# Intro to Effective Field Theory (EFT)



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We are familiar with the idea that one does not need to know the detailed microscopic properties of a system to accurately model it.

e.g. Friction, Thermodynamics, ...

QM e.g. Hydrogen: treat proton as pointlike is good approx since $r_B = 10^{-7}$ cm while $r_p \sim 10^{-13}$ cm

ratio of scales $r_p / r_B \sim 10^{-6}$

EFT makes these types of approximations systemati

We will use toy scalar theory to illustrate how to generate sys expansion $E/M$

$E$ = low energy of experiments + $M$ = heavy mass scale

Termanology

relevant couplings: positive mass dim ($m^2 \phi^2$)

marginal couplings: zero mass dim ($\lambda \phi^4$)

irrelevant couplings: negative mass dim ($\frac{1}{M^2} \phi^6$)

fundamental theory: UV

Effective theory: IR

Two real scalar fields $\varphi$ & $\eta$ w/ $m_\varphi \ll m_\eta$

Want effective description for $E \ll m_\eta$

Impose a $\mathbb{Z}_2$ symmetry $\varphi \to -\varphi$ & $\eta \to +\eta$

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m_\varphi^2\varphi^2 + \frac{1}{2}(\partial\eta)^2 - \frac{1}{2}m_\eta^2\eta^2 - V$$

w/ $V = \frac{\lambda}{4!}\varphi^4 + \frac{g}{2}\varphi^2\eta + \frac{g'}{3!}\eta^3 + \frac{\lambda'}{4}\varphi^2\eta^2 + \frac{\lambda''}{4!}\eta^4$

Path integral to compute $\varphi$ correlation functions:

$$Z[J] = \int \mathcal{D}\varphi\, \mathcal{D}\eta\, \exp\left[i S[\varphi,\eta] + \int J\varphi\right]$$

Note only include source for $\varphi$, since we do not have energy to produce $\eta$

Path integral for EFT

$$Z_{eff}[J] = \int \mathcal{D}\varphi\, \exp\left[i S_{eff}[\varphi] + \int J\varphi\right]$$

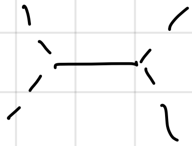
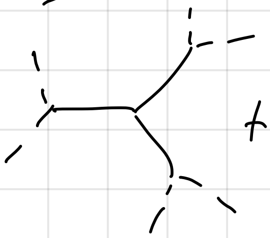
with $\exp\left[i S_{eff}[\varphi]\right] = \int \mathcal{D}\eta\, e^{iS[\varphi,\eta]}$

This could be useful <u>if</u> $S_{eff}$ is "local"

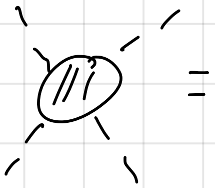
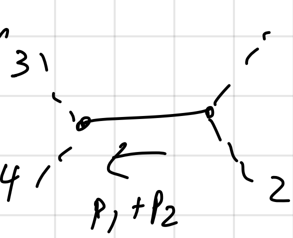
i.e., $\mathcal{L}_{eff}$ is polynomial in fields and derivatives of fields

Call procedure for deriving $S_{eff}$ "integrating out" the heavy field $\eta$.

Let's compute a few terms

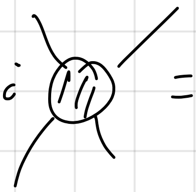
$\Rightarrow i S_{eff}[\varphi] = i S'_{\varphi} +$  $+$  $+ \ldots$

Focus on 4-point:

 $=$  $+ \Big($  $+ \; t + u \Big)$

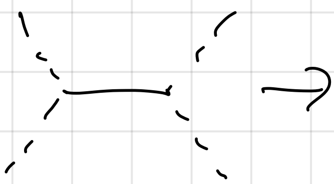
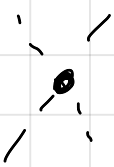
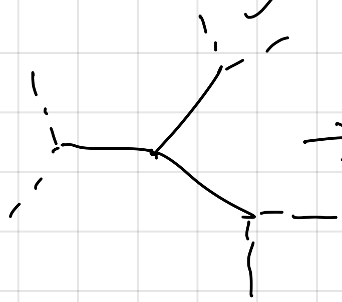
$$= -i\lambda - i g^2 \left[ \frac{1}{s - m_\eta^2} + \frac{1}{t - m_\eta^2} + \frac{1}{u + m_\eta^2} \right]$$

Assume $E << m_\eta$ : $\dfrac{1}{p^2 - m_\eta^2 + i\varepsilon} = -\dfrac{1}{m_\eta^2} - \dfrac{p^2}{m_\eta^4} + \ldots$

$\Rightarrow -i$  $= \lambda + \dfrac{g^2}{m_\eta^2} + \dfrac{g^2}{m_\eta^4}(s+t+u) + \ldots$

$$\Rightarrow \mathcal{L}^{(4)}_{eff} = -\frac{1}{4!}\left(\lambda - \frac{3g^2}{m_\eta^2}\right)\varphi^4 - \frac{g^2}{8 m_\eta^4}\varphi^2 \Box \varphi^2 + \ldots$$

What are we doing? Shrinking heavy line to point:

 $\rightarrow$   $\rightarrow$ 

$\Rightarrow$ local! etc.

## Power Counting



Integrating out $\eta$ generates an $\infty$ # of terms

Can we organize them?

Assume: Fundamental params

$g, g' \sim m_\eta \equiv M$ & $\lambda, \lambda', \lambda'' \sim \mathcal{O}(1)$

$$\Rightarrow \mathcal{L}_{eff} \sim \sum_{n,m} \frac{1}{M^{n+m-4}} \partial^n \phi^m$$ w/ only even powers due to Symmetries

eg at $\mathcal{O}(1/M^2)$: $\phi^6, \partial^2\phi^4, \partial^4\phi^2$

Truncating to $\mathcal{O}(1/M^2) \Rightarrow$ computing amplitudes to accuracy $E^2/M^2$.

