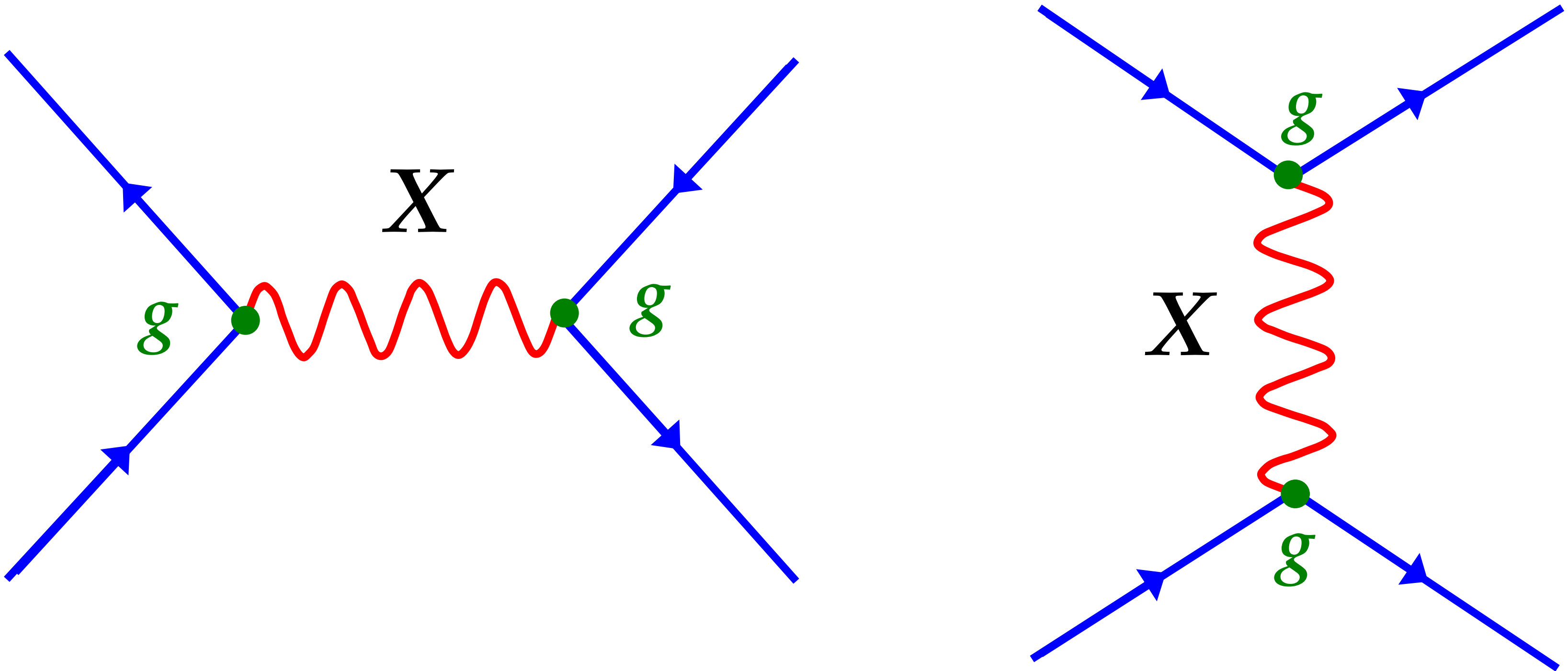
Foundations of the Standard Model

Prof Mark Thomson



3. Aspects of Quantum Electrodynamics

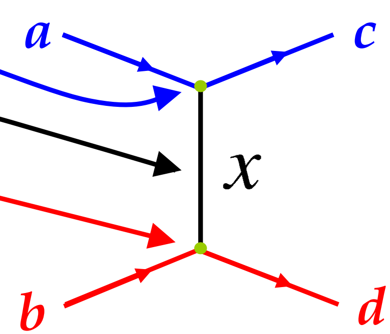
Illustrative Calculation

★ As an example, consider the interaction of an electron and tau lepton by the exchange of a photon. Although the general ideas we applied previously still hold, but now have to account for the **spin of the electron/tau-lepton** and also the **spin (polarization) of the virtual photon**.

- Previously with the example of a simple spin-less interaction we had:

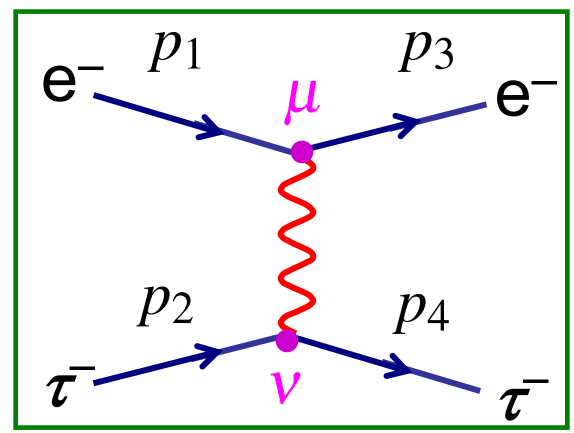
$$M = \langle\psi_c|V|\psi_a\rangle \frac{1}{q^2 - m_x^2} \langle\psi_d|V|\psi_b\rangle$$

$\langle\psi_c|V|\psi_a\rangle = g_a$, $\quad \langle\psi_d|V|\psi_b\rangle = g_b$



★ In QED we could again go through the procedure of summing the time-orderings using Dirac spinors and the expression for $\hat{V}_D$. If we were to do this, remembering to sum over all photon polarizations, we would obtain:

- but first need a description of the interaction vertex for QED



Quantum Electrodynamics (QED)

- **The basic interaction between a photon and a charged particle can be introduced by making the minimal substitution (classical electrodynamics)**

$$\vec{p} \to \vec{p} - q\vec{A}; \quad E \to E - q\phi$$

(here $q =$ charge)

In QM:

$$\vec{p} = -i\vec{\nabla}; \quad E = i\partial/\partial t$$

Therefore, make substitution: $i\partial_\mu \to i\partial_\mu - qA_\mu$

where $A_\mu = (\phi, -\vec{A}); \quad \partial_\mu = (\partial/\partial t, +\vec{\nabla})$

- **The Dirac equation:**

$$\gamma^\mu \partial_\mu \psi + im\psi = 0 \quad \Rightarrow \quad \gamma^\mu \partial_\mu \psi + \boxed{iq\gamma^\mu A_\mu \psi} + im\psi = 0$$

