Cosmological horizons and conserved quantities

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Winter Kindergarten, Karpacz 24.05.2024

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Sections

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Introduction

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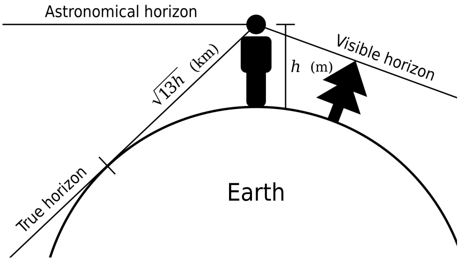
What is a horizon?

Horizon

The apparent curve that separates the surface of a celestial body from its sky when viewed from the perspective of an observer on or near the surface of the relevant body. [*Wikipedia*]



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Einstein Field Equation

Equivalence Principle - inertial frame of reference is locally indistinguishable from free-falling frame. In other words, inertial mass is equal to gravitational mass:

$$m_b = m_g$$

From this principle and using properties:

$$\nabla G_{\mu\nu} = \nabla g_{\mu\nu} = \nabla T_{\mu\nu},$$

where $G_{\mu\nu}$ is Einstein tensor, the geometry of spacetime is described by:

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = \kappa T_{\mu\nu},$$

where Λ is cosmological constant and $\kappa = 8\pi G/c^4$. Solution of these equations is metric.

Energy-momentum tensor

Energy-momentum tensor **T** describes density of four-momentum in each point of the spacetime.

Values of energy-momentum tensor in the rest frame:

T^{00} - density of energy
T^{0k} - flux of energy
T^{k0} - density of momentum
T^{jk} - flux of momentum (pressure and stress)

It is obtained by variation of the Hilbert-Einstein action:

$$\mathcal{S} = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa}(R+2\Lambda)+\mathcal{L}_M\right] \Rightarrow T^{\mu\nu} = \mathcal{L}_M g^{\mu\nu} - 2\frac{\delta\mathcal{L}_M}{\delta g_{\mu\nu}}$$

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Black holes and accelerated observers

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Schwarzschild solution

Assuming:

- $\frac{\partial}{\partial t} g_{\mu\nu} = 0$
- Spherical symmetry
- Empty space $T_{\mu\nu} = G_{\mu\nu} = 0$ without cosmological constant $\Lambda = 0$.
- In limit $r \to \infty$ the solution have to be the Minkowski metric
- In weak field limit, the solution is equivalent to Newtonian potential

The Schwarzschild metric has following form:

$$ds^2 = c^2 \left(1 - \frac{r_s}{r}\right) dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2 d\Sigma^2,$$

where $r_s = \frac{2GM}{c^2}$.

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Event horizon

When $r = r_s$, the Schwarzschild metric is "broken". One call it **event horizon**. In relativistic framework, it is the boundary where time-like vectors become space-like and *vice versa*:

$$\partial_r \text{ is } \begin{cases} \text{space-like} & r > r_s \\ \text{time-like} & r < r_s \end{cases} \qquad \partial_t \text{ is } \begin{cases} \text{time-like} & r > r_s \\ \text{space-like} & r < r_s \end{cases}$$

The Schwarzschild metric is "broken" at $r = 0$ as well and we call it singularity. At $r = r_s$, there is the apparent singularity which may be canceled using different coordinates, e.g. Kruskal-Szekeres coordinates:

$$ds^2 = \frac{4r_s^3}{r} e^{-\frac{r}{r_s}} \left(dT^2 - dR^2\right) - r^2 d\Sigma^2$$

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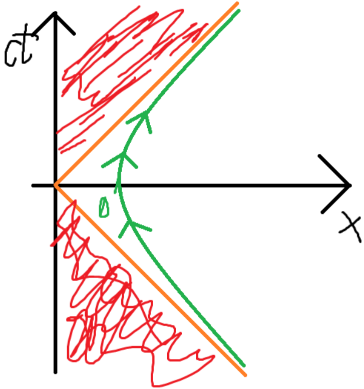
Rindler coordinates and horizon