$\Rightarrow$ Power counting determines accuracy of calculation

Integrate out field using equations of motion

"Semiclassical expansion": evaluate action on a solution to EOM

$$S_{eff}[\phi] = S[\phi, \eta_{cl}] + \mathcal{O}(\hbar)$$

w/ $$\left.\frac{\delta S[\phi, \eta]}{\delta \eta}\right|_{\eta = \eta_{cl}} = 0$$

EOM

$$\Rightarrow \Box \eta + m_\eta^2 \eta + \frac{g}{2}\phi^2 + \frac{g'}{2}\eta^2 + \frac{\lambda'}{2}\eta\phi^2 + \frac{\lambda''}{6}\eta^3 = 0$$

Solve itteratively: $\eta_{cl}^{(1)} = -\frac{g}{2m_\eta^2}\varphi^2 \sim \mathcal{O}\left(\frac{1}{M}\right)$

$$\Rightarrow \eta_{cl} = \underbrace{-\frac{g}{2m_\eta^2}\varphi^2}_{\mathcal{O}(\frac{1}{M})} - \underbrace{\frac{1}{m_\eta^2}\Box\eta_{cl}}_{\mathcal{O}(1/M^3)} - \underbrace{\frac{\lambda'}{2m_\eta^2}\eta_{cl}\varphi^2}_{\mathcal{O}(1/M^3)} - \underbrace{\frac{g'}{2m_\eta^2}\eta_{cl}^2}_{\mathcal{O}(1/M^3)} - \underbrace{\frac{\lambda''}{6m_\eta^2}\eta_{cl}^3}_{\mathcal{O}(1/M^5)}$$

$\Rightarrow$ To go to $\mathcal{O}(1/M^3)$ sub $\eta_{cl}^{(1)}$ into EOM

keeping terms up to $\mathcal{O}(1/M^3)$

$$\Rightarrow \eta_{cl}^{(3)} = \frac{-g}{2m_\eta^2}\varphi^2 + \frac{g}{2m_\eta^4}\Box\varphi^2 + \left(\frac{g\lambda'}{4m_\eta^4} + \frac{g^2 g'}{4m_\eta^6}\right)\varphi^4 + \mathcal{O}\left(\frac{1}{M^5}\right)$$

Sub into $\mathcal{L}_{UV}$

$$\Rightarrow \mathcal{L}_{eff} = \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m_\varphi^2\varphi^2 - \frac{1}{4!}\left(\lambda - \frac{3g^2}{m_\eta^2}\right)\varphi^4$$

$$- \frac{1}{6!}\left(\frac{45\lambda' g^2}{m_\eta^4} - \frac{15 g' g^3}{m_\eta^6}\right)\varphi^6$$

$$+ \frac{g^2}{8m_\eta^4}(\partial_\mu\varphi^2)(\partial^\mu\varphi^2) + \mathcal{O}(1/M^4)$$

which agrees w/ previous diagramatic approach upon integration by parts of $(\partial\varphi^2)^2$ term.

# Simplifying $\mathcal{L}_{eff}$



Two stradigies: (1) integration by parts
(2) field redefinitions

Ex: Classify all possible terms of the form $\partial^2\varphi^n$

Using $\partial_\mu \varphi^r = r\varphi^{r-1}\partial_\mu\varphi$ rewrite operator so each derivative acts on single field.

$\Rightarrow$ Most general operator is linear combo of

$$\varphi^{n-1}\Box\varphi \quad \text{and} \quad \varphi^{n-2}\partial^\mu\varphi\,\partial_\mu\varphi$$

Then
$$\varphi^{n-2}\partial^\mu\varphi\,\partial_\mu\varphi = \frac{1}{n-1}\partial^\mu\varphi^{n-1}\,\partial_\mu\varphi$$
$$= -\frac{1}{n-1}\varphi^{n-1}\Box\varphi + \text{total der}$$

$\Rightarrow$ Only single independent operator for each $n$.

Ex: Field redefinitions (aka "using the equations of motion")

Let $\varphi \to \varphi + f(\varphi)$ and expand in powers of $f(\varphi)$:

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - V \to \frac{1}{2}(\partial\varphi)^2 - V - f(\varphi)\underbrace{\left[\Box\varphi + V'\right]}_{\text{EOM}} + \mathcal{O}(f^2)$$

Ex: Let's simplify our previous example



$$\varphi \to \varphi + c\frac{g^2}{m_\eta^4}\varphi^3$$

$$\Rightarrow \mathcal{L}_{eff} \to \mathcal{L}_{eff} + \frac{cg^2}{m_\eta^4}\varphi^3\left[\Box\varphi + m_\varphi^2\varphi + \frac{\lambda}{3!}\varphi^3\right] + \mathcal{O}(\Lambda^{-4})$$

Taking $c = \frac{1}{2} \Rightarrow$

$$\mathcal{L}_{eff} \to \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m_\varphi^2\varphi^2 - \frac{1}{4!}\Big(\lambda \underbrace{- \frac{3g^2}{m_\eta^2}}_{\mathcal{O}(1)} \underbrace{- \frac{6g^2m_\varphi^2}{m_\eta^4}}_{\mathcal{O}(1/\Lambda^2)}\Big)\varphi^4$$

$$+ \frac{1}{6!}\left[\frac{g^2(45\lambda' - 60\lambda)}{m_\eta^4} - \frac{15g'g^3}{m_\eta^6}\right]\varphi^6 + \mathcal{O}(1/\Lambda^4)$$

We have eliminated the $\partial^2\varphi^4$ term!

$\Rightarrow$ All indirect effects from $\eta$ can be modeled by modified $\varphi^4$ and $\varphi^6$ terms up to $\mathcal{O}(E^2/\Lambda^2)$

This justifies using the classical EOMs to rewrite the $\mathcal{L}$ into a more convenient form.

# Universality

Different UV theories can yield same IR theory.

Call this "universality".

Ex: Let's add $N$ heavy fields $\eta_i$

w/ same $Z_2$ sym $\varphi \to -\varphi$ and $\eta_i \to \eta_i$

$$\mathcal{L} = \mathcal{L}_{kin} - V$$

$$V = \frac{\lambda}{4!}\varphi^4 + \frac{g_i}{2}\varphi^2\eta_i + \frac{g'_{ijk}}{3!}\eta_i\eta_j\eta_k + \frac{\lambda'_{ij}}{4}\varphi^2\eta_i\eta_j + \frac{\lambda''_{ijkl}}{4!}\eta_i\eta_j\eta_k\eta_l$$

Power counting: $M \sim m_{\eta_i} \sim g_i \sim g'_i \gg m_\varphi$

$$\lambda \sim \lambda'_{ij} \sim \lambda''_{ijkl} \sim \mathcal{O}(1)$$

Claim: $$\mathcal{L}_{eft} = \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m_\varphi^2\varphi^2 - \frac{\lambda_{4\,eff}}{4!}\varphi^4 - \frac{\lambda_{6\,eff}}{6!\,M^2}\varphi^6 + \mathcal{O}(1/M^4)$$

w/ $\lambda_{4\,eff}$ + $\lambda_{6\,eff} \sim 1$

and we used int by parts + EOM to elliminate $\partial^2\varphi^4$ term

Using EOMs: $$\eta_{cl\,i} = \frac{-g_i}{2m_{\eta_i}^2}\varphi^2 + \mathcal{O}(1/M^3)$$

$$\Rightarrow \lambda_{4\,eff} = \lambda - \sum_i \frac{3g_i^2}{m_{\eta_i}^2} + \mathcal{O}(1/M^2)$$

Note "decoupling" when $m_{\eta_i} \gg m_{\eta_j}$

# Bottom Up EFT



Leading terms in $\mathcal{L}_{eff}$ consist of all relevant and marginal intersections that are compatible w/ symmetries that are inherited from fundamental theory.