- **The final complication is that we have to account for the photon polarization states.**

$$A_\mu = \varepsilon_\mu^{(\lambda)} e^{i(\vec{p}.\vec{r} - Et)}$$

e.g. for a *real* photon propagating in the z direction we have two orthogonal transverse polarization states

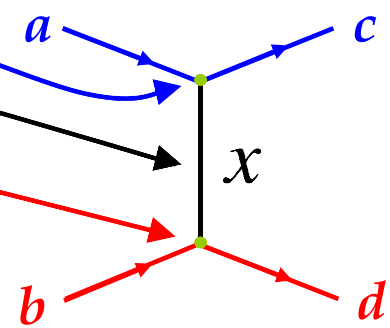
$$\varepsilon^{(1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \varepsilon^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Could equally have chosen circularly polarized states

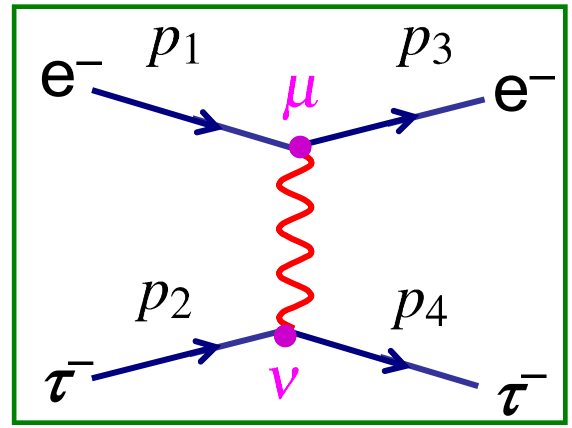
•Previously with the example of a simple spin-less interaction we had:

$$M = \langle \psi_c | V | \psi_a \rangle \frac{1}{q^2 - m_x^2} \langle \psi_d | V | \psi_b \rangle$$

$\langle \psi_c | V | \psi_a \rangle = g_a$ $\qquad$ $\langle \psi_d | V | \psi_b \rangle = g_b$



★In QED we could again go through the procedure of summing the time-orderings using Dirac spinors and the expression for $\hat{V}_D$. If we were to do this, remembering to sum over all photon polarizations, we would obtain:



$$M = \left[u_e^\dagger(p_3) q_e \gamma^0 \gamma^\mu u_e(p_1)\right] \sum_\lambda \frac{\varepsilon_\mu^\lambda (\varepsilon_\nu^\lambda)^*}{q^2} \left[u_\tau^\dagger(p_4) q_\tau \gamma^0 \gamma^\nu u_\tau(p_2)\right]$$

Interaction of e^- with photon

Massless photon propagator summing over polarizations

Interaction of τ^- with photon

• All the physics of QED is in the above expression !

- The sum over the polarizations of the **VIRTUAL** photon has to include longitudinal and scalar contributions, i.e. 4 polarisation states

$$\varepsilon^{(0)} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad \varepsilon^{(1)} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad \varepsilon^{(2)} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad \varepsilon^{(3)} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

and gives:

$$\sum_\lambda \varepsilon_\mu^\lambda (\varepsilon_\nu^\lambda)^* = -g_{\mu\nu}$$

This is not obvious – for the moment just take it on trust

and the invariant matrix element becomes:

$$M = [u_e^\dagger(p_3)q_e\gamma^0\gamma^\mu u_e(p_1)]\frac{-g_{\mu\nu}}{q^2}[u_\tau^\dagger(p_4)q_\tau\gamma^0\gamma^\nu u_\tau(p_2)]$$

- Using the definition of the adjoint spinor $\overline{\psi} = \psi^\dagger\gamma^0$

$$M = [\overline{u}_e(p_3)q_e\gamma^\mu u_e(p_1)]\frac{-g_{\mu\nu}}{q^2}[\overline{u}_\tau(p_4)q_\tau\gamma^\nu u_\tau(p_2)]$$

★ This is a remarkably simple expression ! It can be shown that $\overline{u}_1\gamma^\mu u_2$ transforms as a four vector. Writing

$$j_e^\mu = \overline{u}_e(p_3)\gamma^\mu u_e(p_1) \qquad j_\tau^\nu = \overline{u}_\tau(p_4)\gamma^\nu u_\tau(p_2)$$

$$M = -q_e q_\tau \frac{j_e \cdot j_\tau}{q^2}$$

showing that M is **Lorentz Invariant**

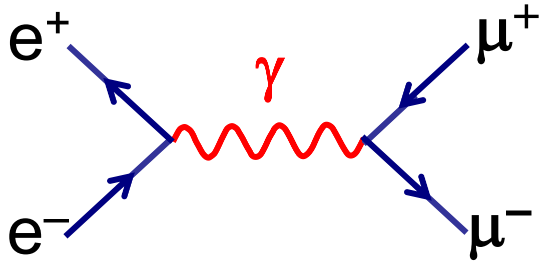
Feynman Rules for QED

- It should be remembered that the expression

$$M = [\overline{u}_e(p_3)q_e\gamma^\mu u_e(p_1)]\frac{-g_{\mu\nu}}{q^2}[\overline{u}_\tau(p_4)q_\tau\gamma^\nu u_\tau(p_2)]$$

hides a lot of complexity. We have summed over all possible time-orderings and summed over all polarization states of the virtual photon. If we are then presented with a new Feynman diagram **we don't want to go through the full calculation again….**

- Fortunately, this isn't necessary – can just write down matrix element using a set of simple rules…



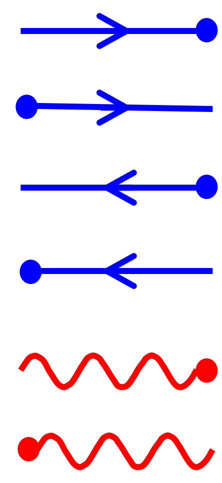
Basic Feynman Rules:

- Propagator factor for each internal line (i.e. each internal virtual particle)
- Dirac Spinor for each external line (i.e. each real incoming or outgoing particle)
- Vertex factor for each vertex

Basic Rules for QED

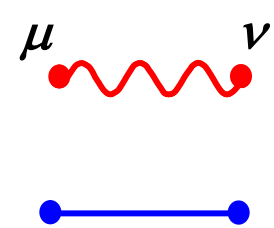
- **External Lines**

spin 1/2	incoming particle	$u(p)$	
	outgoing particle	$\bar{u}(p)$	
	incoming antiparticle	$\bar{v}(p)$	
	outgoing antiparticle	$v(p)$	
spin 1	incoming photon	$\varepsilon^{\mu}(p)$	
	outgoing photon	$\varepsilon^{\mu}(p)^{*}$	