Rindler coordinates are use to describe uniformly accelerating observer in special relativity. If $-\alpha^2$ is spacetime interval of four-acceleration of a body, its four-position in inertial frame of reference may be written as follow (1+1 spacetime):

$$x^\mu = \frac{c^2}{\alpha}\begin{bmatrix}\sinh\left(\frac{\alpha\tau}{c}\right)\\ \cosh\left(\frac{\alpha\tau}{c}\right)\end{bmatrix}$$

It is always space-like:

$$x^\mu x_\mu < 0$$



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The FLRW metric and its horizons

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Assumption of FLRW model

There are two main assumption behind FLRW model. In a big scale (about 300 mln ly) the universe is:

- Homogeneous - invariant with respect to spatial translation (it doesn't matter where we are). Formally, it means that for each point $p \in \mathcal{M}$, there exists isometry ϕ which transform it into another point $\phi(p) = p' \in \mathcal{M}$.
- Isotropic - invariant with respect to spatial rotation (it doesn't matter in which direction we look). Formally, it means that for each point $p \in \mathcal{M}$ and for any vectors from its tangent space $v, w \in T_p\mathcal{M}$, there exist isometry ϕ such that $\phi(p) = p$ and $\phi_* v = w$.

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FLRW metric

FLRW is an acronym created from following surnames: Friedman, Lemaître, Robertson and Walker.
The assumption of homogeneous and isotropic lead to the metric which has maximally symetric spatial slice (maximum number of Killing vectors) that depend on time:

$$g_{\mu\nu} = c^2 dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right],$$

where k is spatial-curvature parameter and $a(t)$ is scale factor.

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Cosmological horizons

One can distinguish three cosmological horizons in FLRW model:

- **Particle horizon** - informational limit which describe maximal distance from where the observer may receive information - boundary of **observable universe**:

$$L_{PH} = ac \int_0^t \frac{dt'}{a}$$

- **Cosmological event horizon** - informational limit which describe maximal distance of how far the observer may send information.

$$L_{EH} = ac \int_t^\infty \frac{dt'}{a}$$

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Cosmological horizons

- **Hubble horizon** - the space boundary beyond which the average velocity of galaxies is greater than c:

$$L_{HH} = \frac{c}{H}$$

For the current time, the numeric values of this horizons are as follow:

$$L_{EH} \approx 16Gly \qquad L_{PH} \approx 45Gly \qquad L_{HH} \approx 14Gly$$

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Hubble Horizon

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Killing horizon of the FLRW metric

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Lie Derivative

Let $\mathbf{X}$ be vector field defined on differential manifold $\mathcal{M}$. Let $\phi^s_{\mathbf{X}} : \mathcal{M} \to \mathcal{M}$ be a diffeomorphism which create one parametric group, so for any $p \in \mathcal{M}$:

$$\phi^s_{\mathbf{X}} \circ \phi^t_{\mathbf{X}}(p) = \phi^{s+t}_{\mathbf{X}}(p)$$

So one may write a differential of any tensor $\mathbf{T}$ with respect to flow of $\mathbf{X}$ in $p \in \mathcal{M}$:

$$\Delta_t \mathbf{T} = (\phi^s_{\mathbf{X}})^* \, \mathbf{T}\,(\phi^s_{\mathbf{X}}) - \mathbf{T}(p)$$

Lie derivative is just infinitesimal version of it:

$$\mathcal{L}_{\mathbf{X}} = \lim_{t \to 0} \frac{\Delta_t \mathbf{T}}{t}$$

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Killing vector field

Killing vector field **K** is a generator of **isometry group** - group whose operations conserve length. Formal definition of **K** looks like below:

$$\mathcal{L}_{\mathbf{K}}\mathbf{g} = 0$$

Writing it down, one obtain:

$$\partial_\mu \left(K^\lambda g_{\lambda\nu}\right) + \partial_\nu \left(K^\lambda g_{\lambda\mu}\right) - 2K^\lambda g_{\lambda\sigma}\Gamma^\sigma_{\nu\mu} = 0,$$

where $\Gamma^\sigma_{\nu\mu}$ is Christoffel symbol (metric connection):

$$\nabla_\nu \mathbf{e}_\mu = \Gamma^\sigma_{\nu\mu}\mathbf{e}_\sigma$$

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Noether theorem

Noether theorem - every continuous symmetry of a physical system has a corresponding conservation law. In General Relativity one may write it as follow:

$$\partial_\mu \left(\sqrt{-g} J^\mu\right) = 0,$$

where J^μ is four-current which is related to conserved quantity. One may write such four-current using Killing vectors and energy-momentum tensor:

$$J^{(b)\mu} = T^\mu_\nu K^{(b)\nu},$$

where b is index which numerate solutions of Killing equation. We consider energy-momentum tensor of perfect fluid:

$$T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) U^\mu U^\nu - p g^{\mu\nu}$$