Impact of higher dimension operators are supressed by powers of $E/M$

Ex: Standard Model

$$\Delta \mathcal{L}_{eff} \sim \frac{1}{M}(LH)^2 \xrightarrow[\langle H\rangle = v]{} m_\nu = \frac{v^2}{M}$$

$\Rightarrow$ "explain" small neutrino masses.

Ex: Baryon + Lepton number are accidental symmetries $\Rightarrow$ proton decay via higher dimension operators

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Interpreting irrelevant interactions

$$\mathcal{L}_{eff} = \frac{\lambda_6}{6!}\varphi^6 \qquad w/ \ [\lambda_6] = -2$$

Assume more fundamental theory w/ scale $M$ and $\mathcal{O}(1)$ couplings

Then $\lambda_6^{-1/2} \overset{?}{\sim} M$

From our example, we had

$$\lambda_6 = \frac{g^2(45\lambda' - 60\lambda)}{m_\eta^4} - \frac{15 g' g^2}{m_\eta^6}$$

So rule of thumb holds for $g \sim g' \sim m_\eta \sim M$
and $\lambda \sim \lambda' \sim 1$

But could have taken couplings small
$\Rightarrow \lambda_6^{-1/2} \gg m_\eta$

Could we have taken couplings large?
$\Rightarrow$ breakdown of pert theory

Can formalize as violating partial wave unitarity to find $M \lesssim 100\, \lambda_6^{-1/2}$

$\Rightarrow$ If we observe higher dim op
$\Rightarrow$ upper bound on new physics scale

Bottom up marginal + relevant Couplings



- marginal: dimensionless ⇒ no info about heavy new physics scale (loops will induce logarithmic sensitivity)

- relevant couplings:

$$\mathcal{L}_{eff} = -\frac{\lambda_3}{3!}\varphi^3$$

Naively if $\lambda_3 \sim M \Rightarrow$ dim analysis $\Rightarrow \mathcal{M} \sim \frac{\lambda_3}{E} \sim \frac{M}{E} \gg 1$

⇒ EFT does not make sense

Must have $\lambda_3 \ll M$. Can be enforced by a symmetry (e.g. $\varphi \to -\varphi$) that is broken by small coupling (spurion).

We will revisit this.

- Mass: $\mathcal{L}_{eff} = -\frac{1}{2} m_\varphi^2 \varphi^2$

Loops will shift this by $\sim \frac{1}{16\pi^2} M^2$

⇒ Hierarchy problem.

Can use symmetry to solve this problem: SUSY or "shift symmetry" $\varphi \to \varphi + c$

Expanded perturbation theory: 

Now we are doing dual expansion

1) loop ($\hbar$) expansion

2) $E/M$ expansion

Clearly distinguished at tree, but

this becomes much more subtle at loop level.

EFTs and the SM:

- $E \ll m_e$: Euler-Heisenberg for photons
- $E \ll \Lambda_{QCD}$: Chiral $\mathcal{L}$ for light mesons
- $E \ll m_W$: Fermi theory for quarks and leptons
- $E \ll \Lambda_{\text{new physics}}$: SMEFT (w/ $\Lambda_{np} \gg v$)

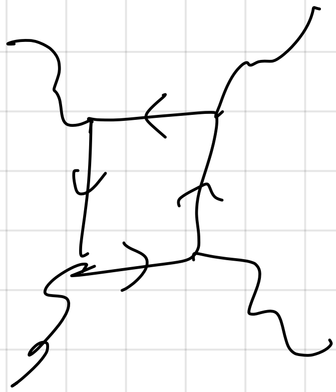
  HEFT (w/ $\Lambda_{NP} \sim v$)

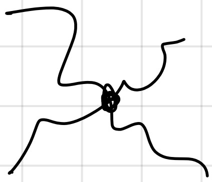
EFTs w/ kinematic restrictions

- HQET: $m_Q \gg \Lambda_{QCD}$ $\quad p \ll m_Q$
- NRQCD: $m_Q \gg \Lambda_{QCD}$ $\quad p \ll m_Q$ $\quad$ $p \sim mv$, $E \sim \frac{1}{2}mv^2$, $p \sim E$
- SCET: $p \ll \sqrt{\tilde{S}}$ w/ $p$ "collinear" or "soft"

# Euler - Heisenberg

Integrate out the electron to generate light-by-light scattering

$i\mathcal{M} =$  Full Theory

$i\mathcal{M} =$  EFT

$$\mathcal{L}_{EFT} = \frac{c}{m_e^4}\left(F_{\mu\nu}F^{\mu\nu}\right)^2 + \frac{c'}{m_e^4}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)^2$$

$$\alpha = \frac{e^2}{4\pi}$$

$$\tilde{F}^{\mu\nu} = \varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$$

Matching calculation $\Rightarrow$ $c = \frac{\alpha^2}{90}$ & $c' = \frac{7\alpha^2}{360}$

Can be used to compute

$$\sigma_{\gamma\gamma\to\gamma\gamma} = \frac{973}{10125\pi}\alpha^4\frac{\omega^6}{m^8}$$

(see Schwartz 33.4.2)

# Fermi Theory



Now let's apply this to the SM at energies $E \ll m_W$. (Let us continue to ignore the fact that QCD becomes non-perturbative at $E \sim 1\,\mathrm{GeV}$)

This will allow us to derive "Fermi Theory" from the top down.