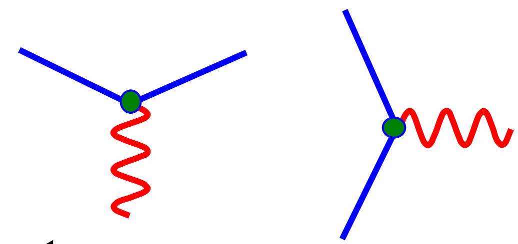
- **Internal Lines (propagators)**

spin 1	photon	$-\dfrac{ig_{\mu\nu}}{q^2}$	
spin 1/2	fermion	$\dfrac{i(\gamma^{\mu}q_{\mu}+m)}{q^2-m^2}$	



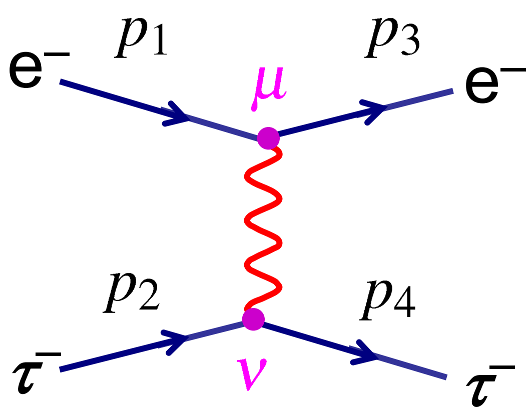
- **Vertex Factors**

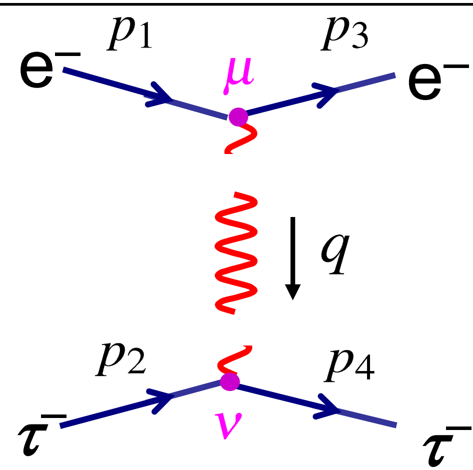
spin 1/2	fermion (charge $-\lvert e \rvert$)	$ie\gamma^{\mu}$



- **Matrix Element** $-iM$ = product of all factors

e.g.





$$\overline{u}_e(p_3)[ie\gamma^\mu]u_e(p_1)$$

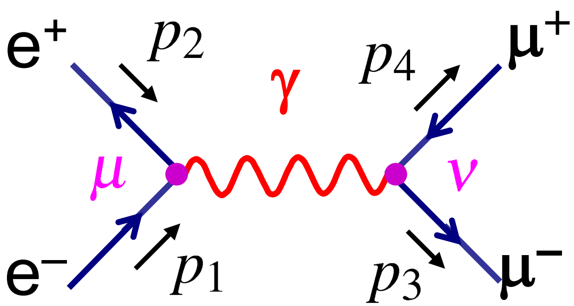
$$\frac{-ig_{\mu\nu}}{q^2}$$

$$\overline{u}_\tau(p_4)[ie\gamma^\nu]u_\tau(p_2)$$

$$-iM = [\overline{u}_e(p_3)ie\gamma^\mu u_e(p_1)]\frac{-ig_{\mu\nu}}{q^2}[\overline{u}_\tau(p_4)ie\gamma^\nu u_\tau(p_2)]$$

- Which is the same expression as we obtained previously

e.g.



$$-iM = [\overline{v}(p_2)ie\gamma^\mu u(p_1)]\frac{-ig_{\mu\nu}}{q^2}[\overline{u}(p_3)ie\gamma^\nu v(p_4)]$$

Note:

- At each vertex the adjoint spinor is written first
- Each vertex has a different index
- The $g_{\mu\nu}$ of the propagator connects the indices at the vertices

The Local Gauge Principle

★ All the interactions between fermions and spin-1 bosons in the SM are specified by the principle of LOCAL GAUGE INVARIANCE

★ To arrive at QED, require physics to be invariant under the local phase transformation of particle wave-functions

$$\psi \rightarrow \psi' = \psi e^{iq\chi(x)}$$

★ Note that the change of phase depends on the space-time coordinate: $\chi(t,\vec{x})$

- Under this transformation the Dirac Equation transforms as

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0 \quad \Rightarrow \quad i\gamma^\mu(\partial_\mu + iq\partial_\mu\chi)\psi - m\psi = 0$$

- To make "physics", i.e. the Dirac equation, invariant under this local phase transformation FORCED to introduce a massless gauge boson, A_μ .

\+ The Dirac equation has to be modified to include this new field:

$$i\gamma^\mu(\partial_\mu - qA_\mu)\psi - m\psi = 0$$

- The modified Dirac equation is invariant under local phase transformations if:

$$A_\mu \rightarrow A'_\mu = A_\mu - \partial_\mu\chi$$

Gauge Invariance

★ For physics to remain unchanged – must have **GAUGE INVARIANCE** of the new field, i.e. physical predictions unchanged for $A_\mu \rightarrow A'_\mu = A_\mu - \partial_\mu \chi$

★ Hence the principle of invariance under local phase transformations completely specifies the interaction between a fermion and the gauge boson (i.e. photon):

