\pi}$

**+ QCD Matrix Element includes an additional "colour factor"**

$$C(ik \rightarrow jl) \equiv \frac{1}{4}\sum_{a=1}^{8}\lambda^a_{ji}\lambda^a_{lk}$$

# Running of $\alpha_s$

**QCD** Similar to QED but also have gluon loops




★ Remembering **adding amplitudes**, so can get negative interference and the sum can be smaller than the original diagram alone

★ Bosonic loops "interfere negatively"

$$\alpha_S(Q^2) = \alpha_S(Q_0^2) \Bigg/ \left[1 + B\alpha_S(Q_0^2)\ln\left(\frac{Q^2}{Q_0^2}\right)\right]$$

with $$B = \frac{11N_c - 2N_f}{12\pi}$$

$N_c$ = no. of colours

$N_f$ = no. of quark flavours

$N_c = 3;\ N_f = 6$ ➡ $B > 0$

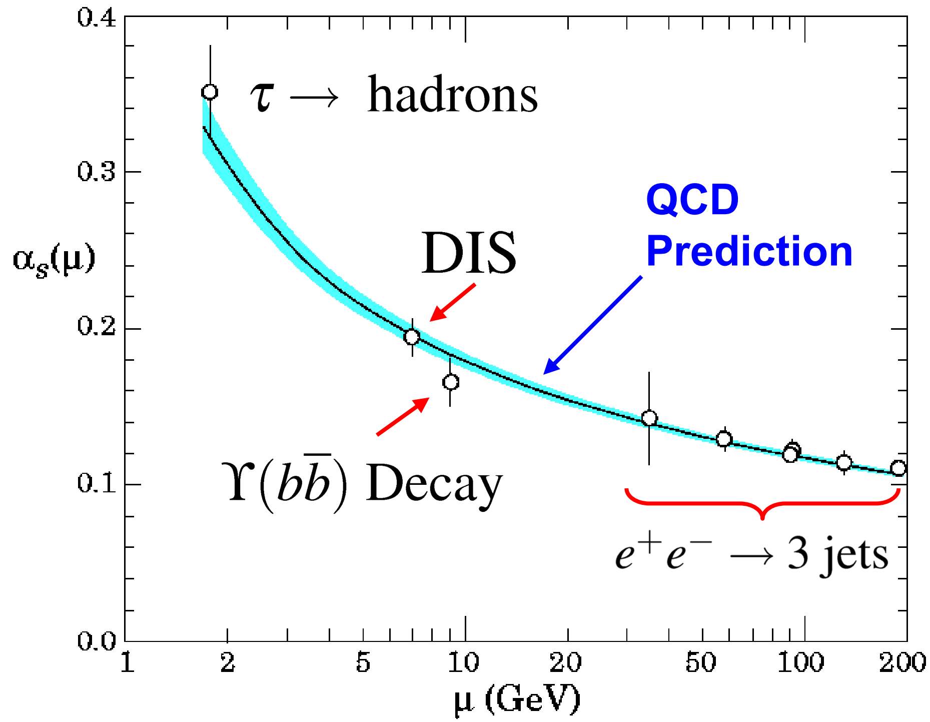
➡ $\alpha_S$ **decreases with** $Q^2$

**Nobel Prize for Physics, 2004**
**(Gross, Politzer, Wilczek)**

★ Measure $\alpha_s$ in many ways:

- jet rates
- DIS
- tau decays
- bottomonium decays
- +…

★ As predicted by QCD, $\alpha_s$ decreases with $Q^2$




$\mu \approx Q$

★ At low $Q^2$: $\alpha_s$ is large, e.g. at $Q^2 = 1\,\mathrm{GeV}^2$ find $\alpha_s \sim 1$

- Can't use perturbation theory ! This is the reason why QCD calculations at low energies are so difficult, e.g. properties hadrons, hadronisation of quarks to jets,…

★ At high $Q^2$ : $\alpha_s$ is rather small, e.g. at $Q^2 = M_Z^2$ find $\alpha_s \sim 0.12$

➡ **Asymptotic Freedom**

- Can use perturbation theory and this is the reason that in DIS at high $Q^2$ quarks behave as if they are quasi-free (i.e. only weakly bound within hadrons)

# Summary

★ Superficially QCD very similar to QED

★ But gluon self-interactions are believed to result in colour confinement

★ All hadrons are colour singlets which explains why only observe

**Mesons** **Baryons**

★ A low energies $\alpha_S \sim 1$

➡ Can't use perturbation theory !

**Non-Perturbative regime**

★ Coupling constant runs, smaller coupling at higher energy scales

$$\alpha_S(100\,\text{GeV}) \sim 0.1$$

➡ Can use perturbation theory

**Asymptotic Freedom**

★ and of course… QCD provides a good description of experimental data at the LHC