Start with the Lagrangian

$$\mathcal{L} = m_W^2 |W_\mu|^2 + \frac{m_Z^2}{2} Z_\mu^2 + \frac{g}{2\sqrt{2}} W_\mu^+ J^{+,\mu}$$

$$+ \frac{g}{2\sqrt{2}} W_\mu^- J^{-,\mu} + \frac{g}{c_W} Z_\mu J^{\mu,0}$$

where "charged current" is

$$J_\mu^+ \equiv \sum_{\text{doublets}}' \bar{\psi}_d \gamma_\mu (1-\gamma^5) \psi_u$$

$$= \bar{l} \gamma_\mu (1-\gamma^5) \nu + \bar{d} \gamma_\mu (1-\gamma^5) u + \text{other families}$$

and "neutral current" is

$$J_\mu^0 \equiv \sum_{\substack{\text{all} \\ \text{fermions}}}' \bar{\psi}_i \gamma^\mu (g_V(i) - g_A(i)\gamma^5) \psi_i$$

This neglects the kinetic terms, since we are only keeping the leading term in the derivative expansion ($\partial^0$).

Neglects W/Z self interactions since this would lead to more powers of $1/m_{W/Z}$. Neglects Higgs interactions, since would either give more powers of $1/m_{W/Z/H}$ or would be proportional to tiny Yukawa couplings.

In this approximation, the EOMs for the W and Z are

$$\begin{aligned} m_W^2 W_\mu^+ + \frac{g}{2\sqrt{2}} J_\mu^+ &= 0 \\ m_Z^2 Z_\mu + \frac{g}{c_W} J_\mu^0 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} W_\mu^+ &= -\frac{g}{2\sqrt{2}\, m_W^2} J_\mu^+ \\ Z_\mu^+ &= \frac{-g}{c_W m_Z^2} J_\mu^0 \end{aligned}$$

Plugging these back into $\mathcal{L}$

$$\Rightarrow \mathcal{L} \supset \frac{g^2}{8m_W^2} J^+ \cdot J^- - 2\frac{g^2}{8m_W^2} J^+ \cdot J^- + \frac{g^2}{2c_W^2 m_Z^2} J_0 \cdot J_0 - \frac{g^2}{c_W^2 m_Z^2} J_0 \cdot J_0$$

$$= -\frac{g^2}{8m_W^2} J^+ \cdot J^- - \frac{g^2}{2W_W^2} J_0 \cdot J_0 \quad \left(\text{used } c_W^2 m_Z^2 = m_W^2\right)$$

We define the "Fermi Constant"

$G_F = 1.2\times 10^{-5}/\text{GeV}^2$

$$\frac{G_F}{\sqrt{2}} \equiv \frac{g^2}{8m_W^2} = \frac{1}{2v^2} \qquad [G_F] = -2$$

$$\Rightarrow \mathcal{L} \supset \underbrace{\sum_{\substack{\text{light}\\ \text{fermions}}} \bar{\psi} i \not{D} \psi}_{\text{QED}} - \underbrace{\frac{G_F}{\sqrt{2}} J^{+}\cdot J^{-}}_{\text{charged current interaction}} - \underbrace{\frac{4G_F}{\sqrt{2}} J^{0}\cdot J^{0}}_{\text{neutral current interaction}}$$

w/ $$J^{\mu,-} = (J^{\mu,+})^{*} = \sum_{\substack{\text{light}\\ \text{families}}} \left(\bar{\nu}\gamma^\mu(1-\gamma^5)e + \bar{u}\gamma^\mu(1-\gamma^5)d\right)$$

$$J^{\mu,0} = \sum_{\substack{\text{light}\\ \text{families}}} \bar{\psi}\gamma^\mu(g_V - g_A\gamma^5)\psi$$

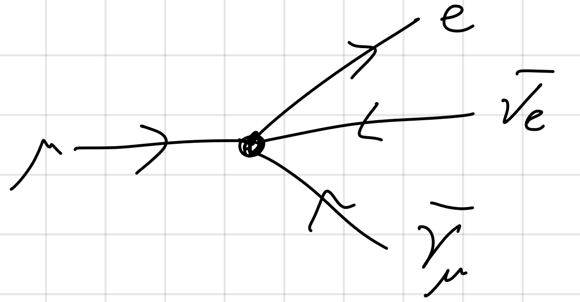
w/ $g_V = \frac{1}{4}\left[2T^3 - 4s_W^2 Q\right]$, $g_A = \frac{1}{4}\left[2T^3\right]$

We see that the weak interactions provide a correction to QED by introducing irrelevant operators.

We can capture a lot of the consequences of electroweak physics using this approximation.

For example, we can compute the decay rates of fermions. For example, the muon decay is determined by

$$\mathcal{L} \supset -\frac{G_F}{\sqrt{2}}\left[\bar{e}\gamma^\mu(1-\gamma^5)\nu_e\right]\left[\bar{\nu}_\mu\gamma_\mu(1-\gamma^5)\mu\right]$$

$\Rightarrow$ 


$$\Rightarrow \Gamma(\mu \to e\bar{\nu}_e\bar{\nu}_\mu) = \frac{G_F^2}{192\pi^3} m_\mu^5$$

Note that this is why the weak interactions are "weak". It is not due to small coupling since $g \sim g'$ in the SM. It is instead due to suppression by mass of heavy particles!

We could have guessed this answer using dimensional analysis:

$\mathcal{M} \sim G_F \Rightarrow \Gamma \sim G_F^2$

$[\Gamma] = 1 \Rightarrow$ assuming $e, \nu_e, \nu_\mu$ massless, only other dimensionful quantity is $m_\mu$

$\Rightarrow \Gamma \sim G_F^2 m_\mu^5$

We also know 2-body $\Gamma \sim \frac{1}{4\pi}$

3-body $\Gamma \sim \frac{1}{4\pi(16\pi^2)} = \frac{1}{64\pi^3}$

$\Rightarrow \Gamma \sim \frac{1}{64\pi^3} G_F^2 m_\mu^5$ is pretty close :)

This explains why the muon lives for a long time. We have $\Gamma_\mu / m_\mu \sim 3 \times 10^{-18}$

Naively expect $\Gamma/m \sim 10^{-2}$, so this is a big supression.

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## SMEFT

Approach to parametrize indirect BSM effects

Write down all $SU(3) \times SU(2) \times U(1)$ invariant operators supressed by heavy scale $\Lambda$.

A dim 5, there is one unique choice:

$$\mathcal{L}_{dim5} = \frac{1}{\Lambda}(H\bar{L}^c)(HL) \to \mathcal{L} \supset \frac{v^2}{\Lambda}\bar{\nu}^c\nu$$

$\Rightarrow$ Majorana neutrino mass.

At dim 6 there are 3045 independent operators. (Must be careful about redundancies)
1512.03433

# EFT for BSM



Why? Want to systemetize experimental search for indirect signals of BSM.