$$i\gamma^\mu(\partial_\mu \psi - qA_\mu)\psi - m\psi = 0$$

⇨ interaction vertex: $i\gamma^\mu q A_\mu$

⇨ **QED !**

★ The local phase transformation of QED is a unitary **U(1)** transformation

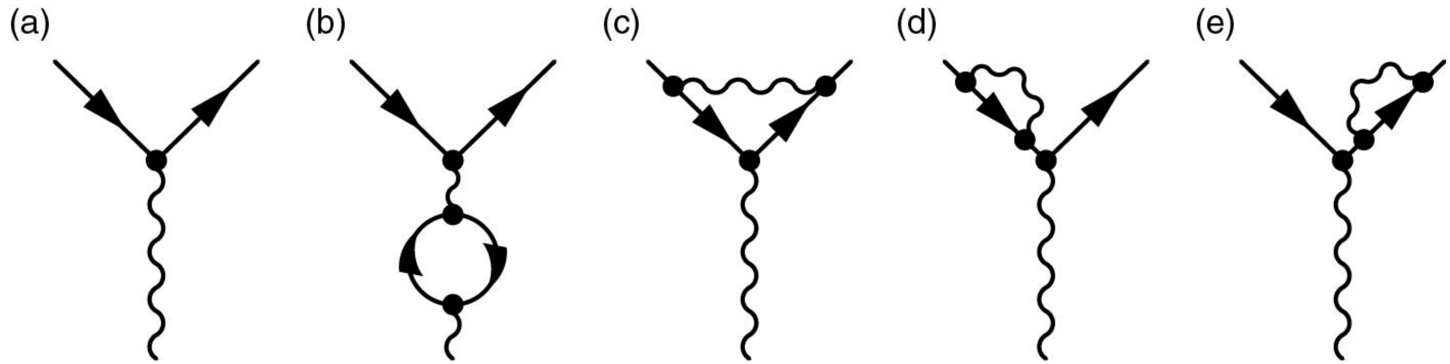
$$\psi \rightarrow \psi' = \hat{U}\psi \quad \text{i.e.} \quad \psi \rightarrow \psi' = \psi e^{iq\chi(x)} \quad \text{with} \quad U^\dagger U = 1$$

★ **LOCAL GAUGE INVARIANCE** lies at the heart of the SM – all forces are associated with a specific **local gauge theory**

We will come back to Gauge Invariance in the last lecture

Renormalisation I

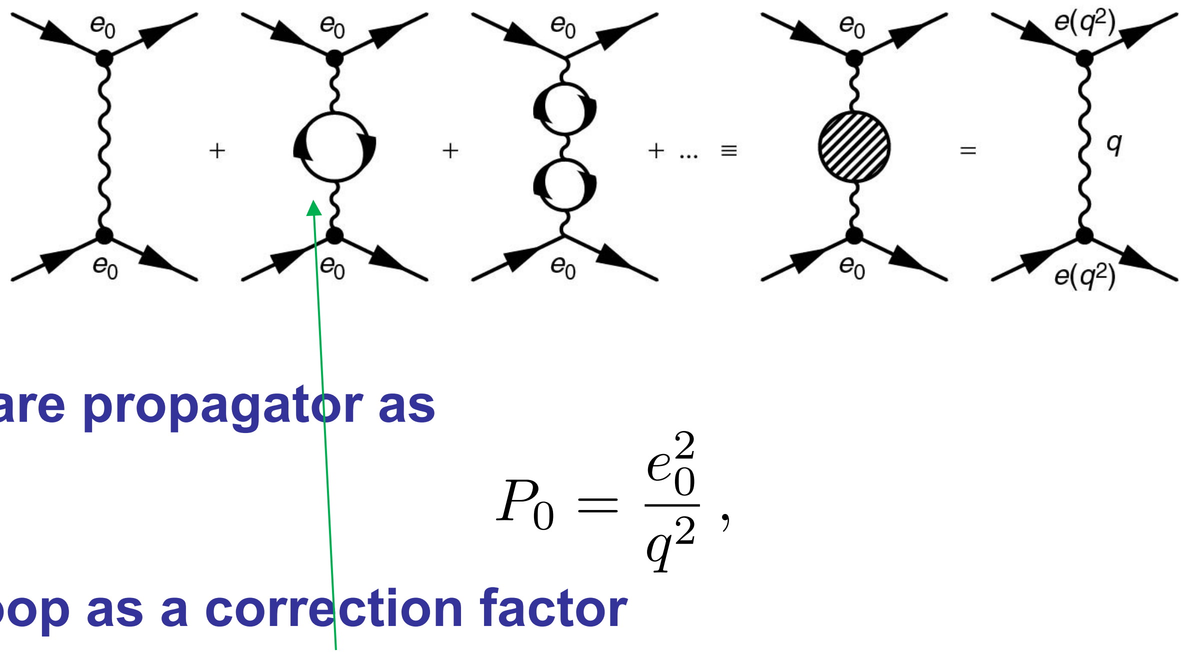
- ★ In principle, we could now use the Feynman rules for QED to write down the matrix element for any QED process
- ★ Also includes corrections to the propagator (loops) and corrections at the interaction vertices



- ★ The loops in the propagator involve integrals over the four-momenta in fermion loops, resulting in infinities that can be absorbed into the definition of the electron charge
- ★ At first sight the vertex corrections might seem even more problematic, since they will depend on the mass of the fermion in questions. Fortunately, in a local gauge theory diagrams such as c), d) and e) exactly cancel - known as a Ward identity.

Renormalisation II

★ **But we still need to deal with the propagator…**



★ **Write the bare propagator as**

$$P_0 = \frac{e_0^2}{q^2}\,,$$

★ **and each loop as a correction factor**

$$P = P_0 + P_0\,\pi(q^2)\,P_0 + P_0\,\pi(q^2)\,P_0\,\pi(q^2)\,P_0 + \;\ldots\,,$$

★ **This is just a geometric series**

$$P = P_0\frac{1}{1-\pi(q^2)\,P_0} = \frac{1}{1-e_0^2\,\Pi(q^2)}$$

Where $\Pi(q^2) = \pi(q^2)/q^2$ **is the one loop self-energy correction**

Renormalisation III – almost there

★ **Since it is an experimental fact that cross sections are finite:**

$$P \equiv \frac{e^2(q^2)}{q^2} = \frac{e_0^2}{q^2}\frac{1}{1 - e_0^2\,\Pi(q^2)}$$

is finite. If we know the physical electron charge at some scale $q^2 = \mu^2$

$$e_0^2 = \frac{e^2(\mu^2)}{1 + e^2(\mu^2)\Pi(\mu^2)}$$

substituting back into the top equation

$$e^2(q^2) = \frac{e^2(\mu^2)}{1 - e^2(\mu^2)\cdot[\Pi(q^2) - \Pi(\mu^2)]}$$

★ **It can be shown that**

$$\Pi(q^2) - \Pi(\mu^2) \approx \frac{1}{12\pi^2}\ln\left(\frac{q^2}{\mu^2}\right)$$