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Space-like Killing vectors for FLRW metric

There are 6 spatial Killing vector (due to maximally symmetric spatial slice) whose conserved currents give the conservation of momentum and angular momentum:

$$\mathbf{K}^{(1)} = \frac{\sqrt{1-kr^2}}{r}\left[r\sin\theta\sin\phi\partial_r + \cos\theta\sin\phi\partial_\theta + \frac{\cos\phi}{\sin\theta}\partial_\phi\right]$$

$$\mathbf{K}^{(2)} = \frac{\sqrt{1-kr^2}}{r}\left[r\sin\theta\cos\phi\partial_r + \cos\theta\cos\phi\partial_\theta - \frac{\sin\phi}{\sin\theta}\partial_\phi\right]$$

$$\mathbf{K}^{(3)} = \frac{\sqrt{1-kr^2}}{r}\left[r\cos\theta\partial_r - \sin\theta\partial_\theta\right] \qquad \mathbf{K}^{(4)} = \sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi$$

$$\mathbf{K}^{(5)} = \cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi \qquad \mathbf{K}^{(6)} = \partial_\phi$$

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Time-like Killing vectors for FLRW metric

There is only one time-like Killing vector for $k = 0$ which depends only on r and t:

$$\mathbf{K}^{(7)} = \partial_t - \frac{\dot{a}}{ac} r \partial_r$$

Four-current which is created with it is related with conservation of energy. But this time-like only in range:

$$r \in \left[0, \frac{c}{\dot{a}}\right]$$

So there exists Killing horizon at $r = \frac{c}{\dot{a}}$.

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Relation between Hubble and Killing horizon

Let's consider real length $L = ar$. Time-like Killing vector for FLRW metric exists only inside of Hubble sphere:

$$|\mathbf{K}^{(7)}| > 0 \Leftrightarrow L < L_{HH}$$

So in this case, Hubble horizon is equivalent to Killing horizon. But one may say about energy conservation only when there exists time-like Killing vector.

Energy in FLRW model may be conserved only inside Hubble horizon.

As the observable universe is greater than the Hubble sphere, not in all universe that one is able to see, one may say about energy conservation.

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Philosophical interpretation

Conservation of energy may be consider only in subset of events, which one may interact with. This subset of events is not equivalent with subset of events which one may observe.

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Horizon problem and inflation

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Horizon problem

There is a huge problem - why regions, which are causally disconnected, are homogeneous? There is no mechanism that set the same initial conditions at each point of the Universe!
But the Universe should have been long enough in such contact because we see it in CMB!

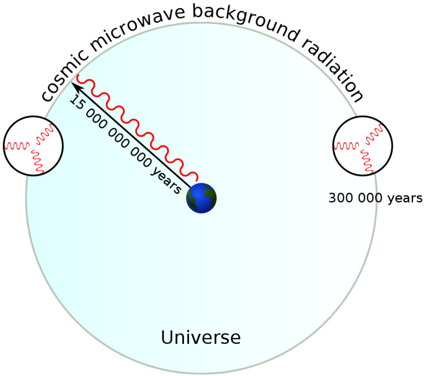

Figure: Wikipedia

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Solution = Inflation

Apart from horizon problem, there are three other ones:

- Flatness problem
- Large-scale structures
- No magnetic monopoles or other topological defects

The most popular solution of this problem is **inflation theory** - theory of exponential growth in the early Universe. It seems to be done due to excitation of hypothetical inflaton field.

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Summary

- Event horizon of a black hole describe hypersurface where where Schwarzschild metric is "singular"
- Rindler Horizon marks subspace of spacetime which is not in causal relation with an accelerated observer.
- Hubble sphere describes subspace of the observable universe where one may consider energy conservation.
- Horizon problem seems to be solved by inflation theory.

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Thank you for attention!

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