Tool of choice is EFT. (See also "primary operators")

* Model dependent! (Model agnostic)

Power counting is a UV hypothesis.

How should we build the EFT?

Work with $v=0$ (SMEFT) or $v \neq 0$ (HEFT)?

Must specify

1) Dofs
2) Symmetries
3) Power counting

| $v=0$ | $v \neq 0$ |
|---|---|
| 1) $H$ | 1) $h, \vec{\pi}$ |
| 2) $SU(2) \times U(1)$ | 2) $U(1)_{EM}$ |
| 3) Mass dimension | 3) Derivatives? |

Let us simplify our lives and assume custodial symmetry

$\Rightarrow SU(2) \times U(1) \to SU(2)_L \times SU(2)_R \cong O(4)$

Custodial sym is only approximate in SM

Explicitly broken by gauging $U(1)_Y \subset SU(2)_R$ and due to fermion mass splittings

## SMEFT

Focus on scalar sector. Let $\vec{\varphi}$ be fundamental of $O(4)$:

$$\vec{\varphi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix} \qquad \text{w/} \quad \vec{\varphi} \to O\,\vec{\varphi} \quad \text{under } O(4)$$

($O$ is $4\times 4$ orthogonal matrix.)

Identify $\quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi_1 + i\varphi_2 \\ \varphi_4 + i\varphi_3 \end{pmatrix}$

$$\mathcal{L}_{SMEFT} = A(|H|^2)\,|\partial H|^2 + \frac{1}{2} B(|H|^2)\left[\partial(|H|^2)\right]^2 - \tilde{V}(|H|^2) + \mathcal{O}(\partial^4)$$

w/ $A, B, \tilde{V}$ are real analytic at origin $|H| = 0$.

Geometrically, $\varphi_i$ are Cartesian coordinates.

# HEFT



Goldstones of $O(4)/O(3)$ $\vec{\pi} \leftarrow$ transform non-linearly

Singlet scalar field $h$

Define $\vec{n} = \begin{pmatrix} n_1 = \pi_1/v \\ n_2 = \pi_2/v \\ n_3 = \pi_3/v \\ n_4 = \sqrt{1 - n_1^2 - n_2^2 - n_3^2} \end{pmatrix}$

Under $O(4)$ $h \to h$ and $\vec{n} \to O\vec{n}$

$\vec{n}(\vec{\pi}) \in S^3$ is 4-component unit vector w/ $\vec{n} \cdot \vec{n} = 1$.

The constrained vector $\vec{n}$ transforms linearly.

The rotations in the 12, 13, and 23 planes act linearly on $(n_1, n_2, n_3)$ and leave $n_4$ invariant. However, if one does eg a 14 rotation (infinitesimal)

$\delta n_1 = \theta n_4$, $\delta n_2 = 0$, $\delta n_3 = 0$, $\delta n_4 = -\theta n_4$

Then the transformation of the unconstrained $\pi$ fields is $\delta \pi_1 = \theta \sqrt{v^2 - \vec{\pi} \cdot \vec{\pi}}$, $\delta \pi_{2,3} = 0$

$\Rightarrow$ non-linear.

$$\mathcal{L}_{HEFT} = \frac{1}{2}\left[\mathbb{K}(h)\right]^2(\partial h)^2 + \frac{1}{2}\left[vF(h)\right]^2(\partial\vec{n})^2 - V(h) + \mathcal{O}(\partial^4)$$

w/ $\mathbb{K}$, $F$, $V$ are real analytic about the physical vacuum $h=0$.

Geometrically, HEFT is like polar coordinates.

* Ultimately, HEFT is description used to do physical calculations, since need to work in physical vacuum.

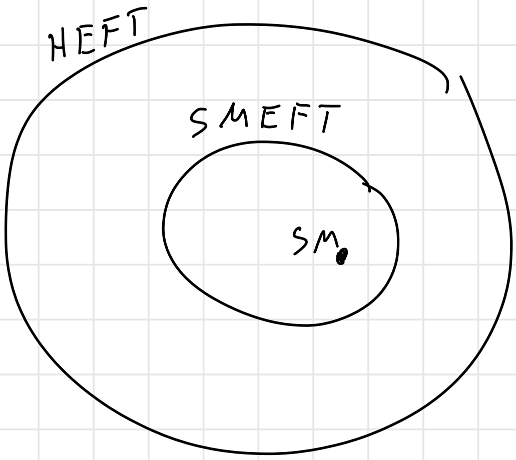
Remember EFT requires truncation of power counting expansion.

Compare $\tilde{V}(H)$ up to dim 6 and $V(h)$ up to 6 fields

$$\tilde{V}(H) = -\mu^2 |H|^2 + \lambda |H|^4 + \frac{1}{\Lambda^2} |H|^6$$

$$V(h) = m^2 h^2 + c_3 h^3 + c_4 h^4 + c_5 h^5 + c_6 h^6$$

Clearly HEFT has larger parameter space then SMEFT.




If we parametrize BSM searches w/ SMEFT, are we potentially missing anything?

Motivates understanding the relationship between HEFT and SMEFT.

Note: preference is to work w/ SMEFT since that is already hard enough.

Also much more natural from model building perspective.

Assume no obstruction to mapping between SMEFT and HEFT:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi_1 + i\varphi_2 \\ \varphi_4 + i\varphi_3 \end{pmatrix} \quad \text{and} \quad \vec{\varphi} = (v_0 + h)\,\vec{n}$$

How to determine $v_0$? ← Revisit

(Note $v$ sets gauge boson masses, etc)

Let's write some $O(4)$ symmetric objects setting $v = v_0$ for simplicity:

$$|H|^2 = \frac{1}{2}\vec{\varphi}\cdot\vec{\varphi} = \frac{1}{2}(v+h)^2$$

$$|\partial H|^2 = \frac{1}{2}(\partial\vec{\varphi})^2 = \frac{1}{2}(\partial h)^2 + \frac{1}{2}(v+h)^2(\partial\vec{n})^2$$

$$(\partial|H|^2)^2 = (\vec{\varphi}\cdot\partial\vec{\varphi})^2 = (v+h)^2(\partial h)^2$$

The using this, we can write (Exercise)

$$\mathcal{L}_{HEFT} = \frac{1}{2}\left[K(h)\right]^2(\partial h)^2 + \frac{1}{2}\left[v\,F(h)\right]^2(\partial\vec{n})^2 - V(h) + \ldots$$

$$= \frac{v^2F^2}{2|H|^2}|\partial H|^2 + \frac{1}{2}(\partial|H|^2)^2\frac{1}{2|H|^2}\left(K^2 - \frac{v^2F^2}{2|H|^2}\right)$$

$$- \tilde{V}(|H|^2) + \ldots$$

(Notice non-analiticity.)