where the difference of two divergent terms is finite

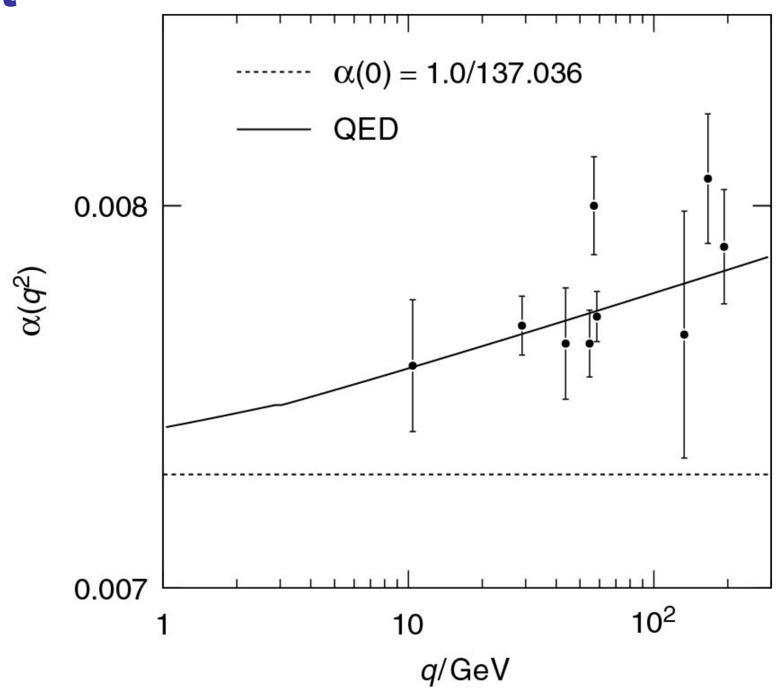
Running Coupling

★ **Consequently**

$$\alpha(q^2) = e^2(q^2)/4\pi = \frac{\alpha(\mu^2)}{1 - \alpha(\mu^2)\frac{1}{3\pi}\ln\left(\frac{q^2}{\mu^2}\right)}$$

and thus the observed strength of the coupling runs with q^2

★ **It can be shown that**

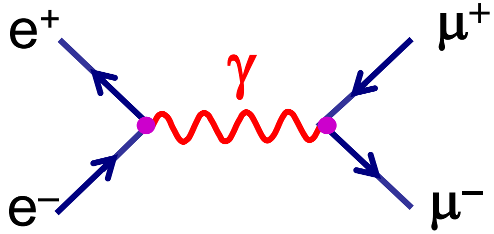


QED Calculations

- **How to calculate a cross section using QED (e.g. $e^+e^- \to \mu^+\mu^-$):**

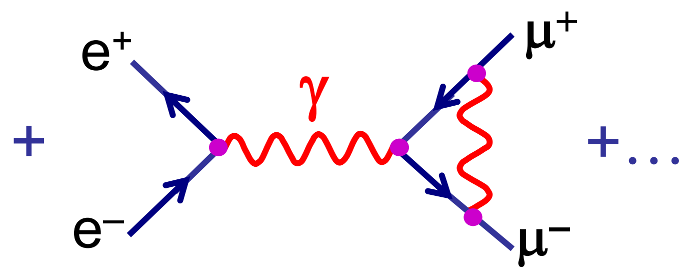
❶ **Draw all possible Feynman Diagrams**

- **For $e^+e^- \to \mu^+\mu^-$ there is just one lowest order diagram**



$$M \propto e^2 \propto \alpha_{em}$$

+ many second order diagrams + …



$$M \propto e^4 \propto \alpha_{em}^2$$

❷ **For each diagram calculate the matrix element using Feynman rules**

❸ **Sum the individual matrix elements (i.e. sum the amplitudes)**

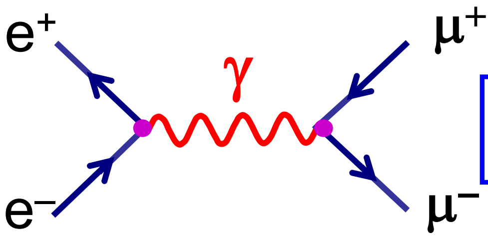
$$M_{fi} = M_1 + M_2 + M_3 + \ldots.$$

- **Note: summing amplitudes therefore different diagrams for the same final state can interfere either positively or negatively!**

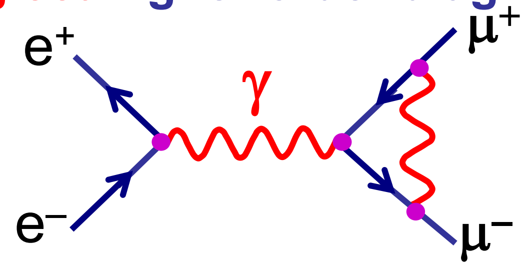
and then square $|M_{fi}|^2 = (M_1 + M_2 + M_3 + ...)(M_1^* + M_2^* + M_3^* + ...)$

➡ this gives the full perturbation expansion in α_{em}

- For QED $\alpha_{em} \sim 1/137$ the lowest order diagram dominates and for most purposes it is sufficient to neglect higher order diagrams.



$M^2 \propto \alpha_{em}^2$



$M^2 \propto \alpha_{em}^4$

❹ Calculate decay rate/cross section using relevant kinematic formula

- e.g. for a decay

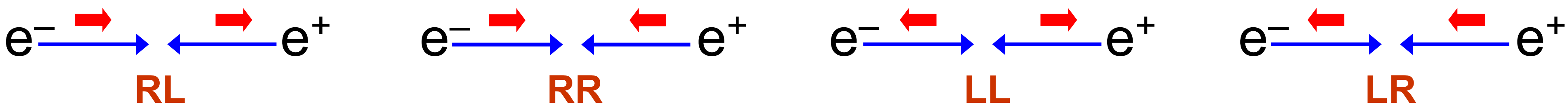
$$\Gamma = \frac{p^*}{32\pi^2 m_a^2} \int |M_{fi}|^2 \mathrm{d}\Omega$$

- For scattering in the centre-of-mass frame

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega^*} = \frac{1}{64\pi^2 s} \frac{|\vec{p}_f^{\,*}|}{|\vec{p}_i^{\,*}|} |M_{fi}|^2$$

Spin in QED (e^+e^- Annihilation)

- In general the electron and positron will not be polarized, i.e. there will be equal numbers of positive and negative helicity states
- There are four possible combinations of spins in the **initial state** !



