Closing the gap in the XXZ chain via magnon interactions

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Want to understand the difference between 1D and higher dimensions.

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This yields:

$$H^{rot}_\alpha = \sum -\alpha \tilde{S}^z_i \tilde{S}^z_{i+1} + \frac{1}{2}(\tilde{S}^+_i \tilde{S}^+_{i+1} + \tilde{S}^-_i \tilde{S}^-_{i+1}) \tag{3}$$

Holstein-Primakoff transformation

Introduction of H-P transformation along z-axis[1]:

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Operators $b_i^\dagger$ and b_i either create or annihilate magnons at ith site. $P(n_i)$ assures the ladder operators have a bounded spectrum[2]. For brevity I will use $P(n_i)b_i = \tilde{b}_i$.

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$$\mathcal{H}_{\alpha,\lambda} = \sum_{i}^{N} -\alpha[\lambda n_i n_{i+1} - \frac{1}{2}(n_i + n_{i+1}) + \frac{1}{4}] + \frac{1}{2}(\tilde{b}_i^\dagger \tilde{b}_{i+1}^\dagger + \tilde{b}_i \tilde{b}_{i+1}) \tag{5}$$

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λ controls the strength of magnon-magnon interactions. Original Hamiltonian obtained for $\lambda = 1$.

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Spin correlations in real and momentum space:

$$S^{aa}(l) = \frac{1}{N} \sum_{i} \langle 0| S_i^a S_{i+l}^a |0\rangle \,,$$
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Spin dynamical structure factor (also known as a spectral function):

$$S^{aa}(k,\omega) = -\lim_{\delta\to 0^+}\frac{1}{\pi}\,\mathrm{Im}\,\langle 0| S^a_{-k}\frac{1}{E_0+\omega+i\delta-H} S^a_k |0\rangle . \qquad (8)$$

In this equations $a \in \{x, z\}$.

Gap in the $S^{xx}(k, \omega)$ spectral function

$$S^{xx}(k, \omega)$$

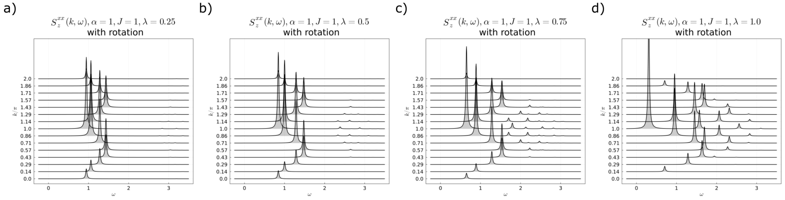

Figure: $S^{xx}(k, \omega)$ for various values of λ. The gap opens when magnon interactions are turned off.

The gap at $k = \pi$ for $\lambda = 1$ is a finite size effect, in the limit $N \longrightarrow \infty$ the spectrum is gapless[3].

[3] Jean-Sébastien Caux et al. "Tracking the Effects of Interactions on Spinons in Gapless Heisenberg Chains". *Physical Review Letters* 106, no. 21 (2011).

Connection between gap and spin correlation

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This gives another tool for tracking the gap, easier to do then observing the spectral functions.

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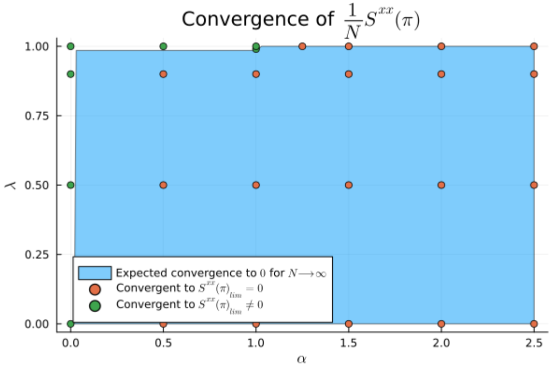

Figure: Calculated convergence of $\frac{1}{N}S^{xx}(k)$. The calculations were done exactly for small systems and then confirmed with DMRG.

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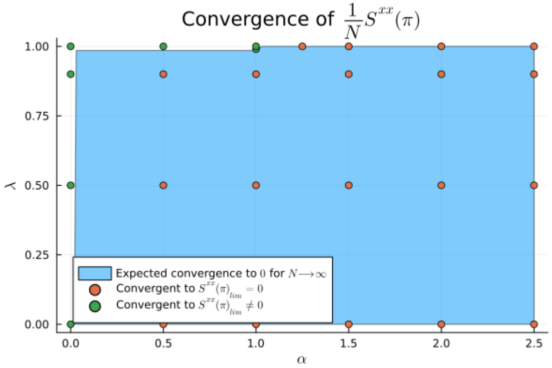

Figure: Calculated convergence of $\frac{1}{N}S^{xx}(k)$. The calculations were done exactly for small systems and then confirmed with DMRG.

We see a critical line along $\lambda = 1$ up to $\alpha = 1$.

Real space spin correlations

Another observable that can be linked to the energy gap is real space correlations–the crucial distinction is power-law vs. exponential decay of correlations.

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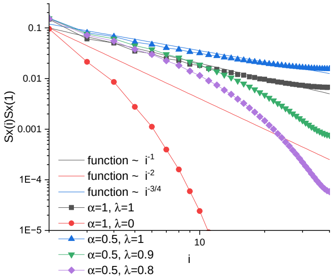

Figure: $S^{xx}(i)$ for range of systems with $\alpha \leq 1$.

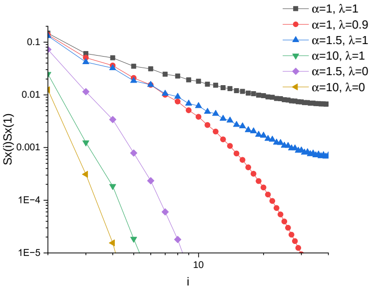

Figure: $S^{xx}(i)$ for range of systems with $\alpha \geq 1$.

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- Come up with a intuitive 'cartoon' picture of the excitations.

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- Magnon-magnon interactions are essential for the closing of the energy gap.
- The crucial part of the Hamiltonian is balance between magnon-magnon interactions and magnon cost.

Acknowledgments

Acknowledgments

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The end

Jordan-Wigner transformation

J-W transformation inconvenient, as it involves so called "Jordan strings"[4]:

$$
\begin{aligned}
n_j &= S^z + \frac{1}{2} \\
a_j &= e^{-i\pi \sum_{k=1}^{j-1} n_k} S_j^- \\
a_j^\dagger &= e^{+i\pi \sum_{k=1}^{j-1} n_k} S_j^+
\end{aligned} \tag{10}
$$

These assure correct anticommutation relations between fermions at different sites, but prove hard to compute in larger systems.

[4] Paul Jordan and Eugen Wigner. "Über das Paulische Äquivalenzverbot". *Zeitschrift für Physik* 47 (1928): 631–651.

Disturbing the balance the other way

Add chemical potential (can be interpreted as magnetic field) to tip the balance between magnon interactions and magnon cost:

$$H_\mu = \sum_i^N -\alpha[\lambda n_i n_{i+1} - \frac{1}{2}(n_i + n_{i+1}) + \frac{1}{4}] + \frac{1}{2}(b_i^\dagger b_{i+1}^\dagger + b_i b_{i+1}) + \mu n_i. \quad (11)$$

$$\alpha = 1$$