# TA: Dimensional Regularization

- $d = 4 - 2\varepsilon \Rightarrow \frac{1}{\varepsilon} + \log\left(\frac{\mu^2}{m^2}\right)$ (Other common convention $d = 4 - \varepsilon \Rightarrow 1/\varepsilon + \log(\mu/m)$ )

Changing space-time dimension

$$\Rightarrow S = \int d^4x\, (\partial_\mu \varphi)(\partial^\mu \varphi) \to S = \int d^d x\, (\partial_\mu \varphi)(\partial^\mu \varphi)$$

$[S] = 0$ in all $d \Rightarrow [(\partial_\mu \varphi)^2] = 4 - 2\varepsilon$

$[\partial] = 1 \Rightarrow [\varphi] = 1 - \varepsilon$

$$\Rightarrow \mathcal{L} \ni \frac{c_4}{4!}\varphi^4 \to \frac{c_4}{4!}\mu^{2\varepsilon}\varphi^4$$

$$\mathcal{L} \ni \frac{c_6}{6!}\varphi^6 \to \frac{c_6}{6!}\mu^{4\varepsilon}\varphi^6$$

<u>We will</u> need to renormalize $\overset{\text{bare}}{C^0} = Z\mu^{n\varepsilon} C^r$ n depends on mass dim of operator

$\tilde{\mu}^2 = \mu^2 (4\pi e^{-\gamma_E})$ is the $\overline{MS}$ scale

We will use $\overline{MS} \Rightarrow$ Counter terms subtract $\frac{1}{\varepsilon}$

- Scaleless integrals vanish in dim reg

$$\boxed{\text{Notation } (d\ell) = \frac{d^d\ell}{(2\pi)^d}}$$

Ex: $I = \mu^{2\varepsilon}\int (d\ell)\,\frac{1}{\ell^4} = \underbrace{\mu^{2\varepsilon}\int (d\ell)\,\frac{1}{\ell^2(\ell^2-m^2)}}_{I_{UV}} - \underbrace{\mu^{2\varepsilon}\int (d\ell)\,\frac{m^2}{\ell^4(\ell^2-m^2)}}_{I_{IR}}$

$I_{UV} = \frac{i}{16\pi^2}\left(\frac{1}{\varepsilon_{UV}} + \log\frac{\bar{\mu}_{UV}^2}{m^2} + 1\right) + \dots$

$I_{IR} = \frac{i}{16\pi^2}\left(\frac{1}{\varepsilon_{IR}} + \log\frac{\bar{\mu}_{IR}^2}{m^2} + 1\right) + \dots$

$\Rightarrow I = \frac{i}{16\pi^2}\left(\frac{1}{\varepsilon_{UV}} - \frac{1}{\varepsilon_{IR}}\right) = 0$ when $\varepsilon_{UV} = \varepsilon_{IR}$, $\mu_{UV} = \mu_{IR}$

- In general, Scaleless integrals vanish in dim reg. This will be critical for the sucess of our EFT formalism

(Note: analytic continuation $\varepsilon_{IR} \to -\varepsilon_{IR}$ since $I_{IR}$ Converges for $d = 4 + 2\varepsilon_{IR}$)

---

- Feynman params $\quad \frac{1}{AB} = \int_0^1 dx\,\frac{1}{(xA + (1-x)B)^2}$

- Feynman params w/ linear denoms (Georgi params)

$$\frac{1}{A^n B^m} = \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)}\int_0^\infty d\eta\,\frac{\eta^{m-1}}{(A+\eta B)^{n+m}}$$

w/ $A$ quadratic, $B$ linear $\quad [\eta] = \text{GeV}$

TA: Renormalization Group Equations $\frac{d}{d\log\tilde{\mu}^2} C_n^r = \gamma_{nm} C_m^r$ 27

If $\gamma =$ const $\Rightarrow \int \frac{1}{C^r} dC^r = \int \gamma \, d\log\tilde{\mu}^2$

anomalous dimension

$$\Rightarrow C^r(\tilde{\mu}_H) = C^r(\tilde{\mu}_L)\exp\left(\gamma \log\frac{\tilde{\mu}_H^2}{\tilde{\mu}_L^2}\right) = \left(\frac{\tilde{\mu}_H^2}{\tilde{\mu}_L^2}\right)^{2\gamma} C^r(\tilde{\mu}_L)$$

When $\gamma_{nm} \neq 0$ w/ $n \neq m \Rightarrow$ operator mixing

Derive equation for $\gamma_{nm}$:

In pert theory $Z = 1 + \mathcal{O}(C^r, \alpha^r)$

Bare Lagrangian must be $\tilde{\mu}$ independent (Callan-Symanzik Eq)

$$0 = \tilde{\mu}\frac{d}{d\tilde{\mu}} C^0 = \tilde{\mu}\frac{d}{d\tilde{\mu}}\left(Z \tilde{\mu}^{n\varepsilon} C^r\right)$$

Ex: $\mathcal{L} \supset \frac{1}{4!} C_4 \phi^4 + \frac{1}{6!}\frac{1}{M^2} C_6 \phi^6$

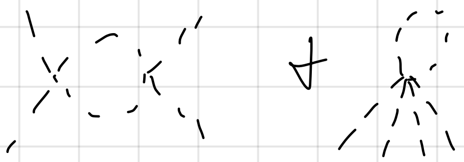
Tree level $0 = \tilde{\mu}\left(\frac{1}{Z_4}\cancelto{0}{\frac{dZ_4}{d\tilde{\mu}}} + \frac{1}{C_4^r}\frac{dC_4^r}{d\tilde{\mu}} + \frac{1}{\tilde{\mu}^{2\varepsilon}} 2\varepsilon \tilde{\mu}^{2\varepsilon - 1}\right)\cancelto{1}{Z_4} C_4^r \tilde{\mu}^{2\varepsilon}$

$$\Rightarrow \frac{dC_4^r}{d\log\tilde{\mu}^2} = -\varepsilon C_4^r \Rightarrow \gamma_{44}^{\text{classical}} = -\varepsilon$$