- Similarly there are four possible helicity combinations in the final state
- In total there are **16** combinations e.g. RL→RR, RL→RL, ….
- To account for these states we need to **sum** over **all 16 possible helicity** combinations and then **average** over the number of **initial helicity states**:

$$\langle|M|^2\rangle = \frac{1}{4}\sum_{\text{spins}}|M_i|^2 = \frac{1}{4}\left(|M_{LL\to LL}|^2 + |M_{LL\to LR}|^2 + \ldots\right)$$

★ i.e. need to evaluate:

$$M = -\frac{e^2}{s} j_e . j_\mu$$

for all 16 helicity combinations ! **But there are tricks…**

★ Fortunately, in the limit $E \gg m_\mu$ only 4 helicity combinations give non-zero matrix elements – we will see that this is an important feature of QED/QCD

CHIRALITY

- The helicity eigenstates for a particle/anti-particle for $E \gg m$ are:

$$u_\uparrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix};\ u_\downarrow = \sqrt{E}\begin{pmatrix} -s \\ ce^{i\phi} \\ s \\ -ce^{i\phi} \end{pmatrix};\ v_\uparrow = \sqrt{E}\begin{pmatrix} s \\ -ce^{i\phi} \\ -s \\ ce^{i\phi} \end{pmatrix};\ v_\downarrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}$$

where $s = \sin\frac{\theta}{2};\quad c = \cos\frac{\theta}{2}$

- Define the matrix

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

- <u>In the limit</u> $E \gg m$ the helicity states are also eigenstates of γ^5

$$\gamma^5 u_\uparrow = +u_\uparrow;\ \ \gamma^5 u_\downarrow = -u_\downarrow;\ \ \gamma^5 v_\uparrow = -v_\uparrow;\ \ \gamma^5 v_\downarrow = +v_\downarrow$$

★ <u>In general</u>, define the eigenstates of γ^5 as LEFT and RIGHT HANDED <u>CHIRAL</u> states $u_R;\quad u_L;\quad v_R;\quad v_L$

i.e. $\gamma^5 u_R = +u_R;\ \ \gamma^5 u_L = -u_L;\ \ \gamma^5 v_R = -v_R;\ \ \gamma^5 v_L = +v_L$

- In the LIMIT $E \gg m$ (and ONLY IN THIS LIMIT):

$$u_R \equiv u_\uparrow;\quad u_L \equiv u_\downarrow;\quad v_R \equiv v_\uparrow;\quad v_L \equiv v_\downarrow$$

★This is a subtle but important point: in general the HELICITY and CHIRAL eigenstates are not the same. It is only in the ultra-relativistic limit that the chiral eigenstates correspond to the helicity eigenstates.

★Chirality is an import concept in the structure of QED, and any interaction of the form $\bar{u}\gamma^{\nu}u$

- In general, the eigenstates of the chirality operator are:

$$\gamma^5 u_R = +u_R; \;\; \gamma^5 u_L = -u_L; \;\; \gamma^5 v_R = -v_R; \;\; \gamma^5 v_L = +v_L$$

- Define the projection operators:

$$P_R = \tfrac{1}{2}(1+\gamma^5); \qquad P_L = \tfrac{1}{2}(1-\gamma^5)$$

- The projection operators, project out the chiral eigenstates

$$P_R u_R = u_R; \quad P_R u_L = 0; \quad P_L u_R = 0; \quad P_L u_L = u_L$$
$$P_R v_R = 0; \quad P_R v_L = v_L; \quad P_L v_R = v_R; \quad P_L v_L = 0$$

- Note P_R projects out right-handed particle states and left-handed anti-particle states
- We can then write any spinor in terms of it left and right-handed chiral components:

$$\psi = \psi_R + \psi_L = \tfrac{1}{2}(1+\gamma^5)\psi + \tfrac{1}{2}(1-\gamma^5)\psi$$

Chirality in QED

- In QED the basic interaction between a fermion and photon is:

$$ie\overline{\psi}\gamma^{\mu}\phi$$

- Can decompose the spinors in terms of Left and Right-handed chiral components:

$$\begin{aligned} ie\overline{\psi}\gamma^{\mu}\phi &= ie(\overline{\psi}_L + \overline{\psi}_R)\gamma^{\mu}(\phi_R + \phi_L) \\ &= ie(\overline{\psi}_R\gamma^{\mu}\phi_R + \overline{\psi}_R\gamma^{\mu}\phi_L + \overline{\psi}_L\gamma^{\mu}\phi_R + \overline{\psi}_L\gamma^{\mu}\phi_L) \end{aligned}$$

- Using the properties of γ^5

$$(\gamma^5)^2 = 1; \qquad \gamma^{5\dagger} = \gamma^5; \qquad \gamma^5\gamma^{\mu} = -\gamma^{\mu}\gamma^5$$

it is straightforward to show

$$\overline{\psi}_R\gamma^{\mu}\phi_L = 0; \qquad \overline{\psi}_L\gamma^{\mu}\phi_R = 0$$

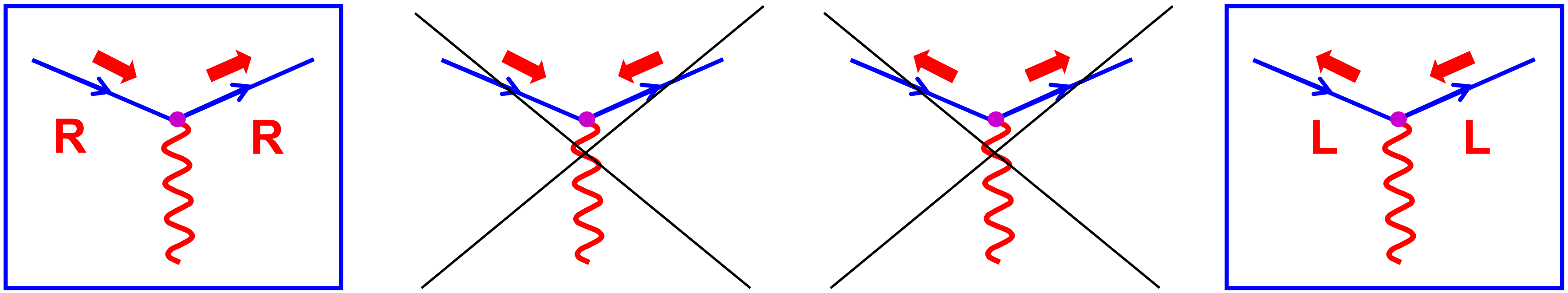
★ Hence only certain combinations of chiral eigenstates contribute to the interaction. This statement is **ALWAYS** true.