Tree level change in dimension of operator w/ $d \neq 4$

Similarly, $\gamma_6^{\text{classical}} = -2\varepsilon$

At one-loop: $Z_4 = Z_4(C_4^r, C_6^r)$



$$\Rightarrow 0 = \frac{d}{d\log\tilde{\mu}^2}\left(Z_4(C_4^r, C_6^r)\,\tilde{\mu}^{2\varepsilon} C_4^r\right)$$

$$= \frac{1}{2}\left(\frac{\partial Z_4}{\partial C_4^r}\frac{\tilde{\mu}}{Z_4}\frac{dC_4^r}{d\tilde{\mu}} + \frac{\partial Z_6}{\partial C_6^r}\frac{\tilde{\mu}}{Z_4}\frac{dC_6^r}{d\tilde{\mu}} + \frac{\tilde{\mu}}{C_4^r}\frac{dC_4^r}{d\tilde{\mu}} + 2\varepsilon\right) Z_4\,\tilde{\mu}^{2\varepsilon} C_4^r$$

Then truncate $\frac{1}{Z_4} = 1$ and use leading sols $\Rightarrow$

$$\frac{dC_4^r}{d\log\tilde{\mu}^2} = \left(\varepsilon (C_4^r)^2 \frac{\partial Z_4}{\partial C_4^r} - \varepsilon C_4^r + 2\varepsilon C_4^r C_6^r \frac{\partial Z_4}{\partial C_6^r}\right)$$

$$\Rightarrow \gamma_{44} = \lim_{\varepsilon\to 0}\left(\varepsilon C_4^r \frac{\partial Z_4}{\partial C_4^r} - \varepsilon\right) \quad \text{and} \quad \gamma_{46} = \lim_{\varepsilon\to 0}\left(2\varepsilon C_4^r \frac{\partial Z_4}{\partial C_6^r}\right)$$

- Practically, one can differentiate fixed order result to get $\gamma$, and then plug back in to resum

- Solving this equation "sums logs"

- Now we have two simultaneous expansions

  - Count $(\alpha \log\lambda)^n$ differently then $\alpha$

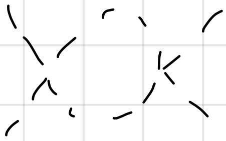
    $(\alpha\log\lambda)^n$ ↑ $N^n LL$ $\qquad$ $\alpha$ ↑ $N^n LO$ $\qquad$ Be careful about double counting

## Summing Logs (no longer careful w/ "r" and $\tilde{\mu}$)

Ex: $\mathcal{L} \supset -\frac{1}{4!} C_4 \phi^4$ is defined at some high scale $\mu_H$

Want to predict $\phi\phi \to \phi\phi$ at $\mu_L^2 \sim m^2$ (threshold)

Tree:  $= -i C_4(\mu_H)$

1-loop: (s+t+u channels)  $= \frac{3i}{32\pi^2} \mu_H^{2\varepsilon} C_4^2 \left( \frac{1}{\varepsilon} + \log \frac{\mu_H^2}{m^2} + \frac{2}{3} \right)$

$$\Rightarrow Z_4 = 1 + \frac{3}{32\pi^2} C_4 \frac{1}{\varepsilon} \Rightarrow \frac{d C_4}{d \log \mu^2} = \frac{3}{32\pi^2} C_4^2$$

$$\Rightarrow C_4(\mu_L) = \frac{C_4(\mu_H)}{1 + C_4(\mu_H) \frac{3}{32\pi^2} \log \frac{\mu_H^2}{\mu_L^2}}$$

Landau pole

$\mu = \Lambda$ when $C_4(\Lambda) \to \infty$

$\Lambda = m \, \mathrm{Exp}\left( 1 \Big/ \frac{3}{32\pi^2} C_4(m) \right)$

$\Rightarrow$ Theory breaks down

Dim transmutation!

Then computing using running coupling @ low scale to 1-loop

$$A = -C_4(\mu_L)\left(1 - \frac{3}{32\pi^2} C_4(\mu_L)\left(\log \frac{\mu_L^2}{m^2} + \frac{2}{3}\right)\right)$$

$$= -C_4(\mu_H)\left\{\left(1 - C_4(\mu_H)\frac{3}{32\pi^2}\log\frac{\mu_H^2}{\mu_L^2}\right) \times \left[1 - \frac{3}{32\pi^2} C_4(\mu_H)\left(\log \frac{\mu_L^2}{m^2}\right)\right] + \dots\right.$$

$$= -C_4(\mu_H)\left\{1 - C_4(\mu_H)\left(\log \frac{\mu_H^2}{m^2} + \frac{2}{3}\right) + \dots\right\}$$

$\Rightarrow$ Reproduces non-improved result.

# Heavy Particle Decoupling

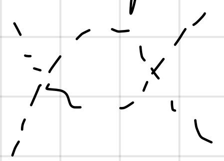
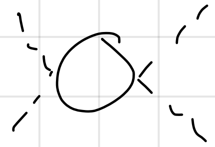
Focus on process $\varphi\varphi \to \varphi\varphi$ @ threshold



$$\mathcal{L}^{Full} \supset -\frac{1}{4}\lambda \varphi^2 \Phi^2 - \frac{1}{4!}\eta \varphi^4$$

Assume $m^2 \ll M^2$

First compute NLO process to uncover apparant non-decoupling

$3\times$  $+\ 3\times$ 

$$\Rightarrow \mathcal{A}^{Full} = \eta + \frac{3}{32\pi^2}\eta^2\left(\log\frac{\mu^2}{m^2} + \frac{2}{3}\right) + \frac{3}{32\pi^2}\lambda^2 \log\frac{\mu^2}{M^2} + \dots$$

Any choice of $\mu$ results in a large log.

RGEs do not solve the problem:

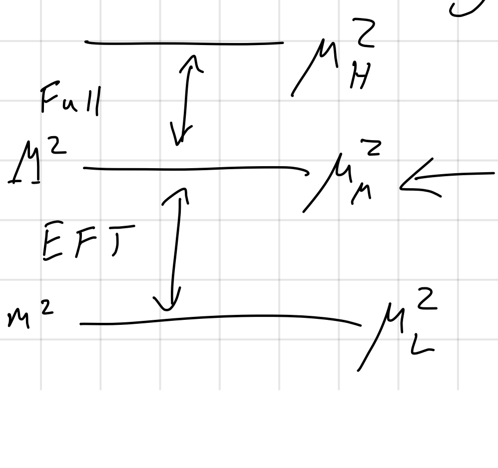
$$\frac{d\eta}{d\log\mu^2} = \frac{3}{32\pi^2}\left(\eta^2 - \lambda^2\right) \qquad \frac{d\lambda}{d\log\mu^2} = \frac{1}{8\pi^2}\lambda^2 + \frac{1}{32\pi^2}\lambda\eta$$

(no mass scales)

Must <u>match</u> onto an EFT at scale $\mu_M$

Need IR of EFT to be same as Full Thy

$\Rightarrow$ Build EFT using $\varphi$ w/ mass $m^2$



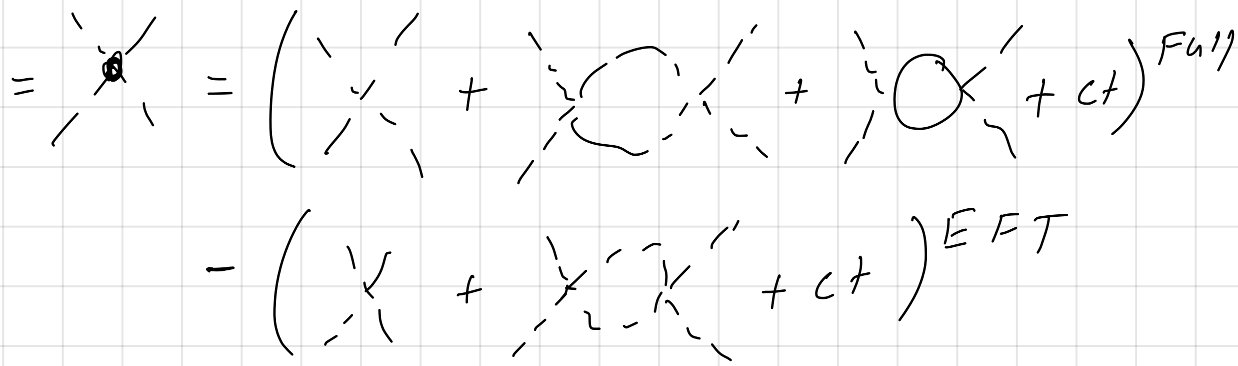

" Intermediate Matching scale "

$$\mathcal{L}^{EFT} \supset -\frac{C_4}{4!}\varphi^4$$

$$A_{match} = \left[A_{Full} - A^{CT}_{Full}\right] - \left[A_{EFT} - A^{CT}_{EFT}\right]$$

mention on-shell w.f. factors

$iA^{Match} =$




At tree level $C_4(\mu_M) = \eta(\mu_M)$

One-loop: $C_4(\mu_M) = \eta(\mu_M) - \frac{3}{32\pi^2}\left(\alpha(\mu_M)\right)^2 \log\frac{\mu_M^2}{M^2} + \dots$

This is BC for RGE within EFT

$$\frac{dC_4}{d\log\mu^2} = \frac{3}{32\pi^2}(C_4)^2 \Leftarrow \text{Heavy particle decoupled}$$

No large logs!

Expanding RGE improved couplings reproduces original $A$

$$iA^{EFT}_{Expanded} = -i\eta(\mu_H) + \frac{3i}{32\pi^2}\left[\eta^2\log\frac{\mu_H^2}{\mu_M^2} + (C_4)^2\left(\log\frac{\mu_M^2}{m^2} + \frac{2}{3}\right)\right]$$

$$+ \frac{3i}{32\pi^2}\alpha^2\log\frac{\mu_H^2}{M^2}$$

# The Hierarchy Problem

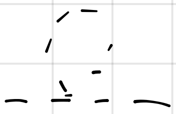
## No quad div in dim reg

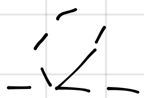
$$\mathcal{L}_{Full} \ni -\frac{1}{2} m^2 \varphi^2 - M^2 \Phi^2 - \frac{1}{4!} \eta \varphi^4 - \frac{1}{4} \lambda \varphi^2 \Phi^2$$

$$\mathcal{L}_{EFT} \ni -m^2 \varphi^2 - \frac{1}{4!} C_4 \varphi^4$$

Matching $A_{match} = \left[ A_{Full} - A_{Full}^{CT} \right] - \left[ A_{EFT} - A_{EFT}^{CT} \right]$

at high scale $\bar{\mu}_H \sim M$

Full  $= \frac{\eta}{2} \mu_H^{2\varepsilon} \int (d\ell) \frac{1}{\ell^2 - m^2} = i \frac{\eta}{32\pi^2} m^2 \left[ \frac{1}{\varepsilon} + \log \frac{\bar{\mu}_H^2}{m^2} + 1 + \mathcal{O}(\varepsilon) \right]$

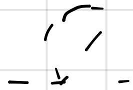
EFT  $= \eta \to C_4$

Use $\varphi\varphi \to \varphi\varphi$ to match at tree-level $-iC_4 = -i\eta$

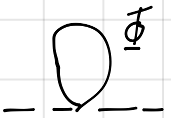
$$\Rightarrow -i m^2_{match} = -i m_r^2 + \frac{i\eta}{32\pi^2} m_r^2 \left( \log \frac{\bar{\mu}_H^2}{m_r^2} + 1 \right) - \left[ \eta \to C_4 \right] = 0$$

So 1-loop matching gives no contribution. Note same prescription on both sides of matching

Then in EFT at $\mu_L$

 w/ $\mu_H \to \mu_L \Rightarrow$ Proportional to $m^2 \Rightarrow$ no tuning problem!

Now what about the Heavy loop?




$$= \frac{1}{2}\lambda \bar{\mu}_H^{2\epsilon}\int (dl)\, \frac{1}{l^2 - M^2} = \frac{i\lambda}{32\pi^2} M^2 \left[\frac{1}{\epsilon} + \log\frac{\bar{\mu}_H^2}{M^2} + 1 + \mathcal{O}(\epsilon)\right]$$

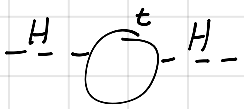
No loop in EFT

$$\Rightarrow -iM^2_{match} = \left(-i m_r^2 + \frac{i\lambda}{32\pi^2} M^2\left[\log\frac{\bar{\mu}_H^2}{M^2} + 1\right]\right) - \left(-i m_r^2\right)$$

Take natural $\bar{\mu}_H = M \Rightarrow \quad m^2_{match} = \frac{-\lambda}{32\pi^2} M^2$

Physical "quadratic divergence"

- Fine tuning at matching scale
- Complain:




$$\sim y_t^2 \int_0^\Lambda \frac{d^4 l}{(2\pi)^4}\, \frac{1}{l^2 - m_t^2} \sim \frac{y_t^2}{16\pi^2}\Lambda^2$$

Not physical! No tuning problem in SM alone

<u>But</u> if another physical scale, will make quad div physical

Always happens in calculable models of Higgs mass Need new symmetry!

Read TASI lectures for context