- For $E \gg m$, the chiral and helicity eigenstates are equivalent. This implies that for $E \gg m$ only certain helicity combinations contribute to the QED vertex ! This is why previously we found that for two of the four helicity combinations for the muon current were zero

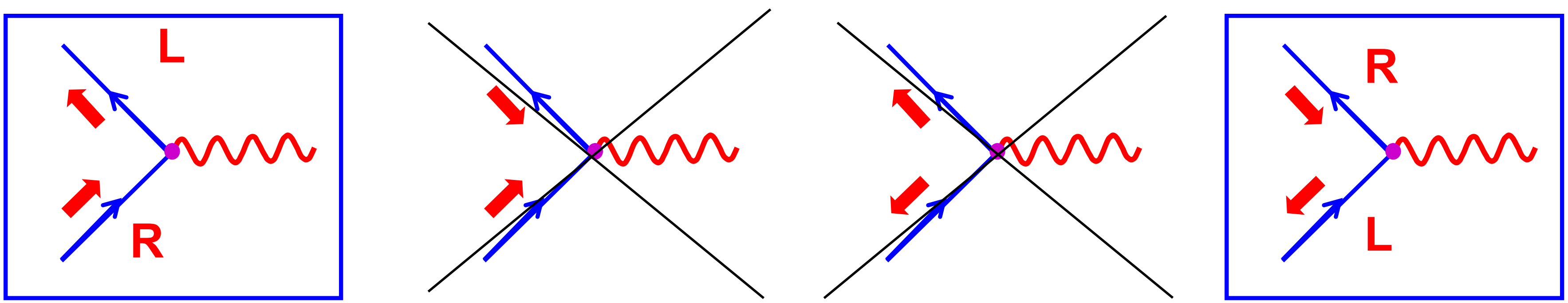
Allowed QED Helicity Combinations

- In the ultra-relativistic limit the helicity eigenstates ≡ chiral eigenstates
- In this limit, the only non-zero helicity combinations in QED are:

Scattering: "Helicity (really chirality) conservation"

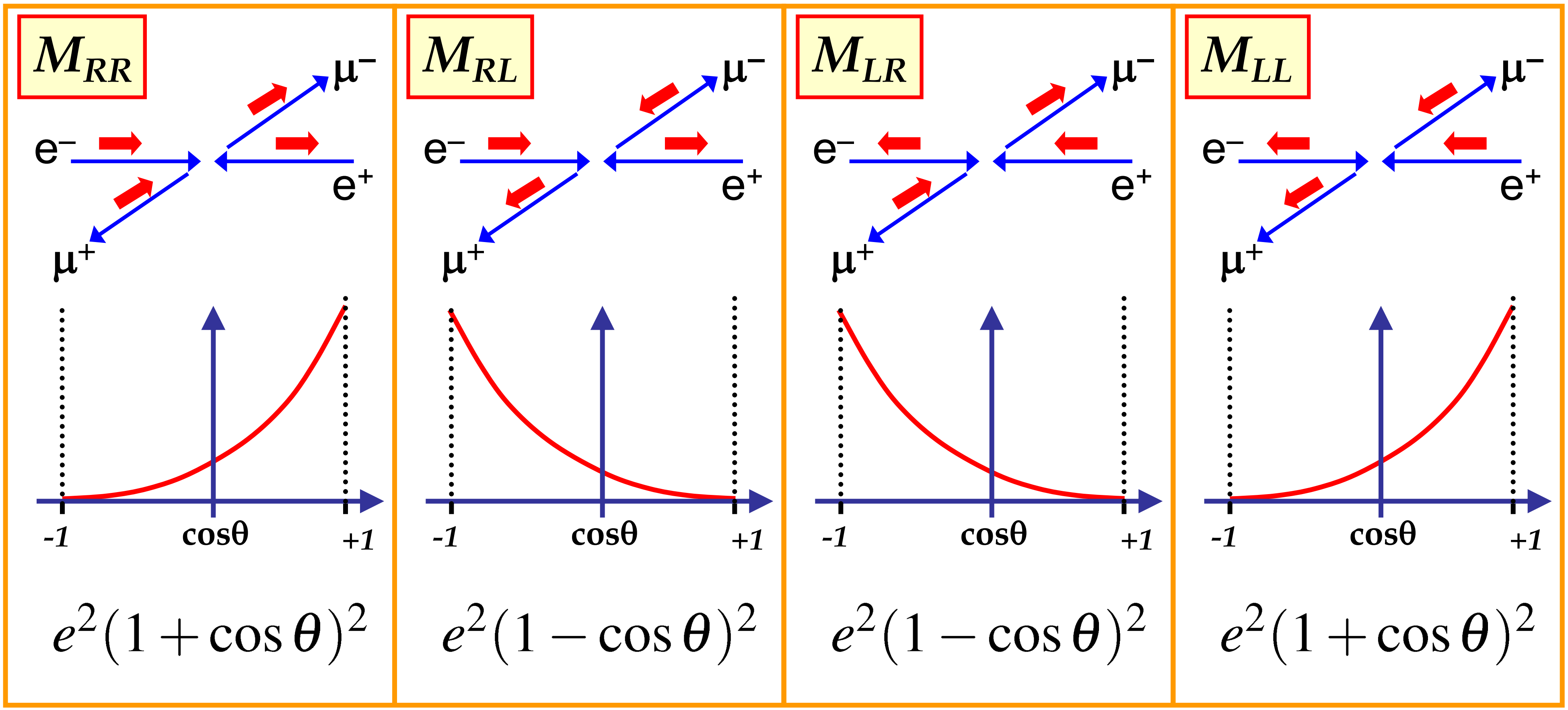


Annihilation:



It is easy to show

$$|M_{RR}|^2 = |M_{LL}|^2 = (4\pi\alpha)^2(1+\cos\theta)^2$$
$$|M_{RL}|^2 = |M_{LR}|^2 = (4\pi\alpha)^2(1-\cos\theta)^2$$



- **Assuming that the incoming electrons and positrons are unpolarized, all 4 possible initial helicity states are equally likely.**

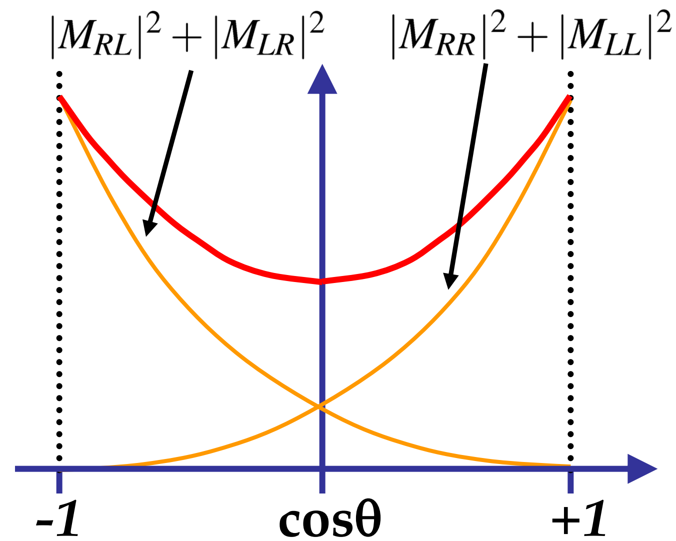
Differential Cross Section

- The cross section is obtained by averaging over the initial spin states and summing over the final spin states:

$$\frac{d\sigma}{d\Omega} = \frac{1}{4}\times\frac{1}{64\pi^2 s}\left(|M_{RR}|^2+|M_{RL}|^2+|M_{LR}|^2+|M_{LL}^2|\right)$$

$$= \frac{(4\pi\alpha)^2}{256\pi^2 s}\left(2(1+\cos\theta)^2+2(1-\cos\theta)^2\right)$$

$$\Rightarrow \quad \boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}\left(1+\cos^2\theta\right)}$$

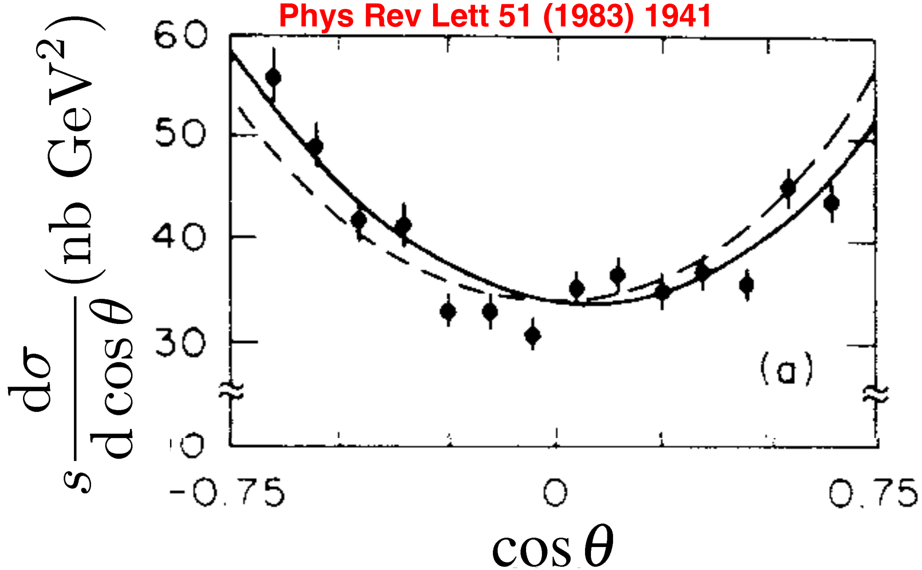


Example:

$e^+e^- \to \mu^+\mu^-$

$\sqrt{s} = 29$ GeV

Mark II Expt., M.E.Levi et al., Phys Rev Lett 51 (1983) 1941



- - - - - pure QED, $O(\alpha^3)$

—— QED plus Z contribution

Angular distribution becomes slightly asymmetric in higher order QED or when Z contribution is included

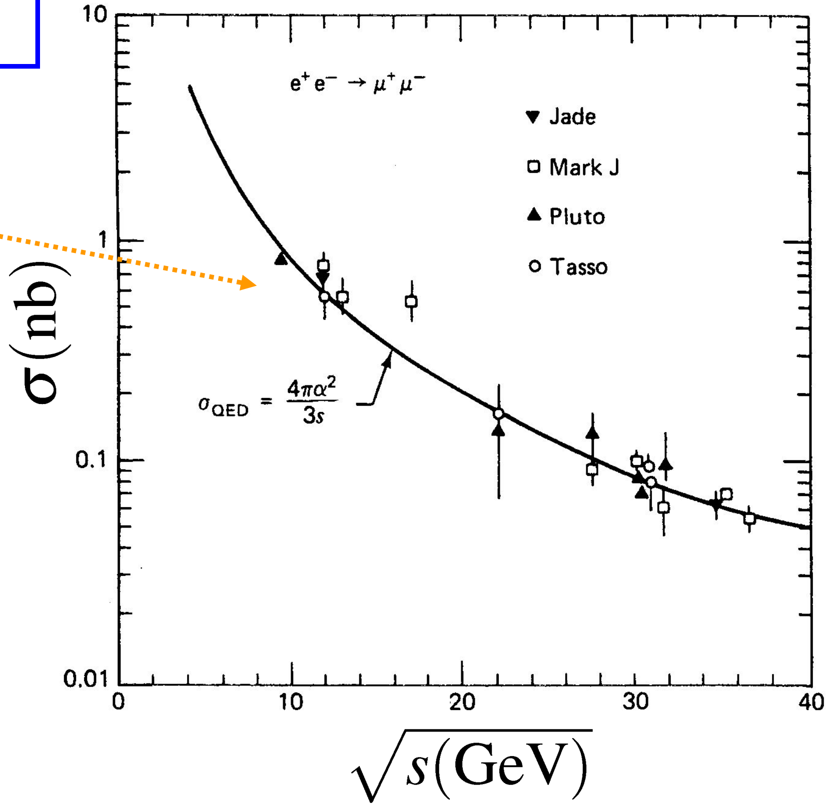
• The total cross section is obtained by integrating over $\theta,\ \phi$ using

$$\int (1+\cos^2\theta)\mathrm{d}\Omega = 2\pi\int_{-1}^{+1}(1+\cos^2\theta)\mathrm{d}\cos\theta = \frac{16\pi}{3}$$

giving the **QED** total cross-section for the process $e^+e^- \to \mu^+\mu^-$

$$\sigma = \frac{4\pi\alpha^2}{3s}$$

★ Lowest order cross section calculation provides a good description of the data !



This is an impressive result. From first principles we can arrive at an expression for the electron-positron annihilation cross section which is good to 1%

Summary

★ Interaction by particle exchange naturally gives rise to **Lorentz Invariant Matrix Element** of the form

★ Derived the basic interaction in **QED** taking into account the spins of the fermions and polarization of the virtual photons

★ The interaction vertex of QED corresponds to a vector interaction, “derived” from a classical electrodynamics approach and from local gauge invariance